A Supermodular Location Game

Two players simultaneously choose possibly random dollar bets $x, y \in [0, 4]$, and each earns payoffs $u(x, y) = 12x - 11x^2 + 12xy^2 - 2xy^3$.

1. Do there exist any strictly mixed Nash equilibria?

   Solution: No. For each possibly random bet Y by the other player, one's own payoff function $u(x, y)$ is strictly concave in x. Thus, any mixture over bets is strictly less preferred than playing the expected bet.

2. Is this a supermodular game?

   Solution: Yes, since the cross partial derivative of payoffs is $u_{xy}(x, y) = (12y^2 - 2y^3)' = 24y - 6y^2 = 6y(4 - y) > 0$ if $y \in (0, 4)$.

3. Find all symmetric Nash equilibria.

   Hint: $6 - 11z + 6z^2 - z^3 \equiv (3 - z)(2 - z)(1 - z)$

   Solution: It suffices to restrict to pure Nash equilibria. The payoff FOC in x is $u_x(x, y) = 2[6 - 11x + 6y^2 - y^3] = 0$, for each y. In a symmetric Nash equilibrium, we have $x = y$, and so the FOC implies $0 = 6 - 11y + 6y^2 - y^3 = (3 - y)(2 - y)(1 - y)$. The three symmetric Nash equilibria are both betting $y = 1$ or $y = 2$ or $y = 3$.

4. Find all stable symmetric Nash equilibria. (Be careful!)

   Solution: Notably, all three equilibria $y = 1, 2, 3$ are stable.

   - if $x \gtreqless 3$ the marginal return to higher x is $u_x(x, y) \lesseqgtr 2[6 - 11 \times 3 + 6 \times 9 - 27] = 0$
   - if $x \gtreqless 2$ the marginal return to higher x is $u_x(x, y) \lesseqgtr 2[6 - 11 \times 2 + 6 \times 4 - 8] = 0$
   - if $x \gtreqless 1$ the marginal return to higher x is $u_x(x, y) \lesseqgtr 2[6 - 11 + 6 - 1] = 0$

   In other words, we do not alternate stable and unstable equilibria. The standard logic applies in finite action games.

5. What strategies are strictly dominated? Intuit which strategies are rationalizable. Argue formally, if you can, using monotonicity of a partial derivative of $u(x, y)$.

   Solution:

   - Every $x > 38/11 \approx 3.454$ is dominated, since $u_x(x, y) < 0$ on $[38/11, 4]$ — true given $u_x(x, y) = 2[6 - 11x + 6y^2 - y^3]$. For then $u_x(x, 4) = 2[6 - 11x + 32] < 0$ on $[38/11, 4]$ and $u_x(x, y)$ is increasing on $[0, 4]$, given $u_{xy}(x, y) = 6y(4 - y) > 0$.
   - Every $x < 6/11 \approx 0.545$ is dominated, since $u_x(x, 0) = 2[6 - 11x] > 0$ on $[0, 6/11]$, and $u_x(x, y)$ is increasing on $[0, 4]$

   The rationalizable strategies are $[1, 3]$. For later rounds of deletion, we continue to eliminate more strategies, moving right in $[0, 1)$ and moving left $(3, 4]$. For instance, on the second round:

- $x > 3.307$ is dominated. For $y^2(6-y) = (38/11)^2(6-38/11) \approx 30.377$ at $y = 38/11$, and $u_x(x, 38/11)/2 = [6-11x] + 30.377 = 0$ when $x \approx 36.377/11 \approx 3.307$.
- $x < 0.693$ is dominated. For $u_x(x,4) = 2[6-11x+32] < 0$ and $y^2(6-y) = (6/11)^2(6-6/11) \approx 1.623$, and $u_x(x, 6/11)/2 = [6-11x] + 1.623 = 0$ when $x \approx 7.623/11 \approx 0.693$.

This continues until we hit the Nash equilibria $y = 1$ from below and $y = 3$ from above. Proof omitted. Needless to say, iterated elimination of strictly dominated strategies coincides with rationalizability for two player games.