Econ 809 Problem Set 1

1. Assume a partial order on integers, where a<=b if a is a factor of b. What do we call the meet and join of two numbers?

Answer: Greatest common divisor and least common multiple

2. Assume a partial order of set inclusion on the set of all subsets of a finite set. Assume each element is a college you might apply to, k=1,2,…,N. Assume the function f(S) is the probability of gaining admission to some college s into the set $S \subset N$.

(a) What is the smallest sublattice that contains any three given schools i,j,k?
(b) Assume admission to college k has chance p_k, and admission events are independent. Is f supermodular, submodular or neither? Prove your claim succinctly!

Answer to (a): all subsets, *ordered by set inclusion*
Answer to (b): I confess my only argument here is to check for diminishing returns

$$(**)\ v(S+j) - v(S) \geq v(T+j) - v(T) \text{ if } S \subset T$$

and use Proposition 5.1 in Lovász (1982), noted in class and the paper. To see why (**) is true, notice that if T has higher ranked schools than j than S then one would take those schools over j. But conversely, S canNOT have any higher ranked schools than j than T does, since $S \subset T$.

3. Prove that the 0-1 indicator function I of

(a) intervals (-infty,a] or [a,infty) are log-supermodular on the reals
(b) rectangles [0,a] × [0,b] are log-supermodular in the northeast partial order in $[0,1]^2$

Solution of (a): This follows by considering cases for indicator of I(x,a) of either $x \leq a$ or $x \geq a$ for values a'<a". For the only way log-supermodularity fails is if I(x,a') =I(x,a")=1 and I(x,min (a',a")) =0 or I(x,max (a',a")) =0, which is impossible for either of those indicator functions.

Part (b) also proceeds by showing that the contrary inequality is impossible. Yuxin Jin's solution:

(b). The northeast partial order is defined by:

$$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1,\ x_2 \leq y_2$$

Given a rectangle $\Omega = [0,a] \times [0,b] \subset [0,1]^2$, we check whether the indicator function satisfies:

$$I_\Omega(x \vee y) \cdot I_\Omega(x \wedge y) \geq I_\Omega(x) \cdot I_\Omega(y)$$

If $x, y \in \Omega$, then both $\max(x_1, y_1) \leq a$ and $\max(x_2, y_2) \leq b$, so $x \vee y \in \Omega$.
Similarly, $\min(x_1, y_1) \leq a$ and $\min(x_2, y_2) \leq b$, so $x \wedge y \in \Omega$. Hence the inequality holds.
Therefore, the indicator function of a rectangle is log-supermodular in the northeast partial order.

4. Prove that $|x-y|^\gamma$ is submodular if $\gamma \geq 1$.

Solution: It is a theorem in this literature that if $g(x)=x^\gamma$ is convex, then $f(x,y)= g(x-y)$ is SBM – eg if differentiable, it follows by taking the cross partial derivative. Hereby I thank Yuxin Jin.

5. Suppose every player i=1,2,….,n+1 picks a random time t according to a cdf F(t) over $[0, \infty)$. Assume n players independently make their time choice. If your time is the kth highest you get payoff x_k, where $x_k - x_{k-1}$ changes sign $m<n$ times. Show that every player's expected payoff changes sign at most m times in t.

Solution: This was critical in my 2008 paper "Caller Number Five and related timing games". Sadly, I see that I need to assume a continuous cdf. I will screenshot the solution:

Consider a symmetric continuous strategy G. If $G(t)=g$,[9] then the expected payoff of a player who stops at time t, when all others stop according to G, is

$$\phi(g):=\sum_{k=0}^{N}\binom{N}{k}g^k(1-g)^{N-k}v(k+1).$$

Then in the appendix:

Lemma 4 (Variation diminishing property of expected rank rewards). *The slope-sign variations are ranked* $\mathrm{SV}(\Delta v)\geq \mathrm{SV}(\phi')$, *and* $\mathrm{SV}(\Delta v)-\mathrm{SV}(\phi')$ *is an even number. Further, the signs of the first and last slopes of v and ϕ coincide.*

Proof. The derivative of $\phi(g)$ in g can be expressed as follows:

$$\phi'(g)=\sum_{k=1}^{N}\binom{N}{k}kg^{k-1}(1-g)^{N-k}(v(k+1)-v(k)).$$

Assume that $\mathrm{SV}(\Delta v)=m$, i.e. the first difference $\Delta v(k)$ changes its sign m times. Scale ϕ' by $g/(1-g)^N$, and let $a_k:=k\binom{N}{k}(v(k+1)-v(k))$ and $z:=g/(1-g)$. Then

$$\frac{g}{(1-g)^N}\phi'(g)=\sum_{k=1}^{N}k\binom{N}{k}(v(k+1)-v(k))\left(\frac{g}{1-g}\right)^k=\sum_{k=1}^{N}a_kz^k=:P(z).$$

Obviously, $P(z(g))$ and $\phi'(g)$ enjoy the same number of sign variations, i.e. positive real roots of P. By Descartes' Rule of Signs, this number is at most the number of sign changes of its coefficients $a_0,a_1,\dots,a_N$. Also, if smaller, it is smaller by a multiple of 2. Thus $\mathrm{SV}(\Delta v)\geq \mathrm{SV}(\phi')$ and $\mathrm{SV}(\Delta v)-\mathrm{SV}(\phi')$ is even.

Finally, $\phi'(0)=\Delta v(1)$ and $\phi'(1)=\Delta v(N)$, proving the last clause. $\square$