Silent Timing Games

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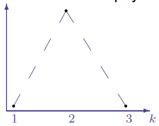
Spring, 2025

Fun Example: “Caller #5 & Related Timing Games”

- We study a “silent timing game”, i.e. a time committed to initially, and thus formally a normal form game
- Eg. 1 Radio stations have call-in shows to win concert tickets.
- Eg. 2 Three firms must decide long in advance when to enter a market with a new product $\Rightarrow$ formally simultaneous move
- Payoffs: prize minus delay costs
- Rank Payoffs (of “Caller Number 2”).
  - First & last firm into market get no prize; 2nd firm gets prize 1
  - If firms tie for a rank, they equally share the prize.
  - Story: Technology is not ready for 1st firm (e.g. 1993 Apple Newton), whereas network lock-in usually hurts the last firm

Caller Number Two: rank payoffs $(0, 1, 0)$



- Exogenous Delay Costs.
  - Entry at time $t \in [0, \infty)$ entails a delay cost t.
  - Delay raises a firm’s costs, since it must pay its idle workforce.
  - Generalizes “all-pay auction” (time = money bid on a good)

Equilibria of the Silent Timing Game

- First, stopping at any time $t > 1$ is a strictly dominated strategy: costs t exceed the prize
- Find all symmetric Nash equilibria, namely, cdf's $G(t)$ on $\mathbb{R}_+$
- We proceed constructively, & need no existence theorem. But:
  - Compact action space: Any time > 1 is dominated by time 0.
  - Since payoffs are continuous in times, Glicksberg applies!
- Equilibrium Case #1: Time-0 Jump
  - All firms jump in at time 0 and get payoff $1/3$.
  - In fact, a jump at any time $t \leq 1/3$ is a Nash equilibrium

Nash Equilibria of Caller Number Two

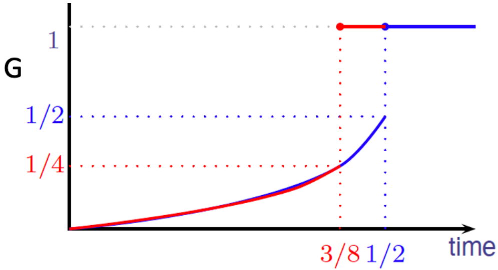
- Equilibrium Case #2: gradual entry followed by a jump.
- It is generally easiest to first solve for the jump size, and then for the continuous play, as we shall see
- With a common entry chance $\bar{G}$, the chances that 0 or 1 or 2 others have entered is (resp.) $(1-\bar{G})^2$, $2\bar{G}(1-\bar{G})$, and $\bar{G}^2$.
- The expected flow payoff reflects that no one enters at that instant:
$$\phi(\bar{G}) = 2\bar{G}(1-\bar{G})$$
- The jump payoff reflects that no one enters at that instant, but either two others enter, or one enters (3-way or 2-way tie):
$$J(\bar{G}) = (1-\bar{G})^2/3 + 2\bar{G}(1-\bar{G})/2$$
- Idea: since the jump occurs at the same time as the limit of times for gradual play, indifference requires equal prize payoffs in the jump and an instant before:
$$2\bar{G}(1-\bar{G}) = (1-\bar{G})^2/3 + 2\bar{G}(1-\bar{G})/2 \quad \Rightarrow \quad \bar{G} = 1/4$$

Gradual Play in Caller Number Two

- Since exiting at time 0 yields pays $\phi(0) = 0$, indifference at time t implies $\phi(G(t)) - t = 0$, or:

$$2G(t)[1 - G(t)] - t = 0 \quad \Rightarrow \quad G(t) = 1/2 - \tfrac{1}{2}\sqrt{1 - 2t}$$

- This is well-defined until $t = 1/2$, when $G(t) = 1/2$.
- *But we have already shown the jump occurs when* $G(t) = 1/4$



Famous Timing Games in Economic Literature

- Thomas Schelling (1978), *Micromotives and Macrobehavior*
  - agent-based model of neighborhood racial tipping
- Al Roth: sorority and employment matching rushes get earlier and earlier each year
- Stock market bubbles often end in a rush.
- Bank runs fall into our fear based rushes, and therefore transpire before the efficient time.
  - Seaman's Savings Bank Run in Financial Panic of 1857

RUN ON THE SEAMEN'S SAVINGS' BANK DURING THE PANIC.

Rushes in Large (Silent) Timing Games

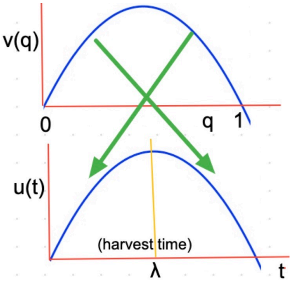
- The last timing game we considered was — until the end — an example of a war of attrition: *a fundamental reason to stop opposed by a strategic incentive to outlast other players.*
- The opposite setting is a pre-emption game, with a *fundamental reason to delay opposed by a strategic incentive to pre-empt (i.e. stop just before) the other players.*
- Rather than payoffs that depend on your rank, payoffs in stopping situations often depend on your stopping quantile Q.
- We now consider a timing game with a unit mass continuum of players and a continuum of actions $[0, \infty)$

★ The cdf Q jumps up iff $\exists$*rush* $\Leftrightarrow$ positive mass of agents stops

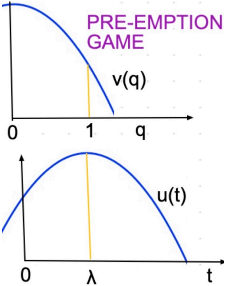
- Key: with continuum of players, no one impacts the game!
- This captures the essence of many economic situations, like:
  - Bank runs (pre-emption)
  - Neighborhoods tipping game, as in Schelling (pre-emption)
  - sorority and employment matching rushes (pre-emption)
  - Stock market bubbles, where mutual fund managers to "beat the average" (war of attrition)

How Payoffs Depend on Time and Quantile

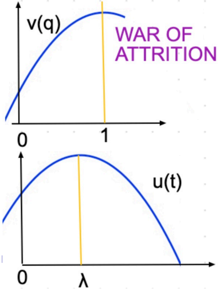
- Assume payoffs $u(t)v(q)$, where
  - fundamental factor $u(t) = 1 + 2\lambda t - t^2 = 1 + \lambda^2 - (\lambda - t)^2$
    - The fundamental peaks at the harvest time $t^* = \lambda$.
  - quantile factor $v(q) = (1 - q/\gamma)(1 + q/\rho)$
    - $v(q)$ peaks at $q^* = (\gamma - \rho)/2$.
    - Crucial: We assume $\rho + 2 > \gamma > \rho$, so that $0 < q^* < 1$.
- Strategy is a cdf quantile function $Q(t)$ (mass stopping by t)
- In a Nash equilibrium $Q(t)$, there are no profitable deviations
- **Lemma**: A rush must occur: $Q(t)$ must jump up at some t.
- Proof: See left picture. If $Q(t)$ continuously rises, indifference across gradual play times is impossible, as $0 < q^* < 1$ (v hump)



$\Rightarrow$

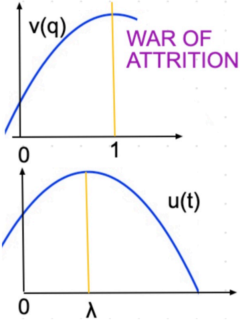


OR



War of Attrition with a Continuum of Players

- Just consider the easy cases: first, just a war of attrition.



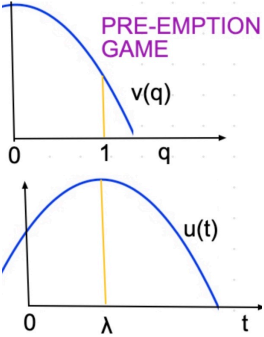
- Inspired by the Caller Number Two game, we conjecture gradual play while fundamentals worsen (and so *no play before the harvest time*), ended by an eventual terminal rush at $\bar{t} > \lambda$
- To avoid a profitable deviation, play starts at the harvest time
- For every time $t \in [\lambda, \bar{t}]$, we have $v(0)u(\lambda) = v(Q(t))u(t)$

$$1 + \lambda^2 = (1 - Q(t)/\gamma)(1 + Q(t)/\rho)(1 + 2\lambda t - t^2)$$

- Finally, solve for $Q(t)$ using the quadratic formula. Find $\bar{t}$.

Pre-Emption Game with a Continuum of Players

- Just consider the easy cases: first, just a pre-emption game.



- We conjecture the opposite equilibrium with an initial rush at $\underline{t} < \lambda$, and then gradual play until the harvest time, while fundamentals improve (and so *no play after the harvest time*)
- For every time $t \in [\underline{t}, \lambda]$, we have $v(1)u(\lambda) = v(Q(t))u(t)$

$$(1-1/\gamma)(1+1/\rho)(1+\lambda^2) = (1-Q(t)/\gamma)(1+Q(t)/\rho)(1+2\lambda t-t^2)$$

- Finally, solve for $Q(t)$ using the quadratic formula. Find $\underline{t}$.

Rushes with a Continuum of Players

- In a quantile jump from p to q, the quantile factor is $\int_q^p \frac{v(x)}{p-q} dx$
- In our war of attrition equilibrium, the end rush $[q_1, 1]$ obeys:

$$\frac{1}{1-q_1} \int_{q_1}^{1} (1 - x/\gamma)(1 + x/\rho) dx = (1 - q_1/\gamma)(1 + q_1/\rho)$$

  - The solution is $q_1 = (3(\gamma - \rho) - 2)/4$, if $\rho + 2/3 \leq \gamma \leq \rho + 2$.
- In our pre-emption equilibrium, the initial rush $[0, q_0]$ obeys:

$$\frac{1}{q_0} \int_0^{q_0} (1 - x/\gamma)(1 + x/\rho) dx = (1 - q_0/\gamma)(1 + q_0/\rho)$$

  - The solution is $q_0 = 3(\gamma - \rho)/4$, if $\rho \leq \gamma \leq \rho + 4/3$.

$\Rightarrow$ For $\rho + 2/3 \leq \gamma \leq \rho + 4/3$, both of these equilibria exist
  - If $\gamma > \rho + 2$, or $q^* > 1$, there is a war of attrition with no rush
  - If $\gamma < \rho$, or $q^* < 0$, there is a pre-emption game with no rush
- Oddness Theorem does not apply, as this is not a finite game!