Supermodular and Submodular Games

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Games with Strategic Complements and Substitutes

- Games with continuous actions can possibly be very complex
- There are two genres of well-behaved such games:

1. Supermodular Games $\iff$ Strategic Complements
    - Higher actions by others encourage higher best replies
2. Submodular Games $\iff$ Strategic Substitutes
    - Higher actions by others encourage lower best replies

Diamond Coconut Model (1982)

- People only eat coconuts, picked from palm trees at a cost.
- One cannot eat a coconut one has picked, but must trade it
- Climbing a coconut tree is worth more with more searchers
- We Have Multiple Equilibria?

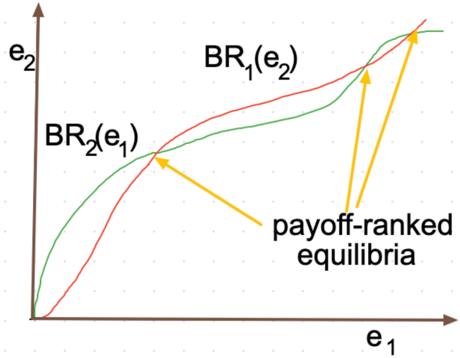
Diamond Coconut Model

- Agents $i = 1, 2, \ldots, I$ exert effort e_i looking for trade partners
- The chance of finding a partner is $e_i \sum_{j \neq i} e_j$
- The effort cost $c(e)$ and marginal cost $c(e)$ are increasing
- Thus, the payoff is $u_i(e_i, e_{-i}) = e_i \sum_{j \neq i} e_j - c(e_i)$
- positive spillovers: one's welfare rises in others' actions

$\Rightarrow$ Multiple Equilibria are Payoff-Ranked

Diamond Coconut Model with Two Players

- Payoff $u_i(e_1, e_2) = e_1 e_2 - c(e_i)$
- Best reply maps: $e_2 = c'(e_1)$ [BR_1] and $e_1 = c'(e_2)$ [BR_2]

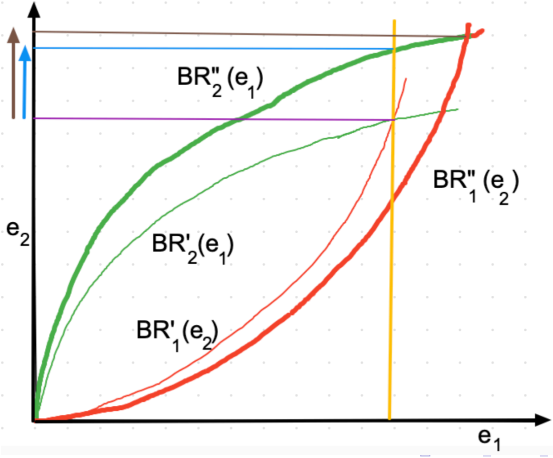


The Amplification Effect of Supermodular Games

- Assume two players, and quadratic marginal costs $c'(e) = e^2$
- Payoff $u_i(e_1, e_2) = \theta e_1 e_2 - c(e_i)$

$\Rightarrow$ i's FOC: $\theta e_j = c'(e_i) = e_i^2$, and so $BR_i(e_j) = \sqrt{\theta e_j}$

- Imagine a parametric shift from θ' to $\theta'' > \theta'$
- Amplification Effect: equilibrium shift $>$ private shift

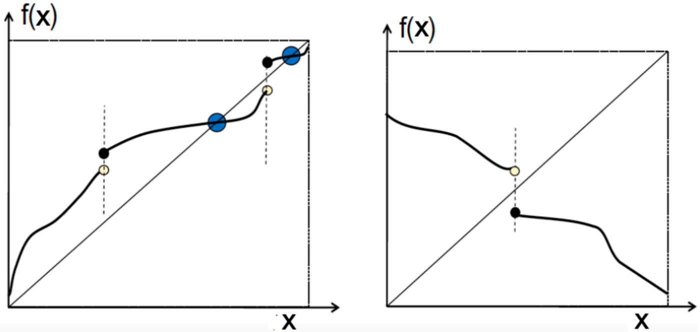


Tarski Fixed Point Theorem

Theorem (Tarski Fixed Point Theorem, 1955)

Let $(X, \succeq)$ be a complete lattice, and $f : X \to X$ a monotone function (i.e. order-preserving w.r.t. to $\succeq$). Then f has a fixed point $f(x) = x$, and the set of fixed points is itself a sublattice of X.

- The proof in wikipedia is quite good!
- Notably, the function need not be continuous: it can jump
- Tarski proved this not just for Euclidean domains, but for partially ordered functions on lattices.



Supermodular Games

- A supermodular game whose payoffs $u_i(s_i, s_{-i})$ have ID $\forall i$

Theorem (Maximum and Minimum Equilibrium)

Consider a supermodular game with continuous payoff functions $u_i(s)$ on a compact domain $\forall i$. Then there exists a maximum and minimum equilibrium.

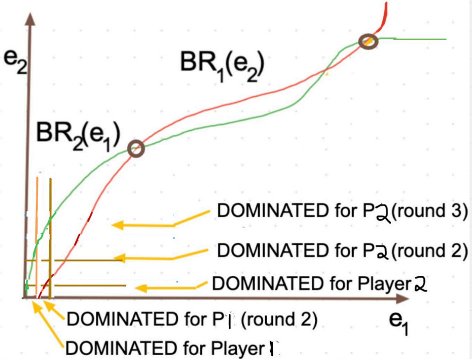
- Proof Step 1: By Topkis, the best response map $BR_i(s_{-i}) = \arg\max u_i(s_i, s_{-i})$ is nonempty, and has monotone max and min elements $\overline{BR}_i(s_{-i})$ and $\underline{BR}_i(s_{-i})$
- Proof Step 2: Apply Tarski Fixed Point Theorem to $f(s) = (\overline{BR}_1(s_{-1}), \ldots, \overline{BR}_I(s_{-I}))$ and $g(s) = (\underline{BR}_1(s_{-1}), \ldots, \underline{BR}_I(s_{-I}))$.

Supermodular Games and Diamond's Coconut Equilibria

Corollary (Iterated Elimination of Dominate Strategies)

In the above supermodular game:

- *Pure strategy equilibria exist*
- *The max and min equilibria are also max and min strategies surviving iterated elimination of dominated strategies.*
- *A game with a unique Nash equilibrium is dominance solvable.*
- *Proof Intuition:*



Differentiated-Good Bertrand Price Competition (1883)

- Given prices $p = (p_1, \ldots, p_I)$ of firms $i = 1, 2, \ldots, I$, demand is

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

where $a_i, b_i, d_{ij} \geq 0$, profits are $\pi_i(p) = (p_i - c_i) D_i(p_i, p_{-i})$

- This is supermodular, because $\frac{\partial^2 \pi_i(p)}{\partial p_i \partial p_j} \geq 0$
- The supermodular games existence proof works with discontinuous demand, as with pure Bertrand pricing $D_i(p_i, p_{-i}) = [a_1 - b_1 p_1] \mathbb{I}_{p_i < \min(p_i | j \neq i)}$

$\Rightarrow$ This was used to prove existence of equilibria in auctions.

Cournot Quantity Competition (1838)

- Demand function $P(q) = A - q_1 - q_2$ of quantities $q = (q_1, q_2)$
- Cost functions $C_1(q_1)$ and $C_2(q_2)$

$\Rightarrow$ Submodular game given profits $u_i(q_1, q_2) = q_i P(q) - C_i(q_i)$, namely,

$$\frac{\partial^2 u_i}{\partial q_1 \partial q_2} = -1 < 0$$

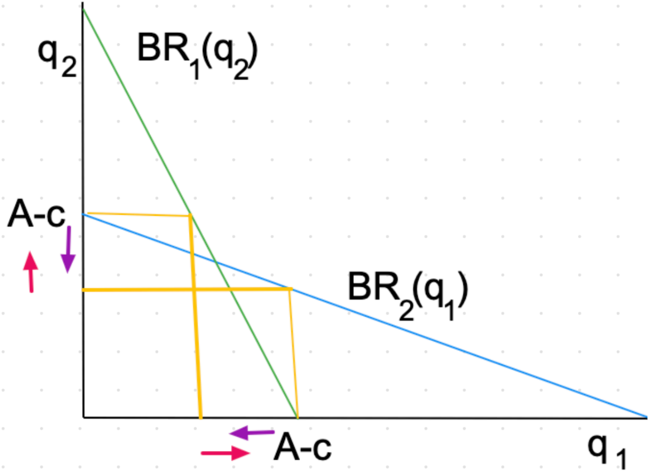
- But it is supermodular if Firm 2's strategy is $s_2 = -q_2$

$\Rightarrow$ Cournot Oligopoly cannot be rendered a supermodular game by this sign swap trick

$\Rightarrow$ Cournot Duopoly survives iterated dominance

Cournot Duopoly and Iterated Dominance

- For linear costs $C_i(q_i) = cq_i$, the FOC is $A - 2q_i - q_j - c = 0$



Submodular Games

- $f(x, \theta)$ has decreasing differences if $f(x, -\theta)$ has ID
- A submodular game, or game of strategic substitutes is one whose payoffs $u_i(s_i, s_{-i})$ have decreasing differences $\forall i$
- Examples of submodular games have a win-lose flavor:
  - Sharing a pie
    - Cournot quantity competition shares demand
    - Bargaining over a pie (example to come)
  - Displacing effort in group projects or preventing accidents:
    - Vigilance in avoiding contagious diseases
    - Vigilance in auto accident prevention
- Supermodular games involve win-win games (coordination, cooperation, matching), or lose-lose games (competition)
  - win-win
    - Trust games, eg. financial (2008 Financial Crisis, corruption)
    - Price competition in Bertrand competition
  - lose-lose
    - Effort in classes with belled grades
    - Vigilance effort in avoiding counterfeit money

Submodular Games and the Attenuation Effect

- Attenuation Effect: The Equilibrium effect of a parameter change is less than the private effect
- a "shock absorber"

The Attenuation Effect in Cournot Duopoly

- Demand function $P(q) = A - q_1 - q_2$ and marginal cost $c > 0$
- The FOC is $A - 2q_i - q_j - c = 0$
- What happens if demand rises: $\Delta A > 0$

$\Rightarrow$ Private Effect: $q_2 = (A - c - q_1^*)/2 \Rightarrow \Delta q_2 = \frac{1}{2}\Delta A$

$\Rightarrow$ Equilibrium effect: $q^* = (A - c)/3 \Rightarrow \Delta q^* = \frac{1}{3}\Delta A$