Blackwell's Theorem with the Original Wonderful Proof

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- $\Omega = \{\omega_1, \ldots, \omega_n\}$, states of the world
- $X = \{x_1, \ldots, x_N\}$, experiment/signal outcomes
- $(\mu_1, \ldots, \mu_n)$, probability measures representing an experiment, represented by the Markov matrix of probability densities $P_{n\times N} \equiv [p_{ij}]$, namely where $\sum_{j=1}^{N} p_{ij} = 1$ and $0 \le p_{ij} =$ chance of x_j in state ω_i
- $A \subset \mathbb{R}^n$, action space, assumed for simplicity to be the vector of payoffs: $a \in A$ is the n-vector of losses/payoffs in each state, i.e. $a_i =$ loss in state ω_i
- $f : X \to A$, the decision function, i.e. $f(x_j) \in A$ is the action taken after outcome x_j
- $v_i(f) \equiv \int_X f_i(x) d\mu_i(x) \equiv \sum_{j=1}^{N} p_{ij} f_i(x_j) =$ expected loss/payoff from f in state ω_i
- $B(P, A) \subset \mathbb{R}^n$, the range of all loss vectors $v(f) = (v_1(f), \ldots, v_n(f))$

Define

⋆ Experiment $P_{n\times N_1}$ is *more informative than* experiment $Q_{n\times N_2}$ [written $P \supset Q$], if any payoff vector attainable with P is also attainable with Q, i.e. $B(P, A) \supseteq B(Q, A)$ *for all* compact convex subsets $A \subset \mathbb{R}^n$.

⋆ Experiment P is *sufficient for* Q [written $P \succ Q$], if $PM = Q$ for some Markov matrix M, i.e. $q_{ij} = \sum_{k=1}^{N_1} p_{ik} m_{kj}$ for all $j = 1, \ldots, N_2$ and $i = 1, \ldots, n$

Proposition (Blackwell's Theorem) *$P \succ Q$ iff $P \supset Q$.*

Proof: ($\Rightarrow$) Assume $P \succ Q$. To show that $P \supset Q$, it suffices that any point in $B(Q, A)$ attainable with a decision function g is attainable under P. In fact, this is possible using the decision function $f(x_k) = (\sum_{j=1}^{N_2} m_{kj} g(y_j))$. Why? The payoff under P in state ω_i is

$$v_i(f) = \sum_{k=1}^{N_1} p_{ik} f_i(x_k) = \sum_{k=1}^{N_1} p_{ik} \sum_{j=1}^{N_2} m_{kj} g_i(y_j) = \sum_{j=1}^{N_2} q_{ij} g_i(y_j) = v_i(g)$$

($\Leftarrow$) Assume $P \supset Q$. The inclusion $B(P, A) \supseteq B(Q, A)$ holds for any $A \subset \mathbb{R}^n$ compact and convex, so in particular let A be the convex hull of the rows of the $N_2 \times n$ matrix D (i.e. the payoffs in each state after each outcome). Hence, the decision function f in problem (Q, A) selects the jth row of D when x_j is observed has value $v_i(f) = \sum_{j=1}^{N_2} q_{ij} d_{ji} = (QD)_{ii}$.

Since $P \supset Q$, there is a decision function g for (P, A) that selects $a^j \in A$ when x_j is observed, with $v_i(g) = \sum_{j=1}^{N_1} p_{ij} a_i^j = v_i(f)$ for all i. Since $a^j \in A$, there is a Markov matrix $\hat{M} \equiv [\hat{m}_{jk}]$ with $a_i^j = \sum_{k=1}^{N_2} \hat{m}_{jk} d_{ki}$. For simply let $\hat{m}_j$ be the vector of *Barycentric coordinates* of a^j in the polytope A. Then $P\hat{M}D$ and QD have the same diagonal entries:

$$v_i(g) = \sum_{j=1}^{N_1} p_{ij} a_i^j = \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} p_{ij} \hat{m}_{jk} d_{ki} = (P\hat{M}D)_{ii}$$

Consider the game pitting the decision-maker against nature. Nature chooses the payoff matrix D and the decision-maker chooses the Markov matrix M. The game is constant-sum, and nature receives the payoff $\Pi(D, M) = tr[(PM - Q)D]$, where $tr[C]$ denotes the trace of the matrix C (sum of the diagonal entries). WLOG, assume that all entries in D lie in $[0, 1]$, since any other matrix can be so normalized. Since the strategy sets are compact and convex in $\mathbb{R}^{N_2 n}$ and $\mathbb{R}^{N_1 N_2}$, respectively, and Π is bilinear, the *Minimax Theorem* yields a saddle point (D_0, M_0) for the game: $\Pi(D, M_0) \leq \Pi(D_0, M_0) \leq \Pi(D_0, M)$ for all feasible M and D. Picking $M = \hat{M}$, we have $\Pi(D_0, M_0) \leq \Pi(D_0, \hat{M}) = 0$, since $P\hat{M}D$ and QD have the same diagonal entries, and hence the same trace. Thus, $tr[(PM_0 - Q)D] = \Pi(D, M_0) \leq 0$. Since D can be chosen with *arbitrary* entries in $[0, 1]$, and since PM_0 and Q are both $n \times N_2$ Markov matrices (nonnegative entries, with row sums 1), all their entries must be identical. That is, $PM_0 = Q$, and so $P \succ Q$. QED