Economic Theory 713A

Economics of Markets

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February 4, 2025

Wisconsin

The Matching Paradigm as Metaphor Economic Interaction

- Simple model: Only the extensive margin (in or out) matters.

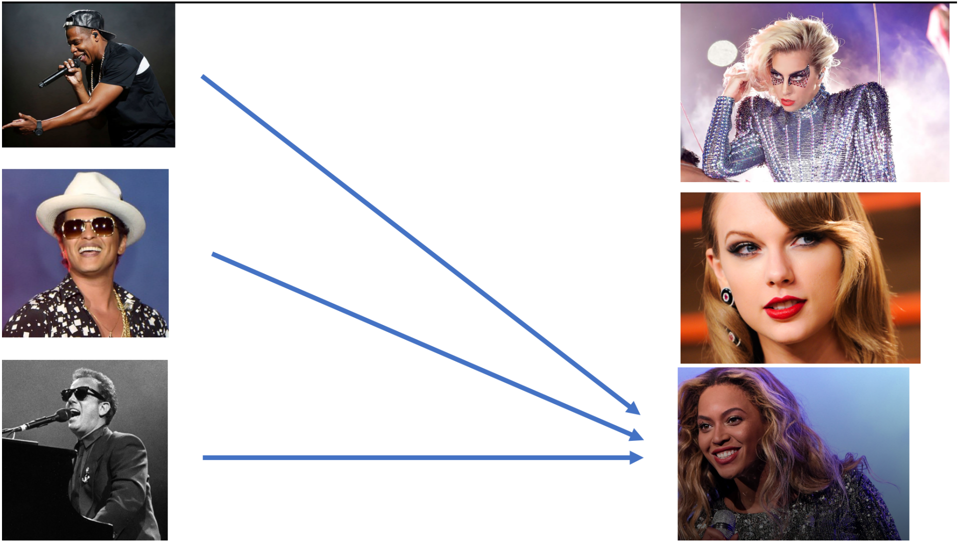
- Pairwise matching models with transferable utility capture in a simple story the economic structures of many settings:
  - assigning tasks to individuals
  - buyers and sellers trading
  - partnerships, and maybe marriages
  - firms hiring workers
- metaphor: two sides of the market are “men” and “women”
- We wish to understand: Who trades with whom? Who pairs with whom? Who marries whom? Who works with whom?

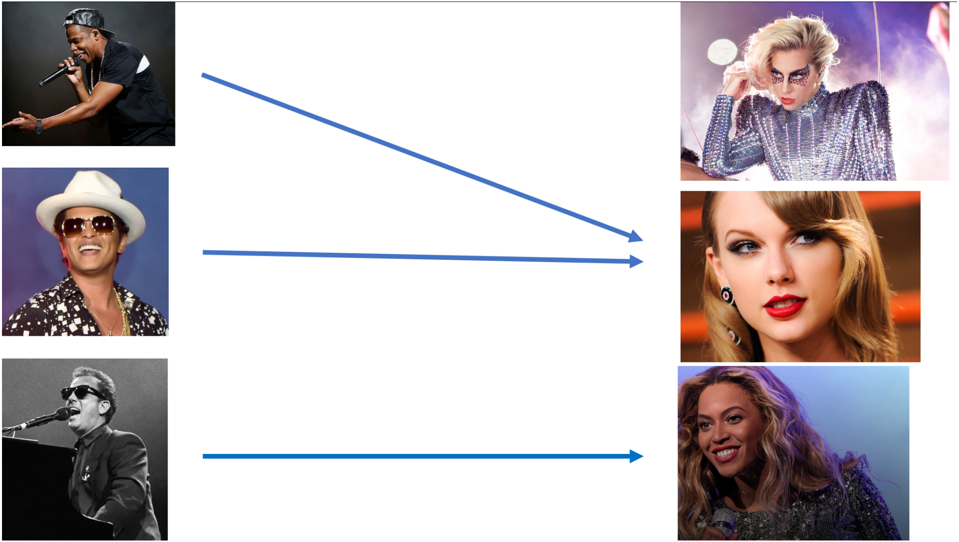
Matching without Transfers: The Girl-Guy Band Contest

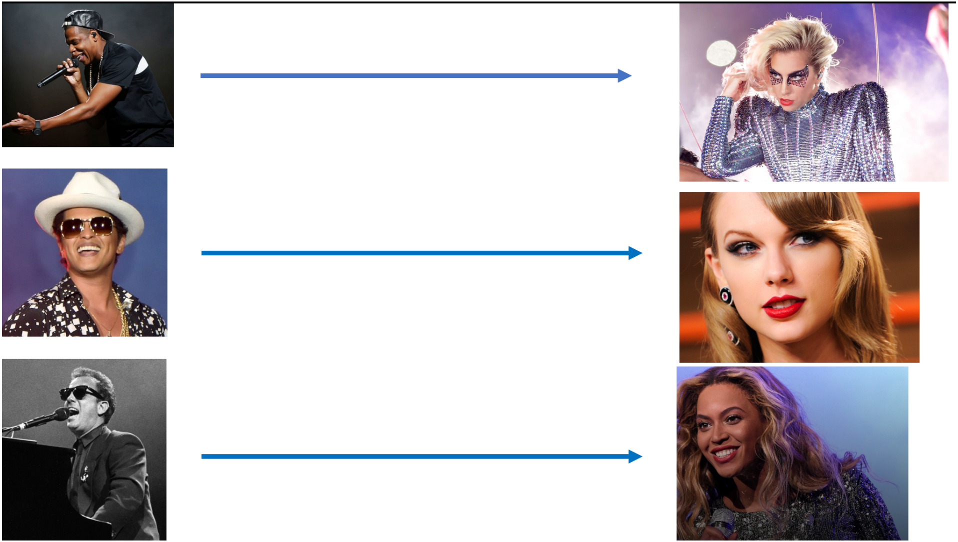
- Contest of Beyonce, Taylor Swift, and Lady Gaga to sing a duet with concert with Billy Joel, Bruno Mars, and Jay-Z
- We first only specify **ordinal preferences**
- Men commonly rank: Beyonce $>$ Taylor Swift $>$ Lady Gaga
- Women commonly rank: Billy Joel $>$ Bruno Mars $>$ Jay-Z

Stable Predictions for Pairwise Matchings

- Matchings must survive new double coincidence of wants
- An assignment is *unstable* if there are men, say Alan and Bob, respectively matched to women Alice and Bea, such that Bob prefers Alice to Bea and Alice prefers Bob to Alan
- Say that the matching of Bob and Alice blocks the matching.
- A matching is *stable* if it is not unstable, i.e. $\nexists$ blocking pair.







Deferred Acceptance Algorithm (DAA)

- Men have preferences over all women and not matching, and women have preferences over all men and not matching

1. All men start unengaged and women start with no suitors.
2. Each unengaged man *proposes* to his most-preferred woman (if any) among those he has not yet proposed to, if better than staying single.
3. Each woman gets *engaged* to the most preferred among all her suitors, including any prior engagements, if she prefers matching with him to remaining single.
4. Rinse and repeat until no more proposals are possible. Engagements become matches.

"I have to go. I'm getting a better call."

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MATHEMATICAL MONTHLY

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Proposition (Gale & Shapley, American Math Monthly, 1962)

(a) *Then the DAA stops in finite time.*
(b) *Given an equal number of men and women, if matching with someone beats remaining single, then everybody matches.*
(c) *The DAA matching is stable, i.e. a stable matching exists.*
(d) *Given strict preferences, the DAA yields a unique matching.*

Proof of Gale-Shapley Th'm: WHY does DAA ⇒ Stability?

- At each iteration, one man proposes to some new woman
- Let Alice and Bob be married, but not to each other.
- **Claim**: *After the DAA, Alice and Bob cannot prefer each other to their match partners.*
  - If Bob prefers Alice to his match partner, then he must have proposed to Alice before his match partner.
  - If Alice accepted, yet ends up not married to him, then she must have dumped him for someone she prefers
    $\Rightarrow$ Alice doesn't prefer Bob to her current partner.
  - If Alice rejected Bob's proposal, then she was already engaged to someone she prefers to Bob. □
- The contradiction proves the theorem!
- The paper's theorem includes many-to-one school matching
- Gale-Shapley (1962) was the 2nd *market design* paper — after Vickrey (1961), introducing second price auctions

Gale-Shapley Theorem

- The band matching example was trivial: When n men and n women had the same preference ranking, it ends in n rounds.

Theorem (Gale and Shapley, Itoga (1978))

With n men and n women, the DAA ends in at most $n^2 - 2n + 2$ steps

- **Lemma**: *In the DAA, one man (say Joey) gets his worst woman and $n-1$ men end up with their second worst.*
  - So there are n proposals in round 1.
  - We then proceed with one proposal per round:
  - $n-1$ more rounds for Joey's courtship (Lemma)
  - plus $n-2$ rounds each more for the other $n-1$ men
  - for a total of $1 + (n-1) + (n-1)(n-2) = n^2 - 2n + 2$ rounds.
- Exercise: Illustrate this for the cases $n = 2$ and $n = 3$.
- Proof is in Itoga (1978), on canvas

- Al Roth found that the DAA was used to match interns to hospitals.
- This was a major reason for:

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

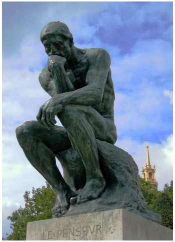
Photo: U. Montan
Alvin E. Roth
Prize share: 1/2

Photo: U. Montan
Lloyd S. Shapley
Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

- Sad Note: David Gale died in 2008.

Thinker: The Stable Roommates (i.e. Unisex) Problem



	Ann	*Beth*	*Cathy*	*Dee*
Ann	–	1	2	3
Beth	2	–	1	3
Cathy	1	2	–	3
Dee	1	3	2	–

- These are the *ranks* of each person over partners
- Show there is no stable allocation. Proof on wikipedia.
- Hint: If a stable allocation exists, someone rooms with Dee.
- Crucially, the DAA does not apply to the unisex model!

Gale and Shapley's Ranking of Stable Matchings

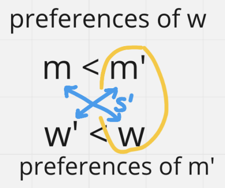
- Assume matching by women x and men y (from XX and XY)
- The set of stable matchings is nonempty.
- x is a **valid partner** of y if they pair in some stable matching.
  - **Male optimal**: each man pairs with best valid partner.
  - **Male pessimal**: each man pairs with worst valid partner.
  - Similarly define **woman-optimal** and **woman-pessimal**.

Proposition (Male Optimality of DAA)

The DAA finds a male-optimal / female-pessimal stable matching.

- *As in Rubinstein's bargaining model, there is a proposer advantage.*

Off Line: Tricky Proof that DAA is Male Optimal



- Proof by contradiction: If the DAA matching S is not male optimal, then a valid partner rejects some man, since men propose in order

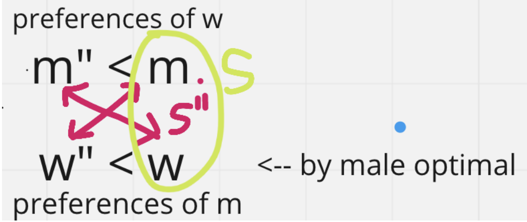
(★) Let m and w be the *first* such rejection in S

- This happens because woman w chose some man $\boxed{m' \succ_w m}$
- (m, w) paired in a stable matching S', since (m, w) is valid
- In stable matching S', let man m' pair with woman w', say
  - Note: m' was not rejected by a valid woman in S before (★)
  - If $w' \succ_{m'} w$ then m' offers to w' first, and must have been rejected if he was available to w, negating "1st" proviso in (★)

  $\Rightarrow$ $\boxed{w \succ_{m'} w'}$

$\Rightarrow$ m' and w form a blocking pair in S'

Off Line: Tricky Proof that DAA is Female Pessimal



- The proof is by contradiction
- Let m and w pair in the DAA matching S, and assume (for a contradiction) that m is not the worst valid partner for w

$\Rightarrow$ $\exists$ a stable matching S'' with w paired to m'', and $\boxed{m \succ_w m''}$

- In matching S'', let man m pair with woman w'', say
- $\boxed{w \succ_m w''}$ by male-optimality

$\Rightarrow$ m and w form a blocking pair in S''

3 Stable Matchings, but DAA Logic Can Only Get Two

	x_1	x_2	x_3
y_1	5,5	6,2	2,6
y_2	2,6	5,5	6,2
y_3	6,2	2,6	5,5

- The default DAA yields the male-optimal and female pessimal matching, where men earn 6 and women 2.
- In the DAA', women do the proposing, rather than men.

$\Rightarrow$ By the above reasoning, DAA' yields the female-optimal and male pessimal matching, where women earn 6 and men 2.

- A third stable matching yields payoffs of 5 for everyone.

Unique Stable Outcomes

- DAA': women do the proposing, rather than men.

Corollary (Uniqueness)

DAA and DAA' yield the same matching if and only if there is a unique stable matching.

- Since DAA and DAA' yield a stable matching, if the stable matching is unique, DAA and DAA' land at same matching
- If DAA and DAA' land at the same matching, then it is both optimal and pessimal for men, and so is unique. □

Proposition (Roth, 1982)

DAA is incentive compatible for men, and DAA' for women.

- Proof is omitted, since it is game theory.
- *But women might gain by misreporting their types in DAA.*
- OFFLINE Example:
  - Man A prefers [X] to Y to Z, and Man B prefers [X] to Z to Y
  - Man C prefers [Y] to X to Z
  - Woman X prefers [C] to A to B, and Woman Y prefers A to C.
- DAA: Men A & B propose to #1 woman X, and Man C to Y
- X retains A, and B proposes to Z next. Proposals end.
- In the end, X is matched to A
- Machiavellian Deviation by X:
  - X sneakily accepts B's proposal.
  - Then A proposes to Y.
  - Y leaves C for A.
  - Then C proposes to X
  - X ends up matched to top choice C

Cardinal Preferences

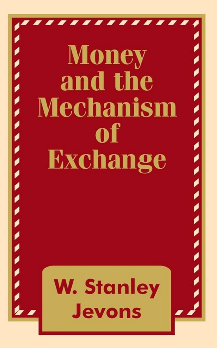
- Start with nontransferable payoffs (all in millions of dollars).
- This might be an organizational rule, eg. NCAA rules used to forbid payoffs to athletes.

♂ \ ♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	6,21	12,12	**18,3**
Bruno Mars	4,14	**8,8**	12,2
Jay-Z	**2,7**	4,4	6,1

- Men commonly rank: Beyonce > Taylor Swift > Lady Gaga
- Women commonly rank: Billy Joel > Bruno Mars > Jay-Z
- DAA ends in three periods!

Matching with Transfers

- Assume cardinal payoffs (or cardinal utility) is money.
- Every man and woman cares only about total money
- This is a special case of quasilinear utility, or utility $U(a, z) = u(a) + z$, where a is a real action and z is money
- Quasi-linear utility precludes income effects on the action
- All fields assume quasilinear utility as a default
- Jevons (1875): Money solves the "double coincidence of wants"



Transfers and Bribery

Lady Gaga's Corrupt Thought:

- Gaga schemes to match up with Billy Joel. To do this, she
  - bribes Billy more than his loss of $18 - 6 = 12$ to accept her,
  - pays Beyonce more than her loss of $3 - 1 = 2$, and
  - collects from Jay-Z less than his gain $6 - 2 = 4$ from match with Billy
- These bribes on net cost as much as $12 + 2 - 4 = 10$. But Lady Gaga gains $21 - 7 = 14$ by matching with Billy Joel.
- We start with this matching

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	6,21	12,12	**18,3**
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Making Matching Immune to Bribery

- The bribery scheme's profitability only depends on total match payoffs

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Jay-Z	$6+21=27$	$12+12=24$	$18+3=21$
Bruno Mars	$4+14=18$	8+8=16	$12+2=14$
Billy Joel	$2+7=9$	$4+4=8$	$6+1=7$

Making Matching Immune to Bribery

- The bribery scheme's profitability only depends on total match payoffs
- Now, the cardinal strength of each party's preference matters.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	27	24	21
Bruno Mars	18	16	14
Jay-Z	9	8	7

Making Matching Immune to Bribery

- A matching is **immune to bribes** if there is no set of matched individuals for whom a profitable re-matching exists.
- An **efficient** matching maximizes the sum of payoffs.

My Theorem *An efficient matching is immune to bribes.*

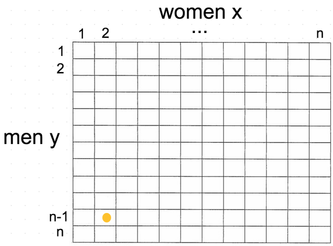
- Proof: If some bribery scheme is profitable, then rematching those people raises total match output.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	**27**	24	21
Bruno Mars	18	**16**	14
Jay-Z	9	8	**7**

- The sum of payoffs is now $27 + 16 + 7 = 50 > 46 = 9 + 16 + 21$

Efficient Matching

- Matching Sudoku: Efficiently match n men to n women.
- $=$ Place exactly one dot in every row and column



- Obviously, an efficient matching exists. But what is it?
- Problem: There are $n! = 1 \times 2 \times \cdots \times n$ possible allocations.
- E.g. there are 10^{158} pairings of 100 men and 100 women.
  The number of electrons in the universe is estimated at 10^{80}.

Historical Background: “Transportation Problem” (1781)

666· MÉMOIRES DE L’ACADÉMIE ROYALE

MÉMOIRE

SUR LA

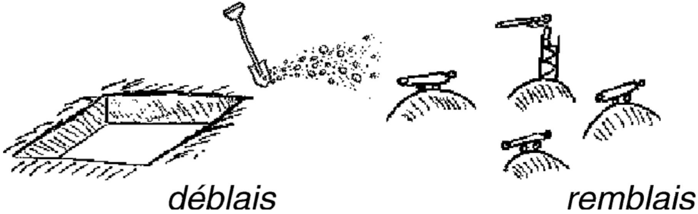
THÉORIE DES DÉBLAIS

ET DES REMBLAIS.

Par M. MONGE.

1781 — Transportation Problem: How Best to Move Dirt

- Transportation problem: a classic resource allocation problem
- The cost $c(x, y)$ of moving dirt from a cut (déblais) x to to a fill (remblais) y depends on the distance, roads, etc.
- Assign unit dirt piles $x_i \in \{x_1, \ldots, x_n\}$ to holes $y_j \in \{y_1, \ldots, y_n\}$ to minimize the sum of transportation costs $c(x_i, y_j)$?

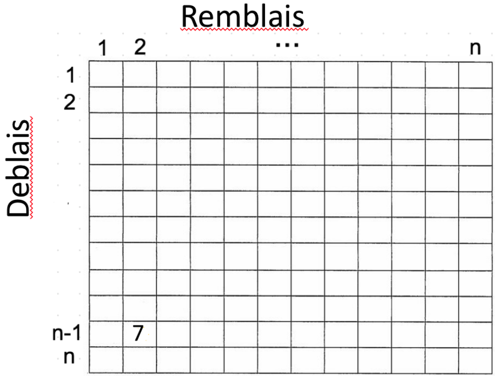


- What is the cheapest way to transport all dirt from each déblais to some remblais, while omitting no déblais and overfilling no remblais?
- As formulated, this is an impossible combinatorics exercise.

1781 — The Transportation Problem

- Start with an $n \times n$ matrix of costs $[c(x_i, y_j)]$
- E.g: It costs 7 to move the dirt in déblais $n-1$ to remblais 2
- Solve the *minimization* $\sum_{i=1}^{n} c(x_i, y_{i\text{'s partner}})$
  - Maximizing payoffs is the same as minimizing negative payoffs
- The problem is doomed with combinatorial math methods.

⇒ Lesson: *Need to reformulate the story to make it solvable!*



1957: Transportation Problem as the Assignment Problem

- 160 years passes and linear programming is invented in WWII, by many in USA (e.g. Dantzig) and Kantorovich in Russia
- *The TU matching story is so great (i.e. general) it also captures the assignment model (& other economic models!)*

Assignment Problems and the Location of Economic Activities

Tjalling C. Koopmans; Martin Beckmann

Econometrica, Vol. 25, No. 1 (Jan., 1957), 53-76.

$$\text{Plants}\ \begin{matrix}1\\2\\3\\4\end{matrix}\ \overset{\text{Locations}}{\overset{\begin{matrix}1 & 2 & 3 & 4\end{matrix}}{\begin{bmatrix}25 & \mathbf{20} & 5 & 19\\ 18 & 3 & \mathbf{0} & 12\\ \mathbf{22} & 4 & 2 & 12\\ 16 & 7 & -2 & \mathbf{10}\end{bmatrix}}} \qquad \text{Plants}\ \begin{matrix}1\\2\\3\\4\end{matrix}\ \overset{\text{Locations}}{\overset{\begin{matrix}1 & 2 & 3 & 4\end{matrix}}{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}}$$

3. AN EQUIVALENT LINEAR PROGRAMMING PROBLEM

This problem is obtained by blandly ignoring the indivisibilities of plants, and admitting the assignment of *fractional plants* to locations in our model even though this is supposed to be meaningless from a realistic point of view.

- Give polyhedron intuition why some vertex is optimal

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975
Leonid Vitaliyevich Kantorovich, Tjalling C. Koopmans

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The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975

Leonid Vitaliyevich Kantorovich
Prize share: 1/2

Tjalling C. Koopmans
Prize share: 1/2

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Koopman's Idea: Convexify the Feasible Matchings Space

- Choices or Actions
- *Finitely* many women x and men y (from XX and XY)
  - $m(x, y) = 1$ if x is matched to man y, and $m(x, y) = 0$ if not.
  - So a woman x remains single if $m(x, y) = 0$ for all $y \in Y$.
- Matching Space $\mathcal{M} = [m(x, y)]$ are all *feasible* matchings
  - $\mathcal{M}$ is *symmetric*: $m(x, y) = m(y, x)$ for all x, y
  - $\mathcal{M}$ is *convex* provided:
    - A fraction $m(x, y) \geq 0$ of woman x matches with man y
    - Or, with a continuum mass of men and women of finitely many types $\{x_i, y_j\}$, a mass $m(x, y)$ of types x and y match.
  - $\mathcal{M}$ is *bounded* (*no overmatching* any man or woman)
    - Finite world: for every x, $m(x, y) = 1$ for at most one y, and for every y, $m(x, y) = 1$ for at most one x.
    - Convex world: $\sum_y m(x_0, y), \sum_x m(x, y_0) \leq 1 \ \forall x_0, y_0$

Socially Efficient Matching with Transferable Utility

- $h(x, y)$ = match payoff of man x and woman y
  - Normalize unmatched payoff to zero: $h(x, \varnothing) = h(\varnothing, y) = 0$
- A **(socially) efficient matching** $[m^*(x, y)]$ maximizes the sum of all match outputs $\sum_x \sum_y m(x, y)h(x, y)$ over $m \in \mathcal{M}$

Proposition (Existence)

An efficient matching $m \in \mathcal{M}$ exists.

- Proof: By Weierstrass Theorem, the maximum of a continuous function (the sum) on a compact set exists
- Compactness is trivial with finitely many types.
  - With a type continuum, we need weak-* topology. (hard)

Rear View Mirror on Last Class

I. Ordinal preferences in a matching model of 'men' & 'women'
- DAA leads to stable matching (no blocking pairs)
  - Men are willing to report preferences to a DAA machine.
  - Women can sometimes game these algorithms
- With more than one stable matching, we claimed (no proof):
  - Men all agree ranking stable matchings. So do women.
  - DAA gives the male optimal and female pessimal matching

II. We shifted to cardinal preferences with monetary transfers.
- Our stable allocation might be destabilized by bribes.
- Efficient allocation (max match payoff sum) cannot be bribed.
- Transportation Problem: impossibly hard as combinatorics but solvable with convexity and linear programming (in progress)

Decentralizing the Matching Market with Middlemen

- Payoffs: We derive **wages** $v(x)$ & $w(y)$ of women x & men y
- *Middlemen* compete in wages, earning *profits* for (x, y) match:

$$h(x,y) - v(x) - w(y)$$

- Free exit of middlemen $\Rightarrow$ profits ≥ 0 for all actual matches

$$v(x) + w(y) \leq h(x,y) \text{ if } m(x,y) > 0$$

  - I.e. No one is forced to stay in a market (obvious IR constraint)
- Free entry of middlemen $\Rightarrow$ profits ≤ 0 for all matches

$$v(x) + w(y) \geq h(x,y) \text{ for any } (x,y)$$

  - i.e. No profitable opportunity goes unexploited!
  - Without free entry, middleman market is not competitive
- A **competitive equilibrium** (m, w, v) satisfies feasibility and:

$$\Rightarrow \quad v(x) + w(y) \begin{cases} \geq h(x,y) & \text{for all women and men } x, y \\ = h(x,y) & \text{if } x, y \text{ are matched.} \qquad (\star\star) \end{cases}$$

$\Rightarrow$ Unmatched x or y earn zero wage: $v(x) = 0$ or $w(y) = 0$ $(\star)$

  - This is intuitive now. We will prove it soon.

Coordinated Middlemen?

- Lady Gaga arranged the bribes, but anyone could have!
- Middlemen — real or metaphorical — determine prices
- Matching is decentralized! Everyone picks the maximum wage offer
- Google is a massive middleman making huge profits due to barriers to entry of middlemen

Google loses massive antitrust case over its search dominance

UPDATED AUGUST 5, 2024 · 7:40 PM ET

By The Associated Press

Competitive Equilibrium is Efficient

Proposition (First Welfare Theorem of Matching)

If (m, v, w) is a competitive equilibrium, then m is an efficient matching

- Proof: By contradiction, let (m, v, w) be an inefficient competitive equilibrium $\Rightarrow \exists$ feasible matching $\hat{m} \in \mathcal{M}$ with a higher payoff: (2)

$$\begin{aligned} \sum_x v(x) + \sum_y w(y) &\geq \sum_y \sum_x h(x,y)\hat{m}(x,y) && (1) \\ &> \sum_y \sum_x h(x,y)m(x,y). && (2) \\ &= \sum_y \sum_x [v(x) + w(y)]m(x,y) && (3) \\ &= \sum_x v(x) + \sum_y w(y) && (4) \end{aligned}$$

- Inequality (1)
  - $\Leftarrow$ free entry: For $v(x) + w(y) \geq h(x,y)$ for all (x,y)
  - $\Leftarrow$ feasibility: $1 \geq \sum_x \hat{m}(x,y)\ \forall y$ and $1 \geq \sum_y \hat{m}(x,y)\ \forall x$
- Inequality (3) $\Leftarrow$ Free exit (★★)
- Equality (4) $\Leftarrow$ Complementary slackness (later on)

(CS) $v(x) = 0$ if $\sum_y m(x,y) < 1$ and $w(y) = 0$ if $\sum_x m(x,y) < 1$

  - Eg if x and y are unmatched, the constraint does not bind

Contrast to a Stable Matching without Transfers

	Y_1	Y_2
X_1	2,0	**0,7**
X_2	**0,7**	2,0

	Y_1	Y_2
X_1	2	**7**
X_2	**7**	2

- At left, are the male and female optimal stable outcomes.
- The male optimal one is efficient (highest total payoffs)
- But stability only reflects ordinal, *and not cardinal*, preferences.
- Let's see how market wages force the efficient matching
- Middlemen compete to offer 7 for the matches (X_2, Y_1) and (X_1, Y_2)
- If outside options are zero, competitive wages $v_1, v_2, w_1, w_2 \geq 0$ obey:

$$\begin{array}{ll} v_1 + w_1 \geq 2 & v_1 + w_2 = 7 \\ v_2 + w_1 = 7 & v_2 + w_2 \geq 2 \end{array}$$

- Crucially, *there are many competitive equilibrium wages*
  - Example: $v_1 = 5, v_2 = 0, w_1 = 7, w_2 = 2$

Trading Houses (Shapley and Shubik, 1971)

- A good economic story allows many interpretations of its formulation!
- Men and women can be metaphors for buyers and sellers
- *The transferable utility matching model captures trading among buyers (men) and sellers (women)!*
- $I \geq 1$ sellers (homeowners) and $J \geq 1$ prospective buyers.
- $c_i > 0$ is opportunity cost of i-th seller for his house (i.e. his value)
- $\xi_{ij} > 0$ is value of j-th buye for seller i's house
- If $\xi_{ij} > c_i$, and seller i sells his house to buyer j for price p_i, then seller i's payoff is $p_i - c_i$ and buyer j's is $\xi_{ij} - p_i$ (quasilinear utility).
- If $\xi_{ij} < c_i$, then seller i cannot profitably sell his house to buyer j
- *Match payoff is the gain from trade*

$$h_{ij} = \max\{0, \xi_{ij} - c_i\}$$

Primal Problem: Maximizing Total Gains from Trade

- Let seller i sell share $m_{ij} \geq 0$ of house i to buyer j (time share?)
- Two types of ("time share") constraints on every share $m_{ij} \geq 0$:
  1. No house is oversold
  2. No buyer buys more than one house.
- The Social Planner solves output maximization primal problem:

$$\max_{(m_{ij})} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} m_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} m_{ij} \leq 1 \quad \forall i \in \{1, \ldots, I\}$$

$$\text{and} \quad \sum_{i=1}^{I} m_{ij} \leq 1 \quad \forall j \in \{1, \ldots, J\}$$

Dual Problem

Lemma

The dual problem to output maximization is cost minimization:

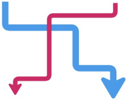
$$\min_{v_i, w_j} \sum_{i=1}^{I} v_i + \sum_{j=1}^{J} w_j \quad s.t. \quad v_i + w_j \geq h_{ij} \; \forall i,j \quad and \quad v_i, w_j \geq 0 \; \forall i,j$$

- *A great story is mathematically solvable.*
- We argue that the primal and dual problems have the same value
  - ⇒ The efficient matching also yields the cheapest way to afford all match output subject to competitive equilibrium free exit constraint
  - ⇒ Very loose intuition: Two ways of measuring output — gross national product and gross national income — coincide at the optimum.
- What are prices?
  - In the competitive market, selfish incentive devices.
  - But in the planner's problem, they are measures of social value

Linear Programming (LP) Duality

- **Primal problem**:

$$\max\{pz | Az \leq q, z \geq 0\}$$



- **Dual problem:**

$$\min\{uq | uA \geq p, u \geq 0\}$$

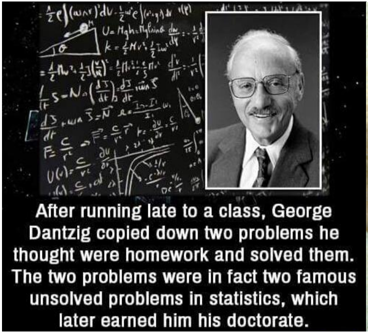
- Question: Find the matrix A for the Shapley and Shubik model
- **Theorem:** These two problems have the same values.
  - Proof logic:
  - Primal feasibility $\Rightarrow Az \leq q$ and dual feasibility $\Rightarrow p \leq uA$.
  - $\Rightarrow$ **weak duality**: $pz \leq uAz \leq uq \ \forall u, z \geq 0$
  - $\Rightarrow$ *primal value* $\leq$ *dual value*
  - Reverse direction (strong duality) is *much harder to show*

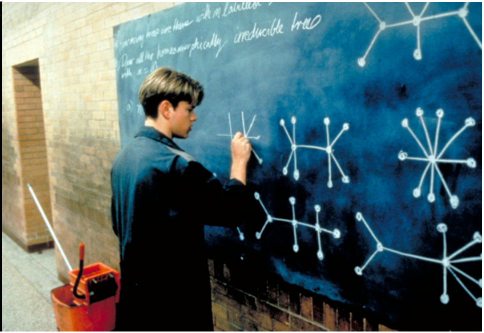
Linear Programming Duality as Deja Vu

- Flashback: von Neumann's Minimax Theorem (Saddle Point)
- George Dantzig, "A Theorem on Linear Inequalities," 1948
- This is the first formal proof of linear programming duality
  - Air Force *Later Tucker asked me, "Why didn't you publish it?" I replied, "Because it was not my result; it was von Neumann's. All I did was to write up, for internal circulation, my own proof of what von Neumann had outlined to me.*
- von Neumann and Dantzig:

Ideal "PhD Conquer the World" Mindset

- Good Will Hunting (1997), written by Ben Affleck and Matt Damon
- George Dantzig became a janitor at MIT





Primal and Dual with Two Buyers and Two Sellers

- Example with $I = J = 2$ buyers and sellers

$$q' = (1, 1, 1, 1)$$
$$h' = (h_{11}, h_{12}, h_{21}, h_{22})$$
$$m' = (m_{11}, m_{12}, m_{21}, m_{22})$$
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Primal Problem: $\max_{m \geq 0} \sum_i \sum_j h_{ij} m_{ij} = \; h'm \quad \text{s.t.} \quad Am \leq q$
- Dual Problem:

$$\min_{w,v \geq 0} \{v_1 + v_2 + w_1 + w_2\} = \min_{v,w \geq 0} (v, w) \cdot q \quad \text{s.t.} \; (v, w)A \geq h$$

Multipliers and Complementary Slackness Conditions

- *Primal*: $\max\{pz | Az \leq q, z \geq 0\}$ vs. *Dual:* $\min\{uq | uA \geq p, u \geq 0\}$
- Imagine a *fictitious zero sum game* with payoff

$$\mathcal{L}(z,u) = pz + uq - uAz \qquad [= pz + u(q - Az) = uq + (p - uA)z]$$

- By the 1928 *Minmax Theorem*, this game has *saddle point*:

$$\max_{z\geq 0}\min_{u\geq 0}[pz + uq - uAz] = \min_{u\geq 0}\max_{z\geq 0}[pz + uq - uAz] \qquad (\bigstar)$$

$$\Rightarrow \max_{z\geq 0}\min_{u\geq 0}[(p - uA)z + uq] = \min_{u\geq 0}\max_{z\geq 0}[pz + u(q - Az)]$$

- Let's intuit complementary slackness (CS):
  - A finite saddle point requires vector inequalities $p - uA \leq 0 \leq q - Az$
  - $\Rightarrow$ maximizer puts 0 weight on $-$ payoffs: $z_\ell = 0$ if $p_\ell - (uA)_\ell < 0$
  - minimizer puts 0 weight on $+$ payoffs: $u_k = 0$ if $q_k - (Az)_k > 0$.
  - Notice that CS $\Rightarrow$ primal value = dual value, given $(\bigstar)$
- Application of complementary slackness in Shapley-Shubik:

$$v_i + w_j \begin{cases} \geq h_{ij} & \text{for all } i,j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases}$$

Multipliers are also Shadow Values!

- *Primal*: $\max\{pz | Az \leq q, z \geq 0\}$
- Social planner's payoff function: $\mathcal{L}(z,u) = pz + u(q - Az)$
- Envelope Theorem $\Rightarrow \frac{\partial}{\partial q}\mathcal{L}(z,u) = u$
  $\Rightarrow$ dq extra constrained resource lifts planner's payoff by $u\ dq$.
- $u =$ shadow value of resource, as it indirectly shows true value
  i.e. marginal value of more slackness in constraint

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Shadow Values in Shapley-Shubik Housing Model

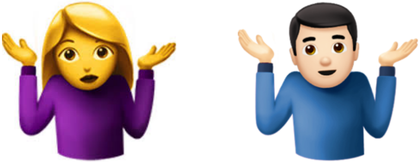
- Application of complementary slackness in Shapley-Shubik:

$$v_i + w_j \begin{cases} \geq h_{ij} & \text{for all } i, j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases}$$

- Intuitive economics of competition yields same inequalities!
- buyer i and seller j trade $\Rightarrow$ gains from trade h_{ij}
- So ε more of i *and* j raises social payoff by εh_{ij}

$\Rightarrow$ All we can say is $v_i + w_j = h_{ij}$

- Who matters more: men or women?



- National political debate: firms vs. workers, buyers vs. sellers
- We cannot separately identify buyers' & sellers' shadow values

1971 — Buyer-Seller Trade: Shapley and Shubik

- Assume three potential home buyers and three sellers

♂\♀	Seller Costs	Buyer Valuations		
		Buyer 1	Buyer 2	Buyer 3
House 1	18	23	26	20
House 2	15	22	24	21
House 3	19	21	22	17

- **Match payoffs are gains from trade, or zero, if negative**

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	$23 - 18 = 5$	$26 - 18 = 8$	$20 - 18 = 2$
Seller 2	$22 - 15 = 7$	$24 - 15 = 9$	$21 - 15 = 6$
Seller 3	$21 - 19 = 2$	$22 - 19 = 3$	$\max(17 - 19, 0) = 0$

1971 — Buyer-Seller Trade: Shapley and Shubik

- Assume three potential home buyers and three sellers

♂\♀	Seller Costs	Buyer Valuations		
		Buyer 1	Buyer 2	Buyer 3
House 1	18	23	26	20
House 2	15	22	24	21
House 3	19	21	22	17

- **Match payoffs are gains from trade, or zero, if negative**

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	5	**8**	2
Seller 2	7	9	**6**
Seller 3	**2**	3	0

- Solve the primal problem by eyeballing (infeasible in a large market)

Solving the Housing Example via the Dual

- Minimize the sum $\sum_i v_i + \sum_j w_j$ of shadow values $v_i \geq 0$ and $w_j \geq 0$:

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 \geq 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 \geq 6 \\ v_3 + w_1 \geq 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- Since the optimum occurs at the red matching, we just solve

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 = 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 = 6 \\ v_3 + w_1 = 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- Inequalities capture how all the nonexistent matches are unprofitable
- a solution: $(v_1, v_2, v_3) = (4, 5.5, 0)$ & $(w_1, w_2, w_3) = (2, 4, 0.5)$
- *These constraints and complementary slackness conditions ensure that Lagrange multipliers are competitive wages*

An Integer Price Solution of the Housing Example

♂ \ ♀	Buyer 1	Buyer 2	Buyer 3	Seller "wage" v_i
Seller 1	5	**8**	2	$v_1 = 4$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 0$
Buyer "wage"	$w_1 = 2$	$w_2 = 4$	$w_3 = 0$	

- Fix solution $(v_1, v_2, v_3) = (4, 5.5, 0)$ & $(w_1, w_2, w_3) = (2, 4, 0.5)$

⇒ home prices are $p_i = c_i + v_i$, or $p_1 = 22$, $p_2 = 20.5$, $p_3 = 19$

- E.g: seller 1 sells his home (cost 18) to buyer 2 (who values it 26) for a seller surplus $v_1 = 4$ and a buyer surplus $w_2 = 4$: from this, we deduce the price $p_1 = 22$
- We increase the price of home 2 to $p_2 = 21$, increasing the surplus of seller 2 to $v_2 = 6$ and reducing the surplus of buyer 3 to $w_3 = 0$.
- So house prices are now $p_1 = 22, p_2 = 21, p_3 = 19$
- How much can we increase or decrease the prices?

Offline: Worst Payoffs ("Wages") for Sellers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 3$
Seller 2	7	9	**6**	$v_2 = 5$
Seller 3	**2**	3	0	$v_3 = 0$
Buyers	$w_1 = 2$	$w_2 = 5$	$w_3 = 1$	

- Buyer 1 does not buy house 1 $\Rightarrow v_1 \geq v_3 + 3$
  - Proof: $w_1 + v_1 \geq 5 = 3 + 2 = 3 + w_1 + v_3$ (Buyer 1 buys house 3)
- Buyer 1 does not buy house 2 $\Rightarrow v_2 \geq v_3 + 5$
  - Proof: $w_1 + v_2 \geq 7 = 5 + 2 = 5 + w_1 + v_3$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least seller payoffs $(\underline{v}_1, \underline{v}_2, \underline{v}_3) = (3, 5, 0)$
- Associated maximum buyer payoffs $(\bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 5, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\underline{v}_1 + \bar{w}_2 = 8$, $\underline{v}_2 + \bar{w}_3 = 6$, $\underline{v}_3 + \bar{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

Offline: Worst Payoffs ("Wages") for Buyers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 5$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 1$
Buyers	$w_1 = 1$	$w_2 = 3$	$w_3 = 0$	

- Buyer 1 does not buy house 2 $\Rightarrow w_1 \geq w_3 + 3$
  - Proof: $w_1 + v_2 \geq 7 = 1 + 6 = 1 + w_3 + v_2$ (Buyer 3 buys house 2)
- Buyer 2 does not buy house 2 $\Rightarrow w_2 \geq w_3 + 3$
  - Proof: $w_2 + v_2 \geq 9 = 3 + 6 = 3 + w_3 + v_2$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least buyer payoffs $(\underline{w}_1, \underline{w}_2, \underline{w}_3) = (1, 3, 0)$
- Associated maximum seller payoffs $(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (5, 6, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\bar{v}_1 + \underline{w}_2 = 8$, $\bar{v}_2 + \underline{w}_3 = 6$, $\bar{v}_3 + \underline{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

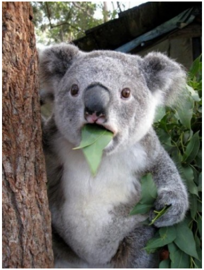
The Welfare Theorems

Welfare Theorems *A competitive equilibrium matching is efficient. Conversely, an efficient matching is a competitive equilibrium, for a suitable set of wages.*

- Proof of ($\Rightarrow$): We already proved this by contradiction
- Proof of ($\Leftarrow$): We use linear programming duality.
  - Maximize output, subject to the linear constraints of not overmatching any man or woman.
  - Call the Lagrange multipliers for these constraints the wages
  - By duality, the maximum total output equals the minimum total wages, subject to all the incentive constraints.
  - These constraints and complementary slackness conditions ensure that Lagrange multipliers are competitive wages
- **Important**. *The dual problem resolves the computational complexity issue — we need only find n wages for men and n wages for women!*

Types in the Becker Marriage Model

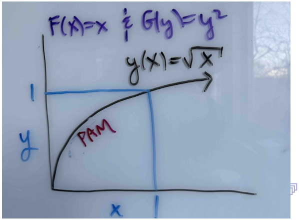
- Allow a finite number of m women (x_i) and n men (y_i)
- We will use calculus to compute wages later on, and so also allow a continuum mass of $\bar{M}$ women and $\bar{N}$ men



- If $m < n$ or $\bar{M} < \bar{N}$, then:
  - men are on the *long side of the market*
  - women on the *short side of the market*
  - If higher types are more productive, the least men will be unmatched
- Cumulative mass functions:
  - $M(x)$ gives the mass of women of type $x' \leq x$
  - $N(y)$ gives the mass of men of type $y' \leq y$

Assortative Matching

- Allocation question: who matches with whom?
- Assortative matching with finitely many types:
  - *positive (PAM)*: k-th highest man & woman pair for all $k = 1, \ldots, n$
  - *negative (NAM)*: woman 1 with man m, woman 2 with man $m - 1$, etc. and men $m + 1, \ldots, n$ unmatched
- Now consider the continuum analogues:
  - Pure matching: Use notation man $y(x)$ is the partner of woman x
  - PAM if $\bar{M} - M(x) = \bar{N} - N(y(x))$ for all matched women x.
  - NAM if $\bar{M} - M(x) = N(y(x))$ for all matched women x.
- The mass of men and women might even differ. If $\bar{M} = \bar{N}$, then
  - PAM: q-th highest quantile man & woman match
  - NAM: q-th highest quantile man matches q-th lowest quantile woman



Assortive Matching?

- Becker (1973), "A Theory of Marriage: Part I"

b) Assortive Mating

We now consider the optimal sorting when *M* and *F* differ in a trait, or set of traits, such as intelligence, race, religion, education, wage rate, height, aggressiveness, tendency to nurture, or age. Psychologists and sociologists have frequently discussed whether likes or unlikes mate, and geneticists have occasionally assumed positive or negative assortive mating instead of random mating. But no systematic analysis has developed that predicts for different kinds of traits when likes or unlikes are motivated to mate.[26] Our analysis implies that likes or unlikes mate when that

- I put the @ into Assortive:
    - My 2000 paper, "Assortative Matching and Search"

Assortative Matching with Nontransferable Payoffs

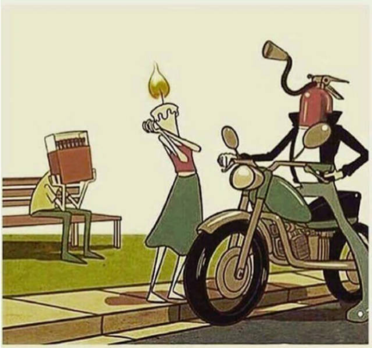
- $f(y|x)$ = payoff of woman x matched with man y
- $g(x|y)$ = payoff of man y matched with woman x
- f and g are *comonotone* if $\forall y_2 > y_1$ and $x_2 > x_1$, we have:

$$[f(y_2|x) - f(y_1|x)] \cdot [g(x_2|y) - g(x_1|y)] > 0 \ \ \forall x, y$$

- The opposite inequality is *reverse comonotone*
- Ignore weak monotonicity (with a natural definition)
- If f and g are differentiable, then both partial derivatives (in first arguments) have the same sign if comonotone
- **Theorem:** The unique stable matching with NTU is PAM if f and g are comonotone, and NAM if reverse comonotone.

Gentle Proof of NTU Sorting Proposition

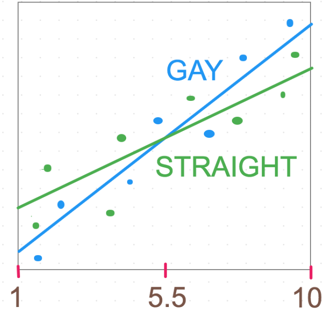
- Assume comonotonicity without PAM in a stable matching
- Then $\exists x' > x$ and $y' > y$ with matches (x, y') and (x', y)
- Claim: either (x', y') or (x, y) is a blocking pair
  1. If $f(y'|x') > f(y|x') \Rightarrow g(x'|y') > g(x|y')$
  2. If $f(y'|x) < f(y|x) \Rightarrow g(x'|y) < g(x|y)$



Positive Sorting is an Empirical Fact

Fun Application (Yale undergrad, 2006): The Dating Market

- Data Source 1: Facebook (Meta?) Dating Market early on
- Data Source 2: Online beauty contest, like www.rate-my-photo.com



Rear View Mirror on Last Class

- Transferable utility ⇒ efficiency is well-defined: maximize payoff sum
- 1st Welfare Theorem: competitive equilibria are efficient
- 2nd Welfare Theorem: efficiency ⇒ competitive equilibrium
  - Proof via linear programming duality theory (Minmax Th'm)
  - shadow values (Complementary slackness) act as competitive prices
  - Computationally, with many men and women, it is easier to find competitive equilibria than compute efficient matchings
  - Shadow values may be:

Eg. 1. wages in the employment model

Eg. 2. consumer and producer surplus in the trading model

Eg. 3. payoffs and rents in the location assignment model

1973 — Becker's Marriage Model

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At left is **positive assortative matching (PAM)**
- Comonotone payoffs: men prefer higher women x and vice versa

⇒ *The stable matching without transfers is PAM.*

- Assume we indexed men or women oppositely.
- Then payoffs are reverse comonotone, and NAM is stable

1973 — Becker's Marriage Model

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At right, we assume payoffs are transferable (TU)
- Now, **negative assortative matching (NAM)** arises
- Why? Matches all profit from higher men, but the matches that profit most from higher men are those with lower women.
- This forces downward sorting.
- For instance, rematching the two sorted pairs $(1, 1)$ and $(2, 2)$ as $(1, 2)$ and $(2, 1)$ changes output by $(18 + 8) - (16 + 9) = 26 - 25 = 1$

Pairwise Efficiency and Efficiency

- Stability with NTU: Can two unmatched people break their matches, to match with each other, & improve their welfare?
  - ⇒ The losses of the dumped partners do not matter
- TU **pairwise efficiency**: Can two matches break, re-match differently, and improve their welfare?
  - ⇒ All losses matter: cardinal strength of the preferences matters
- A matching m is **pairwise efficient with TU** if for all matched pairs (x_1, y_1) and (x_2, y_2):

$$h(x_1, y_1) + h(x_2, y_2) - h(x_1, y_2) - h(x_2, y_1) \geq 0$$

- An **efficient** matching maximizes the sum of all match outputs, and so rematching any set of couples cannot help.

Lemma

Any efficient matching $m \in \mathcal{M}$ is pairwise efficient.

- The converse of this lemma is false

Pairwise Efficiency $\not\Rightarrow$ Efficiency

- With NTU, our target is stability: no pairwise blocking.
- But pairwise efficiency does not suffice for TU efficiency:

	y_1	y_2	y_3
x_1	3	3	0
x_2	0	3	3
x_3	2	0	3

- The pairwise efficient green matching has a lower total payoff than the pairwise efficient cyan matching, and is inefficient.
- Q: What bribery scheme would unravel the green matching?

TU — Strategic Substitutes Drives Negative Sorting

♂ \ ♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

Cross Partial Payoff Differences (Synergies)

	12	23
23	$18 + 24 - 27 - 16 = -1$	$16 + 21 - 14 - 24 = -1$
12	$9 + 16 - 18 - 8 = -1$	$8 + 14 - 16 - 7 = -1$

- **Strategic substitutes**:
  - all cross partial differences of match payoffs are negative
  - pairwise efficiency $\Rightarrow$ positive sorting is not locally efficiency
- **Strategic complements**:
  - all cross partial differences of match payoffs are positive
  - pairwise efficiency $\Rightarrow$ negative sorting is not locally efficiency

TU — Strategic Substitutes Drives Negative Sorting

NTU Matching

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

TU Matching

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- **Left**: payoffs are men get $2xy$ and women get $y(10-3x)$.
  - Men's payoffs $2xy$ increases in women's type x
  - Women's payoffs $y(10-3x)$ increases in men's type y
  - $\Rightarrow$ PAM is the stable allocation without transfers
- **Right**: match payoffs are $2xy + y(10-3x) = 10y - xy$.
  - Cross partial derivative is -1
  - $\Rightarrow$ strategic substitutes
  - $\Rightarrow$ NAM

Becker (1973): Assortative Matching with Transfers

- Match payoff $h(x, y)$ is *(strictly) supermodular* [SPM] if
$$h(x', y') + h(x, y) \geq (>)\ h(x', y) + h(x, y') \tag{5}$$
for any women $x' \geq x$ and men $y' \geq y$ (also: *complements*)
- $h(x, y)$ is *(strictly) submodular* if the reverse inequality holds
- For twice differentiable match payoffs, this says $h_{12}(x, y) \geq 0$

Proposition (Becker's Marriage Model)

(a) If $h(x, y)$ is supermodular (SPM), then PAM is efficient.
If $h(x, y)$ is strictly SPM, then PAM is uniquely efficient.
(b) If $h(x, y)$ is submodular (SBM), then NAM is efficient.
If $h(x, y)$ is strictly SBM, then NAM is uniquely efficient.
(c) If $h(x, y)$ is modular (SPM & SBM), any matching is efficient.

- Proof (by Buz Brock): Assume strictly supermodular (SPM)
- If matching is not PAM, then matching is not pairwise efficient, and so not efficient

Example: Matching with and without Transfers

PAM

♂\♀	$x=1$	$x=2$	$x=3$
$y=3$	6,21	12,12	**18,3**
$y=2$	4,14	**8,8**	12,2
$y=1$	**2,7**	4,4	6,1

NAM

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- Women earn $f(y|x) = y(10-3x)$ and men earn $g(x|y) = 2xy$

$\Rightarrow$ $\frac{\partial f(x|y)}{\partial x} = 10 - 3x > 0$ (women prefer higher men)
$\frac{\partial g(y|x)}{\partial y} = 2x > 0$ (men prefer higher women)

$\Rightarrow$ unique stable matching is PAM

$\Rightarrow$ Hence, the DAA delivered PAM

- With transfers, strictly submodular match payoffs
  $h(x,y) = f(x|y) + g(y|x) = 10y - xy$ since $h_{xy} < 0$
  $\Rightarrow$ unique efficient matching is NAM
  - This is an unusual function that is increasing in x, y and yet with a negative cross partial (since domain is bounded)

How to Compute Competitive Wages with PAM or NAM

- To use calculus, we will assume a continuum of types
  - SPM match payoffs: $h(x,y) = x^2 y$
  - Types of women x and men y uniformly distributed on $[0,1]$
- Since $h_{xy} = 2x > 0$, PAM is the efficient outcome (by Becker)
- Let $w(x)$ and $v(y)$ be the competitive wage functions
- If a middleman matches x and y, paying them their wages, his profits are:

$$\pi(x,y) = x^2 y - w(x) - v(y)$$

- Exercise: Use Topkis' Theorem to prove sorting is a competitive eq'm.
- With free entry by middlemen, competition forces a zero profit max at $y = x$ (competitive equilibrium, by welfare theorem):

$$\begin{aligned} \pi_x &= 0 \Rightarrow \left[2xy = w'(x)\right]_{y=x} \Rightarrow w'(x) = 2x^2 \\ \pi_y &= 0 \Rightarrow \left[x^2 = v'(y)\right]_{x=y} \Rightarrow v'(y) = y^2 \end{aligned}$$

- *Men compete with men, and women compete with women.*
- Aside: This proof applies the "revelation principle" that allows you to solve for bidding strategies in a sealed bid first price auction FPA

Outside Options and the Wages of Men vs. Women

- Evaluating these at the efficient matches, (x, x) and (y, y),

$$w(x) = \frac{2}{3}x^3 + \beta$$
$$v(y) = \frac{1}{3}y^3 + \delta$$

- By zero profits, $\pi(x, x) = 0 \; \forall x$, and so $\beta + \delta = 0$ because

$$0 = x^2 \cdot x - w(x) - v(x) = x^3 - \frac{2}{3}x^3 - \frac{1}{3}x^3 - (\beta + \delta)$$

- If unmatched people earns zero, then $\beta = \delta = 0$
- A *dowry* $\delta > 0$ — a fixed transfer that women pay men — only arises if unmatched women earn a payoff at most $-\delta < 0$
- A *bride price* $\beta > 0$ — a fixed transfer that men pay women — only arises if unmatched men earn a payoff $-\beta < 0$
- If unmatched men and women earn negative payoffs, then a dowry or bride price reflects a social norm (a Nash equilibrium)

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992

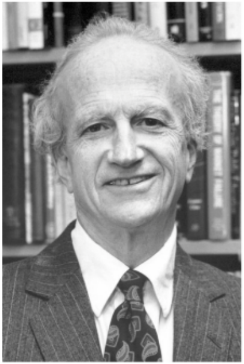


Photo from the Nobel Foundation archive.

Gary S. Becker

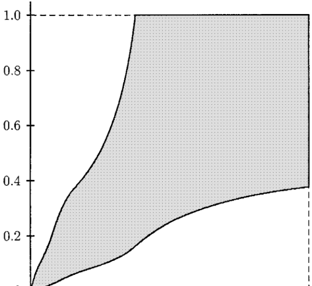
Prize share: 1/1

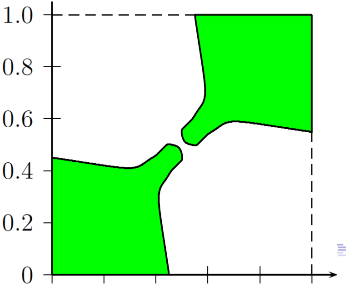
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992 was awarded to Gary S. Becker "for having extended the domain of microeconomic analysis to a wide range of human behaviour and interaction, including nonmarket behaviour."

Advanced Theory Topic: Assortative Matching and Search

- We don't see perfect PAM in reality. Find a less wrong model!
- $\nexists$ stock exchange for marriage partners or firm-worker pairs
- One must search for partners, which takes time!

$\Rightarrow$ Shimer-Smith (2000): Even given SPM, higher types might settle for lower parters since the cost of search (willingness to wait) is higher.
  - With search frictions, PAM requires that $\log h_x(x, y)$ is SPM
- Eg: $h(x, y) = e^{xy}$ and $h(x, y) = (x + y - 1)^2$ are SP $\Rightarrow$ PAM: $y(x) = x$.
- With search frictions $\Rightarrow$ matching sets:





Advanced Topic: The Comparative Statics of Sorting

- We don't see perfect PAM in reality. Find a less wrong model!
- What if we SPM or SBM fail and thus PAM or NAM fail?
- The transportation problem unsolved $\Rightarrow$ *we cannot characterize efficient matching — except in PAM or NAM extreme cases!*
- Idea: derive the comparative statics of the efficient matching
- **synergy**: any *cross partial difference* of match outputs

$$h(x_2, y_2) - h(x_2, y_1) + h(x_1, y_2) - h(x_1, y_2) \qquad \text{for } x_2 \geq x_1, y_2 \geq y_1$$

- Special cases: PAM/NAM iff synergy is everywhere $+/-$

The Comparative Statics of Sorting

Axel Anderson Lones Smith

Advanced Topic: Sorting Need not Rise in Synergy

- **Increasing Sorting Theorem**

  Sorting is "higher" with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

  $$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

    shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

- Example: Synergy rises at each stage, but sorting does not

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**14**	18
$y = 2$	5	2	**14**
$y = 1$	**1**	5	9

Matrix of Cross Differences

8	−8
−7	8

Advanced Topic: Sorting Need not Rise in Synergy

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  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

- Example: Synergy rises at each stage, but sorting does not

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	16	**24**
$y = 2$	**5**	3	16
$y = 1$	1	**5**	9

Matrix of Cross Differences

9	−5
−6	9

Advanced Topic: Sorting Need not Rise in Synergy

- **Increasing Sorting Theorem**

  Sorting is "higher" with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

$$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

- Example: Synergy rises at each stage, but sorting does not

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**20**	30
$y = 2$	5	6	**20**
$y = 1$	**1**	5	9

Matrix of Cross Differences

10	-4
-3	10

Advanced Topic: Sorting Need not Rise in Synergy

- **Increasing Sorting Theorem**

  Sorting is "higher" with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

  $$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

- Example: Synergy rises at each stage, but sorting does not

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	22	**36**
$y = 2$	**5**	7	22
$y = 1$	1	**5**	9

Matrix of Cross Differences

11	-1
-2	11

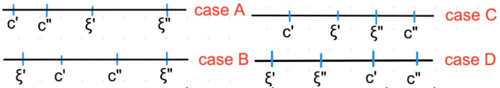
Double Auctions

- *Consider a world with homogeneous houses* (Levittown)
- Buyer j's values all goods at $\xi_j = \xi_{ij}$ for all i
- Sellers still differ by opportunity costs c_i
- Gains from trade: $h(\xi, c) \equiv \max\{0, \xi - c\}$ for buyer ξ and seller c
- The price p_i divides this surplus between matched traders
  - producer surplus: $v_i = p_i - c_i$
  - consumer surplus: $w_j = \xi_{ij} - p_i = \xi_j - p_i$
- We next argue that gains from trade $h(\xi, c)$ is submodular.
  - Math intuition: max preserves SBM, and min preserves SPM

Lemma (Gains from Trade)

Gains from trade h is SBM in (ξ, c).
(a) If $\xi' \leq \xi''$ and $c' \leq c''$, then $h(\xi'', c'') + h(\xi', c') \leq h(\xi'', c') + h(\xi', c'')$.
(b) Equality holds in (a) when two or zero trades should happen

Proof:



1. If two trades should occur (case A), then $h(\xi, c)$ is modular. $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = \xi'' + \xi' - c' - c''$.
2. If one trade should occur, then $h(\xi, c)$ is strictly submodular.

$$B. \quad h(\xi'', c'') + h(\xi', c') = \xi'' - c'' < \xi'' - c' = h(\xi', c'') + h(\xi'', c')$$

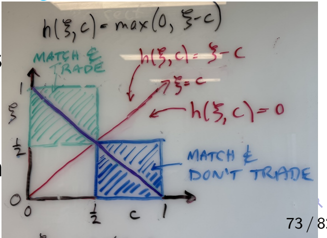
$$C. \quad h(\xi'', c'') + h(\xi', c') = \xi' - c' < \xi'' - c' = h(\xi', c'') + h(\xi'', c')$$

3. If no trades should occur (case D), then $h(\xi, c)$ is modular. $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = 0$.

- Inequalities are strict if $c' < c''$ and $\xi' < \xi''$, since trade surplus falls when the wrong good is traded.

The Supply and Demand Paradigm

- We view a market as a set of pairwise trades of buyers and sellers
- The highest value buyers trade with the lowest cost sellers.
- Rank order buyers: $\xi_1 < \cdots < \xi_k < \xi_{k+1} < \cdots < \xi_N$
- Rank order sellers: $c_1 < \cdots < c_k < c_{k+1} < \cdots < c_N$
  - Wages and prices are usually not unique in the finite marriage model
  - The price is common across all units traded in a double auction
- Since $h(\xi, c)$ is submodular, Becker's Marriage Theorem $\Rightarrow$ NAM $\Rightarrow$ high value buyers trade with low cost sellers (intuitive)
- $h(\xi, c)$ is locally modular for agents trading $(\xi > c)$, and not $(\xi < c)$;
  - *Matching among those trading sellers and buyers is irrelevant.*
  - *Matching among sellers and buyers not trading is irrelevant.*
- Eg: continuum agents with $\xi, c \sim U(0, 1)$
- NAM efficient in shaded green & blue areas $\Rightarrow$ every other matching on those two sets is also efficient, and thus is an equilibrium $\Rightarrow$pairwise matching model$\rightsquigarrow$double auction So a market maker just clears the market.



Competitive Equilibrium in a Double Auction

Proposition (Double Auctions)

(a) *If* $\xi_N < c_1$, *there is no trade. Assume* $c_1 \leq \xi_N$ *henceforth.*
(b) *The* k^* *highest value buyers purchase from the* k^* *lowest cost sellers, where* k^* *is the largest* k *with* $c_k \leq \xi_{N+1-k}$.
(c) *The* **law of one price** *holds, with a common price*

$$p^* \in [\max(c_{k^*}, \xi_{N-k^*}), \min(c_{k^*+1}, \xi_{N+1-k^*})]$$

(d) *A competitive equilibrium is efficient, maximizing the gains from trade.*
(e) *The final allocation is immune to side bribes.*

- Notice that part (c) captures four constraints!
  - The top k value buyers, and bottom k cost sellers want to trade, and the $k+1$st highest buyer or lowest seller does not.
- markets clear: supply balances demand
- To understand typical deviations from the law of one price, we can add search or information frictions to the model

Is There One Price? What is it?

- Proof of (c): If buyer j trades, then the social planner is indifferent across all matches with any (low cost) sellers i, who must trade

$\Rightarrow$ buyer j's shadow value $w_j = \xi_j - p_i > 0$ must not vary in j

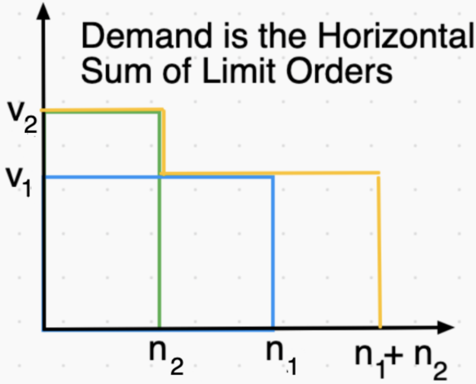
$\Rightarrow$ Seller prices $p_i = p$ cannot vary with i, assuming they trade

- The price p^*:
  - encourages last transaction: $c_{k^*} \leq p^* \leq \xi_{N+1-k^*}$
  - deters another transaction: $\xi_{N-k^*} \leq p^* \leq c_{k^*+1}$
- Hence, crossing of supply and demand determines quantity:

$$\max(c_{k^*}, \xi_{N-k^*}) \leq p \leq \min(c_{k^*+1}, \xi_{N+1-k^*})$$

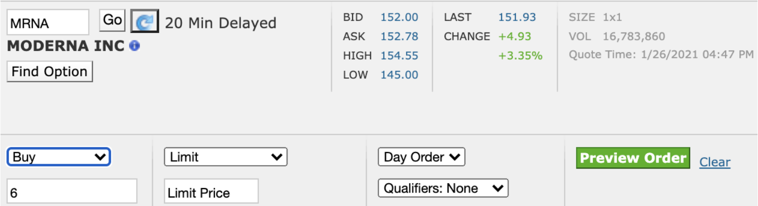
- The competitive price is not pinned down unless the last trade yields no surplus, whereupon the last unit needn't be traded
- A game has a learning dynamic: an impartial *Walrasian auctioneer* finds a competitive equilibrium by raising the price with excess demand and reducing the price with excess supply
- Opening stock market prices are set to clear the market

Beyond Unit Supply and Demand: Limit Orders



- The same can be done to construct the supply curve.

Overnight Market in Stock Exchanges



- To open/close, many stock exchanges use single price double auction
- The buyer must ask for a limit order (my choice) or a market order (limit order with unspecified price)

Offline: Easy Double Auction Example

- Consider 20 traders: buyers 2,4,. . . , 20, and sellers 1,3,. . . ,19
- Buyer valuations are $\xi_i = 2i$ and sellers costs are $c_j = 3j$.
- Ordering the valuations from high to low:

$$40, 36, 32, 28, 24, 20, 16, 12, 8, 4$$

- Ordering costs from low to high:

$$3, 9, 15, 21, 27, 33, 39, 45, 51, 57$$

- An efficient matching clears the market: the high value buyers and low cost sellers $\Rightarrow k^* = 4$ (but actual pairing irrelevant)
- The price p^* encourages the value 28 buyer and cost 21 seller to trade:

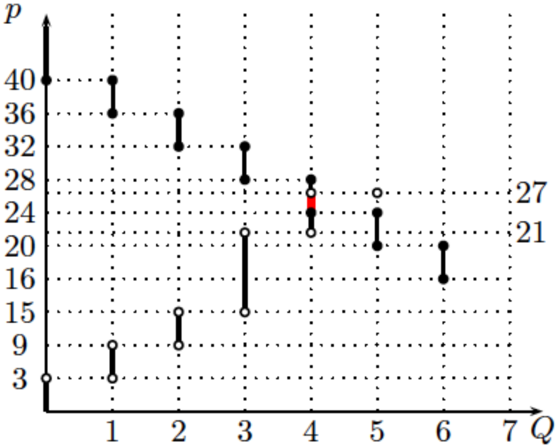
$$21 \leq p^* \leq 28$$

- The price p^* deters the value 24 buyer and cost 27 seller from trading:

$$24 \leq p^* \leq 27$$

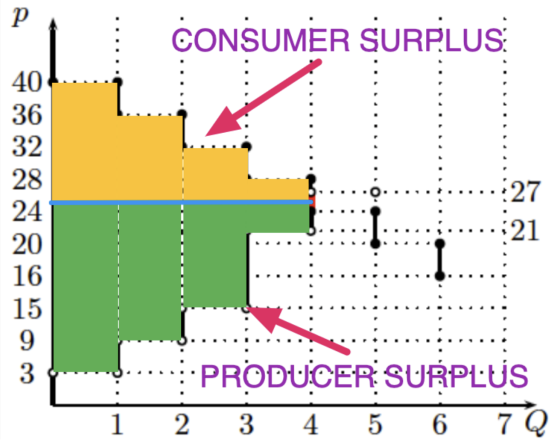
- any price in the interval $[24, 27]$ clears the market

All Positive Gains from Trade are Realized



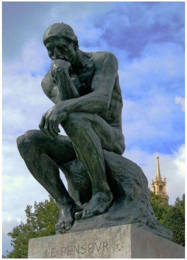
- Okay, I admit my plot is deformed around $[15, 16]$:)
- All traders earn positive surplus: e.g. at $p^* = 25$, the marginal buyer earns $28 - 25 = 3$ and the marginal seller $25 - 21 = 4$

When are Gains from Trade Larger?



- Heterogeneity is good and the source of all gains from trade.
  - If everyone had identical valuations, then no consumer secures consumer surplus at the market clearing price
  - the more heterogeneous are consumers or producers, the larger the total gains from trade.

Thinker Problem About Merging Markets



- $\exists > 50$ Thinkers (!)
- What happens to the price & quantity if we merge markets?
- Important question as world markets merge via trade!
- *Assume an exchange market for a good in cities A and B. Competitive prices are $p_A < p_B$ and quantities are q_A, q_B. Then the markets merge.*
  1. *How does the new competitive price compare to p_A and p_B?*
  2. *How does the new competitive quantity compare to $q_A + q_B$?*
  3. *Is total trade surplus higher or lower after the merger?*
- *Hint: Find examples where quantity traded rises or falls.*

Paternalism Does Not Maximize Efficiency

- Paternalism is imposing your values on another. Examples:
- Volunteer vs. Draft Army (Welfare Theorem Application)
  - A volunteer army maximizes gains from trade: it sets a wage so that the people who most want to serve willingly do so.
  - Milton Friedman's opposition the Draft helped end it in 1973.
    - Old exam Q: how much trade surplus did the draft erase?
- Organ Sale Example: only Iran allows kidney sales
- Scalping Example: Ticket Resale Laws vary (my advisee Axel)
- Regifting Example: Jay Leno's freely gave away Tonight Show tickets to unemployed in Detroit in 2009.
  - People resold tickets on eBay and Leno mocked them.
  - Q: how much trade surplus did resale create?
- Gifting Example: giving gifts usually means value $<$ cost
  - Waldfogel (1993), "The Deadweight Loss of Christmas"
  - Lost surplus was about ten billion dollars per holiday season!