Econ 713 Midterm

UW-Madison

February 8, 2024 (in class)

There are 65 ***points in 75 minutes.*** *Points are at the right.*
Rigorously justify everything with graphs or algebra or a known theorem. *Enjoy!*

1. The payoff for matches for each of three women (x_i) and three men (y_j) are given by the table, where each person's utility function is increasing in her payoff:

	x_1	x_2	x_3
y_1	4,1	1,0	3,3
y_2	1,2	3,2	1,1
y_3	4,4	5,1	2,1

(a) Find all stable matchings with non-transferable payoffs. **[10]**

(b) Choose one stable match in (a). Show that the partner of y_1 has no blocking pairs. **[5]**

Solution:
(a) We run the DAA to check for stable matches. We start by y proposing:

$$\textbf{Stage 1: } y_1 \to x_1,\ y_2 \to x_2,\ y_3 \to x_2,\ \mathcal{M}_y = \{(1,1),(2,2)\}\,.$$
$$\textbf{Stage 2: } y_3 \to x_1,\ \mathcal{M}_y = \{(3,1),(2,2)\}\,.$$
$$\textbf{Stage 3: } y_1 \to x_3,\ \mathcal{M}_y = \{(3,1),(2,2),(1,3)\}\,.$$

The algorithm ends after after 3 rounds. Now, we check if it is unique by letting x propose:

$$\textbf{Stage 1: } x_1 \to y_3,\ x_2 \to y_2,\ x_3 \to y_1,\ \mathcal{M}_x = \{(1,3),(2,2),(3,1)\}\,.$$

The algorithm ends in one step. Since $\mathcal{M}_y = \mathcal{M}_x$ there is a unique stable matching, given by $\mathcal{M} = \{(1,3),(2,2),(3,1)\}$.

(b) In the only stable matching, consider the match (y_1, x_3). No suitor can improve x_3's payoff, and so there are no blocking pairs involving x_3. For y_1, only woman x_1 could improve his payoff. But x_1 is matched with y_3, and so cannot gain by changing partners. All told, there is no blocking pair for (y_1, x_3).

2. Consider a two sided market of tennis players and coaches, both drawn from a unit mass continuum. Players are indexed by their skill $x \in [0,1]$, with mass cdf $F(x) = x$. Coaches are indexed by their experience $y \in [0,2]$, with mass cdf $G(y) = y/2$. The match outcome is given by the performance during the Grand Slam, where the expected monetary prize to a match of player x and coach y is given by

$$h(x,y) = yx^2 - xy^2 + 4xy$$

The outside option for unmatched players is zero, and one for coaches.

(a) Find the efficient matching of this situation. Who matches with whom? **[5]**

(b) Obtain the competitive equilibrium wages $w(x)$ and $v(y)$. **[10]**

Once the Grand Slam ends, the sponsorship renewal season starts. A unit mass of firms indexed by their public image $p \in [0,1]$, with cdf $I(p) = p^2$, looks for a team (x,y) that competed in the Grand Slam to promote their merchandise. Firms with low public image benefit the most from the sponsorship, as the partnership brings new consumers to their brand. In particular, the revenue generated by the promotion of firm p by team (x,y) is given by

$$\pi(p,x,y) = (2-p)h(x,y)$$

(c) Who matches with whom in the sponsorship market? **[5]**

Hint: *Is there a short side in the sponsorship market?*

Solution:
(a) First we observe that payoffs $h(x,y)$ are supermodular: $h_{xy} = 2x - 2y + 4 \geq 0$ for any (x,y). By Becker, PAM is efficient:

$$\begin{aligned}1 - F(x) &= 1 - G(y(x))\\ 1 - x &= 1 - y/2,\\ y(x) &= 2x.\end{aligned}$$

Thus, the efficient matching is $(z, 2z)$ for $z \in [0,1]$.

(b) We set up the middleman's problem $\max_{x,y} \pi(x,y)$, where

$$\pi(x,y) \equiv h(x,y) - w(x) - v(y)$$

Evaluate the FOC at the efficient (PAM) matching:

$$\frac{\partial \pi(x,y)}{\partial x} = 2xy - y^2 + 4y - w'(x) = 0\Big|_{(x,y)=(z,2z)} \qquad w'(z) = 8z$$
$$\frac{\partial \pi(x,y)}{\partial y} = x^2 - 2xy + 4x - v'(y) = 0\Big|_{(x,y)=(z/2,z)} \qquad v'(z) = 2z - \frac{3}{4}z^2$$

Integrating,

$$w(z) = 4z^2 + C_x \quad \text{and} \quad v(z) = z^2 - \frac{z^3}{4} + C_y \qquad (\clubsuit)$$

Next we use the middleman zero profits condition to obtain the relationship between the constants of integration. An equilibrium match produces output $h(z,2z) = 8z^2 - 2z^3$. Middleman earn zero profits if:

$$h(z,2z) - w(z) - v(2z) = 0 \implies C_x + C_y = 0$$

Lastly, we use coaches outside option to fully characterize the constant of integration. The worst player $\underline{z}$ that find a coach solves,

$$h(\underline{z}, 2\underline{z}) = 1 \qquad (\spadesuit)$$

Thus, $v(2\underline{z}) = 1$ identify C_y and $C_x = -C_y$.

[Using Wolfram to solve ($\spadesuit$) we obtain $\underline{z} = 0.3712$, and $C_y = 0.5511 = -C_x$].

(c) Let's index the team by the type of player z in it. From (a), the teams are of the form $h(z,2z)$,

$$\pi(p,z) = h(z,2z)(2-p) = (8z^2 - 2z^3)(2-p)$$

Taking partial derivatives, $\pi_{pz} = z(6z-16) < 0$ for all $z \in [\underline{z}, 1]$ — and so NAM is efficient. From (b), not all players competed in the Grand Slam; the lowest type competing was $\underline{z}$, thus not all the firms will have a team (firms are the long side of the market). Note that $\pi(p,z) > 0$ for all pairs and $\pi_p < 0$ and $\pi_z > 0$. Hence, the output is increasing in teams type and decreasing in firms type. Therefore, the market clear from the bottom of the firms type, which is matched with the best team. Lastly, we consider how the mass of teams is distributed by using the distribution of players. Let $M(z)$ be the mass of teams with players of type $z' \leq z$,

$$M(z) = \begin{cases} 0 & \text{if } z \leq \underline{z} \\ z - \underline{z} & \text{if } 1 \geq z > \underline{z} \\ 1 - \underline{z} & \text{if } z > 1 \end{cases}$$

Thus, NAM implies

$$\begin{aligned} (1-\underline{z}) - M(z) &= I(p) \\ 1 - z = p^2, \ &\Rightarrow p(z) = (1-z)^{1/2} \text{ for } z \in [\underline{z}, 1] \end{aligned}$$

3. There are three firms f_1, f_2, f_3 and three workers x_1, x_2, x_3. The utility of worker i at firm j is $u_i(j) = q_{i,j} + w_j$, where $q_{i,j}$ is the fixed enjoyment that worker i gets from working at firm j, and w_j is the wage that firm j pays. The profits of firm j that hire worker i are given by $\pi_j(i) = v_{i,j} - w_j$, where $v_{i,j}$ is the productivity of worker i at firm j. The table presents the enjoyment and productivity of each match $(q_{ij}, v_{i,j})$:

	f_1	f_2	f_3	$\varnothing$
x_1	5,3	4,2	3,2	0,0
x_2	3,2	4,3	5,2	0,0
x_3	4,2	5,2	3,3	0,0
$\varnothing$	$0,\eta_1$	$0,\eta_2$	$0,\eta_3$	

where $\varnothing$ is the unmatched state. Workers accept at most one job, and firms hire at most one worker, and unmatched firms outsource the vacancy and earn $\eta_j > 0$. Firms can freely set wages.

(a) Do all workers match with a firm in equilibrium? What conditions ensure that all workers match with a firm? **[5]**

(b) Who matches with whom in an equilibrium in which all workers match with a firm? **[3]**

(c) Let u_i and π_j be the equilibrium utility of worker i and profits of firm j. State all conditions on u_i and π_j that ensure the stability of the equilibrium described in (b)? **[7]**

Each worker i is fully aware she is most productive in firm i, according to the payoff matrix. Suppose that before going to the job market, worker $i \in \{1, 2, 3\}$ can go to college and specialize in the type of work that firm i requires. College tuition costs $t \in [0, 1]$, and raises the worker's productivity in the firm she specializes. The productivity of worker i in firm j after she graduates from a program of cost t is:

$$g_i(j, t) = \begin{cases} v_{i,j} + 2t & \text{if } i = j, \\ v_{i,j} & \text{if } i \neq j. \end{cases}$$

To pay the tuition, the worker takes a student loan that is repaid once the worker accepts a job. Hence, the utility of worker i that accepts a position at firm j and took a student loan t is given by $u_i(j, t) = q_{i,j} + w_j - t$.

Assumption: For the following questions consider that all the workers choose the same level of program $t \in [0, 1]$ and $\eta_j = 0$ for $j \in \{1, 2, 3\}$.

(d) Characterize the allocation of the competitive equilibrium as a function of t. **[7]**

(e) Characterize the welfare in a competitive equilibrium as a function of t. **[3]**

(f) Let $T \subseteq [0, 1]$ be the set of programs that are welfare improving with respect the original situation in (b). Identify T. **[5]**

Solution:

(a) Since firms can freely set wages, we are in a transfer utility environment. So we look for conditions that maximize total output. First suppose each firm matches with a worker, as below:

	f_1	f_2	f_3
e_1	**8**	6	5
e_2	5	7	**7**
e_3	6	**7**	6

For these matches to be optimal over the outside option of the firms, we need that $\eta_1 < 8$, $\eta_2 < 7$ and $\eta_3 < 7$. If all three conditions hold we have a match in which all workers match with a firm. For any condition that does not hold the match involved that firm is removed from the equilibrium. [Arguing that $q_{i,j}$ is non-transferable and/or that $w_j \geq 0$ with the respective conditions is also considered for grading]

(b) If all the condition stated in (a) hold, then the matching between workers and firms is given by $\mathcal{M} = \{(1,1),(2,3),(3,2)\}$.

(c) For the equilibrium in (b) to hold we need $u_i \geq 0, \pi_j \geq \eta_j$, for all $i,j \in \{1,2,3\}$, and:

$$\begin{aligned} &u_1 + \pi_1 = 8, && u_1 + \pi_2 \geq 6, && u_1 + \pi_3 \geq 5. \\ &u_2 + \pi_1 \geq 5, && u_2 + \pi_2 \geq 7, && u_2 + \pi_3 = 7. \\ &u_3 + \pi_1 \geq 6, && u_3 + \pi_2 = 7, && u_3 + \pi_3 \geq 6. \end{aligned}$$

(d) Let $\mathcal{M}(t)$ be the matching in a competitive equilibrium with specialization t and $W(t,\mathcal{M})$ the total match output for any given matching $\mathcal{M}$. Efficiency of competitive markets implies that,

$$\mathcal{M}(t) \in \arg\max_{\mathcal{M}^f} \mathcal{W}(t,\mathcal{M}). \qquad (\clubsuit)$$

where $\mathcal{M}^f$ is the set of feasible matching. The payoff matrix as a function of t is:

	f_1	f_2	f_3
e_1	$8+t$	$6-t$	$5-t$
e_2	$5-t$	$7+t$	$7-t$
e_3	$6-t$	$7-t$	$6+t$

The case in (b) correspond to $t = 0$,

$$\mathcal{M}_0 = \{(1,1),(2,3),(3,2)\} \quad \text{and} \quad \mathcal{W}(t,\mathcal{M}_0) = 22 - t.$$

Note that the match output of $\mathcal{M}_0$ is decreasing in t. Thus, if there is another feasible match $\mathcal{M}_1 \in \mathcal{M}^f$ such $\mathcal{W}(t,\mathcal{M}_1) > \mathcal{W}(t,\mathcal{M}_0)$ the allocation of the competitive equilibrium will change. We look for a feasible match that is increasing in t,

$$\mathcal{M}_1 = \{(1,1),(2,2),(3,3)\} \quad \text{and} \quad \mathcal{W}(t,\mathcal{M}_1) = 21 + 3t.$$

Next, we solve the transition point of the economy

$$\mathcal{W}(t,\mathcal{M}_1) = \mathcal{W}(t,\mathcal{M}_0) \Rightarrow 21 + 3t = 22 - t \implies \tilde{t} = 1/4.$$

Hence, the competitive equilibrium change from $\mathcal{M}_0$ to $\mathcal{M}_1$ at $\tilde{t} = 1/4$.

$$\mathcal{M}(t) = \begin{cases} \mathcal{M}_0 & \text{if } t \in [0, 1/4] \\ \\ \mathcal{M}_1 & \text{if } t \in (1/4, 1] \end{cases}$$

(e) The welfare $\mathcal{W}(t)$ in the competitive market is the maximization over all feasible matching,

$$\mathcal{W}(t) = \max_{\mathcal{M}^f} W(t, \mathcal{M})$$

Thus, the welfare in a competitive equilibrium is given by

$$\mathcal{W}(t) = \begin{cases} 22 - t & \text{if } t \in [0, 1/4] \\ \\ 21 + 3t & \text{if } t \in (1/4, 1] \end{cases}$$

Note that the welfare is convex and continuous in t.

(f) The initial situation has welfare $\mathcal{W}(0) = 22$. Welfare is is decreasing over $t \in [0, 1/4]$ and increasing over $[1/4, 1]$. Thus, we solve $\mathcal{W}(t) = 22$ over $t \in [1/4, 1]$,

$$21 + 3t = 22, \; \Rightarrow t = 1/3.$$

Hence, the set of programs that are welfare improving are $T = [1/3, 1]$.