Economics 713: Final Exam

March 9, 2020

Cite any theorems you apply. Rigorously justify everything.

You have 150 minutes for 120 points (+5 Bonus).

1. The Martian **[15pts]**

Matt Damon has been abandoned on Mars, and immediately sets up two firms to help him survive: Farm Co. and Tech Co, from which he retains all profits. Tech Co. converts labor hours h_x into technology x according to $x(h_x) = \sqrt{h_x}$. Farm Co. takes labor hours h_y and outputs yams y according to $y(h_y) = h_y$. Matt Damon has a single unit of time, which he spends sleeping h, working for Farm Co. (h_x) or working for Tech Co. (h_y) Damon receives utility according to:

$$u(h, p, t) = h^\alpha x^\beta y^{1-\alpha-\beta}, \qquad \text{where} \quad \alpha, \beta \geq 0, \quad \text{and} \quad \alpha + \beta < 1$$

(a) Write the production possibility set. [2]

$\{(-h, x, y) \mid 0 \leq h \leq 1, \quad x^2 + y \leq h\}$

(b) Calculate firms' profits and labor demand, and find Matt's demand for h, x, and y. [5]

Normalize the price of good x to $p_x \equiv 1$. Farm Co. has linear technology. Thus, $p_y = w$.

Tech Co. optimizing yields:

$$\pi_x = p_x \sqrt{h_x} - w h_x \implies h_x = \frac{p_x^2}{4w^2} \implies \pi_x = \frac{p_x^2}{4w}$$

Matt Damon's demand is given by:

$$h^* = \frac{\alpha}{w}\left(w + \frac{1}{4w}\right), \quad x^* = \beta\left(w + \frac{1}{4w}\right), \quad y^* = \frac{(1-\alpha-\beta)}{p}\left(w + \frac{1}{4w}\right)$$

(c) Find equilibrium prices and consumption when $\alpha = 0$. [4]

Because we only have two markets, we just need to clear one. Let's focus on the market for tech:

$$\beta\left(w + \frac{1}{4w}\right) = \frac{1}{2w} \implies 2 - \beta = 4\beta w^2 \implies w^* = \frac{1}{2}\sqrt{\frac{2-\beta}{\beta}} = p$$

(d) How does your answer to (c) change when $\alpha > 0$? [4]

The same market clearing condition guarantees:

$$w = \frac{1}{2}\sqrt{\frac{2-\beta}{\beta}}$$

But now Matt Damon reduces his consumption of the linear technology good y and shifts it into sleep h.

2. Stackelberg with Fixed Costs **[30pts]**

Firms 1 and 2 compete by sequentially choosing quantities. They face market demand $P(Q) = a - Q$, where $Q = q_1 + q_2$. Firms have total cost functions:

$$C_1(q_1) = f_1 \qquad \text{and} \qquad C_2(q_2) = f_2$$

Here, f_1 and f_2 are escapable fixed costs, with $f_1, f_2 < a/4$.

Firms act *sequentially*: First, Firm 1 sets q_1, and then Firm 2 sets q_2 after observing q_1.

(a) If the firms act together as a cartel, what quantities will they choose? [3]

The firm with lower fixed costs produces the monopoly quantity $a/2$ so long as it makes profits. Either firm is profitable, since $f_1, f_2 < a/4$, the profit with price $a/2$.

(b) Compute the best response of Firm 2 as a function of the quantity of Firm 1. [7]

When Firm 2 produces, Firm 1's best response is to act as a monopolist on the residual quantity:

$$q_2(q_1) = \frac{a - q_1}{2}$$

If they produce this quantity, Firm 2's profits are given by:

$$q_2\left[a - q_1 - q_2\right] - f_2 = \left(\frac{a - q_1}{2}\right)^2 - f_2$$

Firm 2 will not produce if profits are negative. Thus the function is piecewise:

$$q_2^*(q_1) = \begin{cases} 0 & \text{if} \quad \left(\frac{\alpha - q_1}{2}\right)^2 \leq f_2 \\ \frac{\alpha - q_1}{2} & \text{if} \quad \left(\frac{\alpha - q_1}{2}\right)^2 > f_2 \end{cases}$$

(c) Intuitively explain the shape of Firm 1's profit as a function of q_1. [5]
(Hint: You do not need to calculate the full best-response of Firm 1.)

The piecewise nature of Firm 2's best response means that Firm 1's profits will be discontinuous at the quantity where Firm 2 is deterred from entry.

(d) Suppose $f_1 \leq f_2$. For what f_1 and f_2 do the firms produce at the cartel level? [5]

The cartel outcome is realized if Firm 1's monopoly quantity deters entry from Firm 2. That happens when $f_2 > \frac{a^2}{16}$.

Now assume Firm 2 consists of a vast competitive fringe — specifically, each of a continuum of firms can enter and incur a zero fixed cost, but constant marginal cost $c > 0$ for production. Each is negligible and acts as if it has no impact on the market demand.

(e) What mass of such firms enter and what price emerges? [5]

The competitive fringe takes price as given, as their quantity choice has no impact on the price level. Thus, firms will enter until price is equal to their marginal cost c_2. Firm 1 now chooses quantity knowing that price will ultimately be c_2. Because

Firm 1 has no marginal cost, they will either produce the full quantity: $q_1 = a - c_2$ or nothing at all, depending on their profits:

$$q_1 = \begin{cases} 0 & \text{if } f_1 > (a - c_2)c_2 \\ a - c_2 & \text{if } f_1 \leq (a - c_2)c_2 \end{cases}$$

Thus the mass of entering competitive firms is either 0 or $a - c_2$.

(f) How does your answer to (e) change if Firm 1 has marginal cost $c_1 < c_2$? [5]

The competitive fringe still forces down the price to c. Firm 1 will choose between the same two quantities, though it will be incentivized to drop out for even higher c values (or lower f_1 values):

$$q_1 = \begin{cases} 0 & \text{if } f_1 > (a - c_2)(c_2 - c_1) \\ a - c_2 & \text{if } f_1 \leq (a - c_2)(c_2 - c_1) \end{cases}$$

Thus the mass of entering competitive firms is either 0 or $a - c_2$.

3. Practice Makes (Im)Perfect **[20pts]**

Jonathan's neighbor just bought their son a trumpet, and he really sucks. Unfortunately for Jonathan, his apartment building allows the kid to make as much noise as he wants. If the kid plays trumpet for H hours, the utilities of Jonathan and his neighbor are:

$$u_J(H) = m_J - H^2 \qquad \text{and} \qquad u_N(H) = m_N + 16H - H^2$$

where m is money. Jonathan and his neighbor don't get along, so bargaining is costly. Namely, if the parties negotiate a reduction of Δ hours, Jonathan experiences a disutility of $\alpha\Delta$.

(a) Find the efficient and individually rational levels of trumpet practice. [3]

Efficient level H^* satisfies: $16 = 4H \implies H^* = 4$

Individually rational level H' satisfies: $16 = 2H' \implies H' = 8$

(b) Suppose Jon's condo association decides that the neighbors must pay Jon constant damages τ for every hour of practice. What is the optimal τ? [2]

The optimal level of damages decentralizes the efficient outcome:

$$(16 - \tau) = 2H^* = 8 \implies \tau = 8$$

(c) Now suppose there are no damages charged. If Jonathan bargains with his neighbor, how much will the kid play trumpet, and what transfers will occur? [10]

The two parties should negotiate so long as the cost of bargaining is less than the utility gained by reducing the trumpet playing.

Jon has a cost:

$$u_J = (8 - \Delta)^2 + \alpha\Delta$$

His Marginal cost in Δ is thus:

$$\alpha - 2(8 - \Delta)$$

Setting this marginal cost to the neighbors equal to the marginal benefit of Jonathan:

$$-16 + 2(8 - \Delta) = \alpha - 2(8 - \Delta) \implies \alpha - 16 + 2\Delta = -2\Delta \implies \Delta = \frac{16 - \alpha}{4}$$

This implies that the negotiated noise reduction Δ is:

$$\Delta^* = \begin{cases} 0 & \text{if} \quad \alpha \geq 16 \\ \frac{16-\alpha}{4} & \text{if} \quad \alpha \leq 16 \end{cases}$$

All parties need to receive transfers such that they are better off than before bargaining. Thus, transfers must lie between:

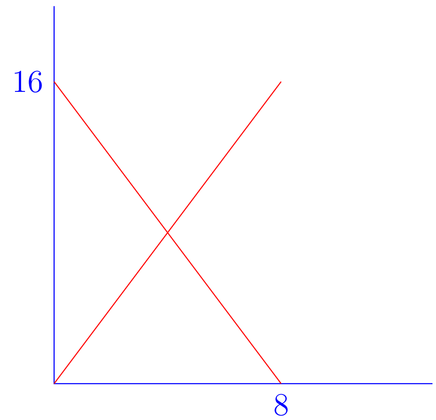
(d) Suppose Jonathan's Condo sets up an internal market for noise-hour permits, and issues 8 such permits to the Neighbor. Compute and graph the induced supply and demand for permits, and find the equilibrium permit price. [5]

For Jonathan, we have:

$$u_J = m - (8-q)^2 - pq \implies p(q) = 16 - 2q$$

For the neighbors, we have:

$$u_N = m + 16(8-q) - (8-q)^2 + pq \implies p(q) = 2q$$



Intersecting these gives the equilibrium price:

$$4 = q$$

4. Enter the Wu-Tang **[15pts]**

The nine members of the Wu-Tang Clan make a new album while also trying to advance their solo careers. Each member i is endowed with w_i units of studio time, to be spent on the group album t_i or on their solo projects s_i. (Negative s_i consumption is allowed.) If the total contribution is $T = \sum_{i=1}^{9} t_i$, the group produces an album of quality: $Q(T) = \sqrt{T}$.

Each member has a utility function:

$$u_i(s_i, Q) = s_i + \alpha_i \log(Q), \qquad \text{where} \qquad \alpha_i > 0 \quad \forall i$$

(a) Find the socially efficient quality of the Wu-Tang Album. [5]

We must use the Samuelson condition:

$$\sum_{i=1}^{9} MRS^i_{Q,s} = \sum_{i=1}^{9} \frac{\alpha_i}{Q} = \frac{1}{\frac{1}{2}T^{-1/2}} = 2\sqrt{T} = MRT_{Q,s}$$

Rearranging and plugging in for $Q = \sqrt{T}$, we get:

$$\sum_{i=1}^{9} \alpha_i = 2T$$

Which implies:

$$T^* = \frac{\sum_{i=1}^{9} \alpha_i}{2}$$

(b) What share of the public good does each member pay in a Lindahl equilibrium? [5]

Consumers solve the problem:

$$\max_Q \left\{ \frac{w_i - p_i Q^2}{\alpha_i} + \ln(Q) \right\}$$

which implies

$$\frac{2p_i Q}{\alpha_i} = \frac{1}{Q} \implies p_i = \frac{\alpha_i}{2T^*} = \frac{\alpha_i}{\sum_{i=1}^{9} \alpha_i}$$

Positive consumption of s_i requires $\theta_i \geq \alpha_i/2$, but this is satisfied by assumption.

Suppose that instead of using the Lindahl prices, the members are required to spend some fraction τ of their studio time on the group album (i.e., τ functions as a wealth tax).

(c) If member i sets this tax rate dictatorially, what tax rate τ_i do they choose? [5]

If individual i imposes tax rate τ, their utility is given by:

$$w_i(1-\tau) + \alpha_i \log\left(\sqrt{\tau \sum_{i=1}^{9} w_i}\right)$$

Optimizing gives:

$$-w_i + \alpha_i \frac{\sqrt{\sum_{i=1}^{9} w_i}}{2\tau\sqrt{\sum_{i=1}^{9} w_i}} = 0 \implies \tau_i^* = \frac{\alpha_i}{2w_i}$$

(d) (Bonus) Members submit these preferred tax rates to a vote. What tax rate is chosen? Is the public good necessarily over- or under-provided, or does it depend? [5]

It depends. The vote will select the median of $\frac{\alpha_i}{w_i}$—call this agent j. But this will always produce $\frac{9\alpha_j}{2}$. The winner of the election can be the lowest or the highest α_j-value.

5. In the Box with "Out of the Box" **[25pts]**

Tony and **V**ivian host the children's **TV** show, "Out of the Box," where they work together to create the ultimate fort out of cardboard boxes. Upon the show's cancellation, Tony and Vivian part ways, and each construct their own separate cardboard box forts. Forts can be built using long boxes (L) and wide boxes (W). Their utility functions are identical and given by: $U(L, W) = \min\{2L + W, L + 2W\}$. Initial endowments are: $(W_V, L_V) = (4, 0)$ and $(W_T, L_T) = (0, 6)$.

(a) In an Edgeworth box, depict the the endowment point and set of socially optimal allocations (the "contract curve"): [2]

The income expansion path is given by setting:

$$2L + W = L + 2W \implies L = W$$

For the complete figure, see below.

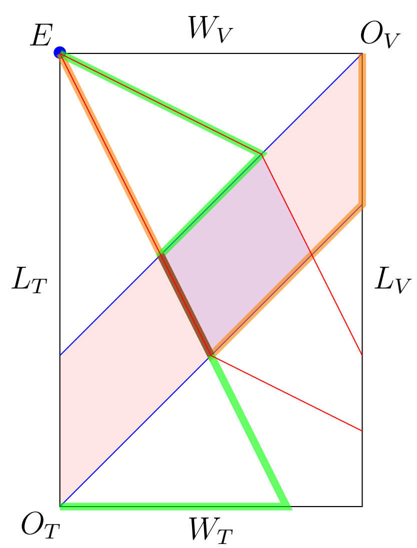
Since consumers have identical preferences, two indifference curves can overlap along an entire line where the goods are substitutable in the same ratios for the two consumers. The shaded region captures all such overlapping regions of the indifference curves.

(b) Define the *core*, and depict the core of this economy. [4]

The core is the set of allocations that are robust to coalitional deviations. In the two player game, it's just the set of allocations that are pareto optimal and individually rational. Highlighted in blue below.

(c) Carefully draw the trade offer curves. Find all competitive equilibria. [6]

The full figure is depicted below:



Contract curve: Red area. Core: Blue area. TOC_T: Orange. TOC_V: Green. Competitive Equilibrium: Gray.

Competitive Equilibria are those allocations that give Tony a utility of 6 and Vivian a utility of 8:

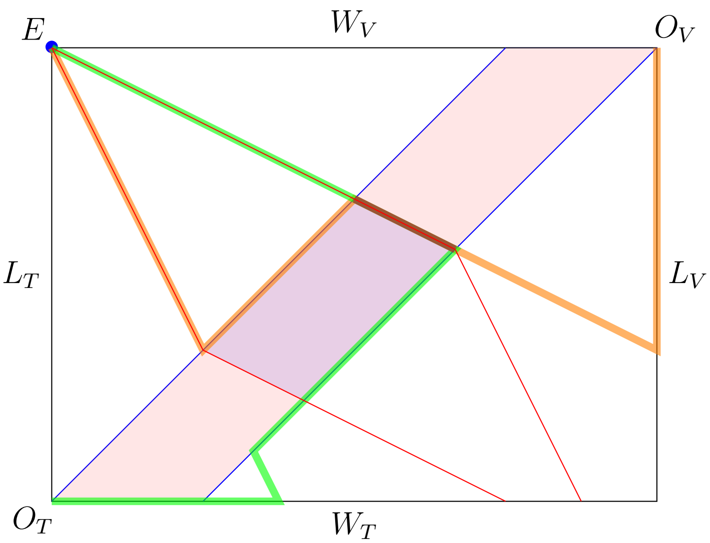
$$CE = \{(W_V, L_V), (W_T, L_T) \mid L_T = 6 - 2W_T,\ L_V = 8 - 2W_V, W_V > 2, W_T > 4/3\}$$

(d) Suppose Vivian's endowment is multiplied by α. How does the competitive equilibrium price ratio change with α? [8]
(Hint: It may be helpful to sketch a new figure.)

For $\alpha < 1.5$, the competitive equilibrium price ratio is identical to that in part (d): $p = 2$. When $\alpha = 1.5$, any price ratio between 2 and 1 gives a competitive equilibrium, while only a price ratio of 1 holds in the competitive equilibrium for $\alpha > 1.5$.

To see this, note that for $\alpha = 1.5$, the income expansion paths overlap perfectly, thus the competitive equilibria coincide with the core.

The full figure for $\alpha = 2$ is depicted below:



Now the competitive equilibria are those allocations where Vivian has utility of 8 and Tony has utility of 12.

(e) Characterize agents' utilities in the core of the 2−replica economy, with two Tony's and two Vivians. Are there allocations in the original core which are blocked in the replica economy? [8]
(Hint: Consider the total utility on the contract curve?)

We have a few types of constraints:

Autarky constraints:

$$u_T^i \geq 6, \qquad u_V^i \geq 4$$

Two-person constraints:

$$u_T^i + u_V^j \geq 14$$

Three-person constraints:

$$u_T^i + u_V^1 + u_V^2 \geq 20, \qquad u_V^i + u_T^1 + u_T^2 \geq 20$$

Grand Coalition:

$$u_V^1 + u_V^2 + u_T^1 + u_T^2 = 28$$

Together, these constraints imply:

$$u_T^i \in [6, 8], \qquad u_V^i \in [6, 8]$$

In the replica economy, allocations from the original core which result in $u_V \in [4, 6]$ are blocked, and thus the core is shrinking. That is, for both Tony and Vivian, the acceptable core allocations in the replica economy are a subset of the core allocations in the non-replica economy.

6. Uncertainty **[15pts]**

Jonathan and Elise are worried about Coronavirus. Jonathan has a probability α of getting sick. Elise only gets sick if Jonathan gets sick. If Jonathan gets sick, then Elise has a probability β of getting sick. Any healthy person is endowed with 2 units of consumption good x, while any sick person is endowed with 1 unit of x. Utility is given by $u(x) = \log(x)$.

Before learning who is sick, Jonathan and Elise trade in a full set of state-contingent consumption claims for three states of the world:

– State 1: Neither have the virus.
– State 2: Only Jonathan has the virus.
– State 3: Both have the virus.

(a) Write the expected utility functions for Elise and Jonathan. [3]

For Jon and Elise, expected utilities are given by:

$$\mathbb{E}u(x) = (1-\alpha)\log(x_1) + \alpha(1-\beta)\log(x_2) + \alpha\beta\log(x_3)$$

(b) Normalize the price of state 1 consumption to $p_1 = 1$, $p_2 = p$, $p_3 = q$. Solve explicitly for p in terms of α and β. [7]

Market one clearing gives:

$$4 = (1-\alpha)\left[4 + 3p + 2q\right] \implies \frac{4\alpha}{1-\alpha} = 3p + 2q$$

Market three clearing gives:

$$2 = \frac{\alpha\beta(4+3p+2q)}{q} \implies 2q(1-\alpha\beta) = \alpha\beta(4+3p) \implies q = \frac{\alpha\beta(4+3p)}{2(1-\alpha\beta)}$$

When paired with the constraint in market 1, this gives:

$$\frac{4\alpha}{1-\alpha} = \frac{3p+4\alpha\beta}{1-\alpha\beta} \implies p = \frac{4\alpha(1-\alpha\beta) - 4\alpha\beta(1-\alpha)}{3(1-\alpha)} = \frac{4\alpha(1-\beta)}{3(1-\alpha)}$$

Thus we can also write $q = \frac{2\alpha\beta}{1-\alpha}$.

Now suppose that Elise is tested for Coronavirus, and so observes her true state.

(c) Find prices in a revealing equilibrium. [5]

Case 1: Elise has Coronavirus.

$$2 = \frac{1}{q}(4 + 2q + 3p) \implies 0 = \frac{4}{q} + \frac{3p}{q} \implies q = \infty.$$

Case 2: Elise doesn't have coronavirus.

$$4 = \frac{1-\alpha}{1-\alpha\beta}(4+3p) \implies 4 - 4\alpha\beta = (1-\alpha)(4+3p) \implies \frac{4\alpha(1-\beta)}{3(1-\alpha)} = p$$

(d) (Bonus) Argue that for arbitrarily chosen α and β, there is generally no concealing equilibrium. [5]

We just need a few inequalities that show a deterministic relationship between α and β. It's easiest to get these by looking at the state of the world where Elise has the virus. There, market clearing for markets 1 and 2 give:

$$4 = (1-\alpha)(2+p+q) \implies \frac{2+2\alpha}{1-\alpha} = p+q$$

$$3 = \alpha(1-\beta)(2+p+q) \implies \frac{3-2\alpha(1-\beta)}{\alpha(1-\beta)} = p+q$$

Thus if we are to have a concealing equilibrium, we must–as a bare minimum–have:

$$\alpha\beta(2+2\alpha) = (2-2\alpha\beta)(1-\alpha) \implies 2\alpha\beta = (1-\alpha)$$

This is not satisfied for generic α and β.