Econ 713 Midterm

UW-Madison

National Dream Day ♣, March 11, 2024

♣ *There are **115 points in 150 minutes.** (One point a minute is doable). Points are at right.*

♣ **Justify everything with graphs or algebra or a known theorem or class logic.** *Enjoy!*

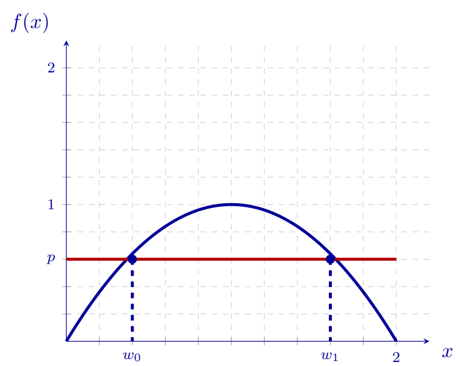
1. Factor meals is advertised as a "*chef-prepared, ready to eat*" Madison meal delivery service. Consumers pay a subscription fee p for a fixed number of meals home delivered. A consumer's value of the service depends of his income: Low income consumers consider the service not worth it, as they can cook their own meals. By the other hand, high income consumers perceive the product as low quality. All told, the net value of a consumer with income w derives at price p is: **[15]**

$$v(w,p) = 1 - (1-w)^2 - p.$$

Consumers subscribe to the service if $v(w,p) \geq 0$. A continuum unit mass of consumers live in Madison, with income between 0 and 2 and has cdf $F(w) = w^2/4$. Find the optimal subscription fee, if the subscription service costs the firm $c \in (0,1)$.

Solution: We solve for the aggregated demand when the subscription is p,

$$1 - (1-w)^2 - p = 0 \quad \Rightarrow w = 1 \pm \sqrt{1-p}$$



Only consumers with wealth $w \in [w_0, w_1]$ subscribe to the service, where $w_0 = 1 - \sqrt{1-p}$ and $w_1 = 1 + \sqrt{1-p}$. Next, we compute the market demand,

$$D(p) = F(w_1) - F(w_0) = \frac{(1+(1-p)^{1/2})^2}{4} - \frac{(1-(1-p)^{1/2})^2}{4} = \sqrt{1-p}.$$

Therefore, the maximization problem of factor meals is

$$\max_p \, (p-c)(1-p)^{1/2}.$$

F.O.C.:

$$(1-p)^{1/2} - \frac{p-c}{2(1-p)^{1/2}} = 0 \quad \Longrightarrow 2(1-p) = p - c \quad \Longrightarrow p = \frac{2+c}{3}.$$

2. A continuum unit mass of people lives in Chicago. Each weekend, residents choose to (g)o out or (s)tay at home. To go out, people must drive downtown, the utility from this activity depend of the traffic T and the driver's impatience θ, which is uniformly distributed between 0 and 1 across Chicagoans. Given impatience θ, the benefit of action $a \in \{g, s\}$ is

$$\beta(a, \theta) = \begin{cases} 1 - \theta T & \text{if } a = g \\ 0 & \text{if } a = s \end{cases}$$

If a mass x of people go out driving, the traffic is $T(x) = x/2 + x^2$.

(a) Characterize the Nash equilibrium. **[5]**

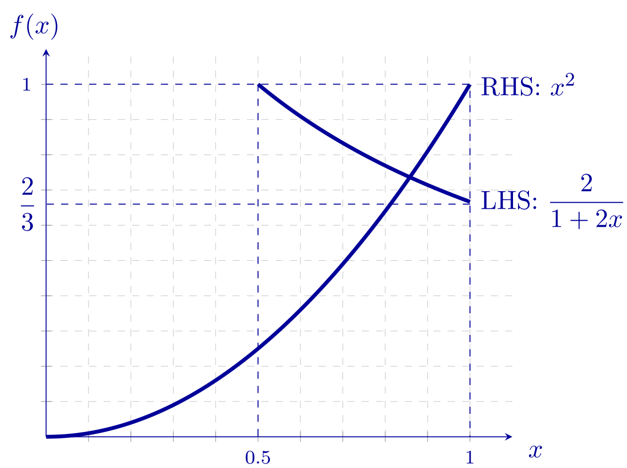
(b) Chicago considers adding bicycle lanes downtown, which offers a non-driving commuting option with benefit $7/8 - x^3$ — where the subtracted term reflects the risk of accidents that cyclists face when x mass of people is driving. Characterize the new equilibrium. How many people stay home now? **[10]**

(c) Chicago's mayor charge a road fee τ on drivers. Obtain the Nash equilibrium masses of drivers and cyclists in Chicago as a function of this fee. **[5]**

(d) Chicago's mayor wants the road fee that maximizes the integral of benefits in the city. Write down the mayor's maximization problem and state the first order condition. **[5]**

Solution: (a) We solve for the marginal type $\overline{\theta}$ indifferent between going out or staying home. Intuitively, all types $\theta \leq x_d$ take the road each weekend, and higher types $\theta > x_d$ stay home. Keep in mind that if type $\overline{\theta}$ is indifferent, then a mass $x = \overline{\theta}$ of drivers are on the road — since the impatience parameter is uniformly distributed on $[0, 1]$. The marginal agent that is indifferent solves:

$$1 - x(x/2 + x^2) = 0 \;\Rightarrow\; \frac{2}{1 + 2x} = x^2.$$



This has a unique solution $x_d \in (1/2, 1)$. There are a mass $x_d > 0$ of drivers on the road, and a mass $1 - x_d > 0$ staying home.

(b) Note that now highly impatience types may choose to bike to go out, since the benefit of using the bike lane does not depend on their type. We look for a threshold θ such that types lower than θ will drive, and higher types will cycle. The type x who is indifferent between driving or cycling solves:

$$1 - x(x/2 + x^2) = 7/8 - x^3 \;\Rightarrow x^* = 1/2$$

Since $x^* < x_d$, every agent now derives a strictly positive benefit from going out. Chicagoans with impatience $\theta \leq 1/2$ drive, meanwhile those with $\theta > 1/2$ use the bicycle lane since $7/8 - 1/2^3 > 0$. All told, fewer drive, but everyone chooses to go out, and social welfare rises.

(c) Agents have three options: drive, bike or stay home. Let τ be the road fee for drivers. The benefit from each option is:

$$\begin{aligned} \text{Drive} &: 1 - \theta(x/2 + x^2) - \tau \\ \text{Bike} &: 7/8 - x^3 \\ \text{Home} &: 0 \end{aligned}$$

Let $\overline{\theta}_d(\tau)$ be the impatience of the agent that is indifferent between driving or staying at home when charged fee τ:

$$\overline{\theta}_d(\tau) \text{ solves: } 1 - x(x/2 + x^2) - \tau = 0.$$

Next, we solve for the impatience $\theta_b(\tau_d)$ of the type indifferent between driving and biking:

$$1 - x(x/2 + x^2) - \tau = 7/8 - x^3 \qquad \Longrightarrow \; \theta_b(\tau) = \frac{(1 - 8\tau)^{1/2}}{2}$$

Since one can always stay, the mass of drivers is:

$$x_d(\tau) = \min\left\{\theta_b(\tau),\; \overline{\theta}_d(\tau)\right\}.$$

For cyclists, the maximum amount of traffic that leaves one indifferent between biking or staying at home is the threshold $\overline{x}$ solving:

$$7/8 - \overline{x}^3 = 0.$$

Thus, the mass of cyclists is:

$$x_c(\tau) = \begin{cases} 1 - x_d(\tau) & \text{if } \overline{x} \geq x_d(\tau) \\ 0 & \text{else} \end{cases}$$

(d) The mayor maximizes the total social benefit choosing the fee τ,

$$\max_{\tau} \int_{0}^{x_d(\tau)} (1 - y(x_d(\tau)/2 + x_d^2(\tau)) - \tau)\, dy + \int_{0}^{x_c(\tau)} (7/8 - x_d^3(\tau))\, dy$$

subject to:

$$x_d(\tau) = \min\left\{\theta_b(\tau),\ \overline{\theta}_d(\tau)\right\}, \quad x_c(\tau) = \begin{cases} 1 - x_d(\tau) & \text{if } \overline{x} \geq x_d(\tau) \\ 0 & \sim \end{cases}$$

[Considering tax revenue as a transfer is also valid] To obtain the FOC we note that choosing a fee that keep people at home is never socially efficient as produce zero benefit, hence $x_d(\tau) = \theta_b(\tau)$ and $x_c(\tau) = 1 - x_d(\tau)$,

$$\frac{\partial \mathcal{W}}{\partial \tau} : \ \frac{9}{8}\sqrt{1 - 8\tau} + 4\tau - \frac{1}{2} = 0.$$

3. In a new three player $k = 1, 2, 3$ game called TE-I-AM — with motto "there is an i in team" — coalitions bargain over a pot of money of size 17. Assume singleton coalition values $v(k) = k$, pairwise coalition values $v(i, j) = i + j + ij$, and the grand coalition has value **[15]**

$$v(1, 2, 3) = 1 + 2 + 3 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 17$$

What is the most and least that each player 1,2,3 secures in this game?

Solution: The core is not empty, since it is a supermodular game.

$$u_1 \geq 1,\ u_2 \geq 2,\ u_3 \geq 3.$$

This gives lower bounds. Meanwhile, subtract each of the pairwise constraints

$$u_1 + u_2 \geq v(1, 2) = 5, \quad u_1 + u_3 \geq v(1, 3) = 7, \quad u_3 + u_2 \geq v(2, 3) = 11,$$

from the grand coalition equality $u_1 + u_2 + u_3 = 17$ to deduce

$$u_1 \leq 17 - 11 = 6,\ u_2 \leq 17 - 7 = 10,\ u_3 \leq 17 - 5 = 12$$

Payoffs are $u_1 \in [1, 6]$, $u_2 \in [2, 10]$, $u_3 \in [3, 12]$

4. Firms $i \in \{1, \ldots, n\}$ sell widgets in a market with demand $D(p) = A - p$. Firms use different technologies to produce the good, however all of them power their processes with electricity, and take its price as given. The electricity required to produce q units of output is q. The cost function of firm i is thus

$$C_i(q) = \frac{b_i q^2}{2} + p_e q,$$

where p_e is the price of electricity which is provided by a monopolist that firms take as given. The marginal cost of the monopolist supplying the electricity is $c > 0$.

(a) Find the derived demand for electricity by widget makers. **[3]**

(b) Solve for the optimal pricing the monopoly electric company. **[5]**

(c) Assume $n = 3$ with $b_1 = b_2 = b_3 = 1$. A long power outage strikes, during which the electric company cannot provide power. Firms $i \in \{1, 2, 3\}$ own generators. The firms can produce electricity at constant marginal cost $\kappa > 0$. But only firm 1's generator is ready to be used, so firm one can send its production to the market ahead of firms 2 and 3. How much does each firm produce? **[12]**

(d) The generators produce noise. Each of them generates a marginal social cost $\text{MSC}(q) = q$ due the noise. In a figure characterize the social efficient output q^s and identify the deadweight loss of this situation. **[5]**

Solution: (a) To find the demand for electricity we solve the retails problem.

Firm i solves:

$$\max_{q_i} \, (A - Q)q_i - \frac{b_i q_i^2}{2} - p_e q_i$$

From FOCs, we obtain:

$$(A - Q) - q_i - b_i q_i - p_e = 0 \qquad \Rightarrow \qquad q_i = \frac{A - p_e - Q}{1 + b_i}.$$

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Let $\beta = \sum_{i=1}^{n} (1 + b_i)^{-1}$. Sum over all firms to obtain Q and thus the demand for electricity $D_e = Q$:

$$D_e(p_e) = \frac{A\beta}{1 + \beta} - \frac{\beta}{1 + \beta} p_e \equiv B(A - p_e)$$

.

(b) The monopoly electric company's maximization problem is $\max_{p_e} \, D_e(p_e)(p_e - c)$.

$$\text{FOC: } D_e(p_e) - Bp_e + Bc = 0,$$

$$B(A - p_e) - Bp_e - Bc = 0, \qquad \Longrightarrow p_e = \frac{A + c}{2}$$

(c) Use backward induction. For any quantity q that firm 1 produces, firms $j \in \{2,3\}$ solves

$$\max_{q_j} (A - q - q_j - q_{-j})q_j - \frac{b}{2}q_j^2 - \kappa q_j$$

FOC

$$\frac{\partial \pi_j}{\partial q_j} = (A - q - q_{-j}) - 2q_j - bq_j - \kappa = 0, \implies q_j = \frac{A - q - \kappa - q_{-j}}{2 + b}.$$

By symmetry of firm 2 and 3,

$$q_j(q_1) = \frac{A - \kappa - q_1}{3 + b}.$$

Next, we solve for firm 1

$$\max_{q_1} (A - q_1 - 2q_j(q_1))q_1 - \frac{b}{2}q_1^2 - \kappa q_1.$$

FOC

$$\frac{\partial \pi_1}{\partial q_1} = A - \kappa - \frac{2(A - \kappa - q_1)}{3 + b} - 2q_1 - bq_1 + \frac{2}{3 + b}q_1 = 0$$

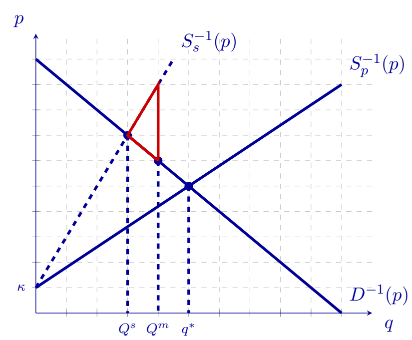
Solving for q_1 and replacing $b = 1$ we obtain the output of each firm and total output,

$$q_1 = \frac{1}{4}(A - \kappa), \qquad q_2 = q_3 = \frac{3}{16}(A - \kappa), \qquad Q^m = \frac{5}{8}(A - \kappa).$$

(d) The social efficient output Q^s follows from $C_i'(q_i) + MSC(q_i) = p$,

$$C_i(q) = \frac{q^2}{2} + \kappa q, \implies S_i(p) = \frac{p - \kappa}{2}.$$

We sum over all firms to get the socially efficient aggregated supply $S_s(p) = 3(p - \kappa)/2$ and we obtain the socially efficient output $Q^s = 3(A - \kappa)/5 < Q^m$, hence market power does not fully correct the externality. To have the full picture we get the aggregated supply without market power $S_p(p) = 3(p - \kappa)$,



5. In a competitive market, goods x and y are produced from labor (ℓ). The technologies are: **[15]**

$$x : f(\ell) = 4\ell \qquad y : g(\ell) = 2\ell$$

Type 1 consumers own firm x, and Type 2 consumers own firm y. All consumers are endowed with labor $\ell_i = 10$, and have preferences over the goods over the goods,

$$u_1(x, y) = x^{1/5}y^{4/5} \qquad \text{and} \qquad u_2(x, y) = xy.$$

If labor is numeraire, find the prices p_x and p_y.

Solution: In equilibrium the prices will be equal to the marginal rate of transformation, so $p_y/p_x = 2$. Labor is paid the value of its marginal product which is 4 in units of x. Thus, the price of x in numeraire is $1/4$, and the price of y is $1/2$.

6. Consider an economy with three consumers $i \in \{1,2,3\}$ and two goods x and y. Each consumer has Leontief preferences, but they differ in their taste for good x:

$$u_1(x,y) = \min\{x,y\}, \quad u_2(x,y) = \min\left\{\frac{x}{2}, y\right\}, \quad u_3(x,y) = \min\left\{\frac{x}{4}, y\right\}.$$

The initial endowment of the consumers is $e_1 = (100,0)$, $e_2 = (0,100)$ and $e_3 = (100,0)$. For the following questions use good x as the numeraire.

(a) Obtain the Walrasian equilibrium using the excess demand approach. **[5]**

(b) Show that consumer 1 can improve her situation by given away a part $\delta > 0$ of her endowment to consumer 2. Characterize δ. **[10]**

(c) Provide an explanation/intuition of your result in (b). **[5]**

Solution: (a) Let x be numeraire and p the price of y. Each consumer chooses optimal bundles in the proportions:

$$\text{Consumer 1}: x = y, \qquad \text{Consumer 2}: x = 2y, \qquad \text{Consumer 3}: x = 4y.$$

Therefore, the Marshallian demand for y is given by,

$$y_1(p) = \frac{100}{1+p}, \quad y_2(p) = \frac{100p}{2+p}, \quad y_3(p) = \frac{100}{4+p}.$$

Equilibrium is a zero of the excess demand function for market y is, The equilibrium price solves:

$$\frac{100}{1+p} + \frac{100p}{2+p} + \frac{100}{4+p} - 100 = 0$$

So the Walrasian equilibrium is price vector $(1,p) = (1,2)$ and the allocations of good x and y are $(100/3, 100, 200/3)$ and $(100/3, 50, 50/3)$.

(b) If Consumer 1 gives $\delta > 0$ to consumer 2, then the new excess of demand zero of p solves:

$$\frac{100-\delta}{1+p} + \frac{100p+\delta}{2+p} + \frac{100}{4+p} - 100 = 0 \quad \Rightarrow \quad p(\delta) = \frac{200-4\delta}{100+\delta}.$$

Provided $\delta \in (0,50)$, this improves the welfare of consumer 1 since his utility is his y demand, which is:

$$\frac{100-\delta}{1+p(\delta)} > \frac{100}{3} \quad \Leftrightarrow \quad \delta > 0.$$

(c) Consumer 1's demand for good y falls as it has given away part of her income, reducing the price of good y. When the price of good y falls consumer 2, the exporter of good y, is worse off. Note that given some units of good x to consumer 2 increase his demand of good y, but since consumer 2 preference dictates $x = 2y$, this increase is overshadow by the fall on demand due consumer 1 as her preferences over good y are stronger ($x = y$). On the other hand, consumer 3 is better off due the fall in price of good y and their demand for both goods increases. By a similar logic as before, due the preferences of consumer 3 ($x = 4y$) the increase in demand for good x overcome the increase in demand of good y, pushing the relative price of good y further down. Altogether, the demand of good y fall enough to improve the terms of trade for consumer 1 even after losing part of her endowment. This is known as the transfer paradox, first described by Leontief (1936)[1], a more recent and formal treatment can be found in Balasko (2014)[2]; Rao (1992)[3] provide a full characterization for the three consumer case with Leontief preferences.

[1] Leontief, Wassily (1936), "Note on the pure theory of capital transfers." In Explorations in Economics; Notes and Essays Contributed in Honor of F.W. Taussig, 84–92, McGraw-Hill, New York. [436].

[2] Balasko, Y. (2014), The transfer problem: A complete characterization. Theoretical Economics, 9: 435-444.

[3] Rao, M. On the transfer and advantageous reallocation paradoxes. Soc Choice Welfare 9, 131–139 (1992).