An Economic Theory Masterclass

Part I: Matching Foundations of Economics

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February 19, 2024

The Matching Paradigm as Metaphor Economic Interaction

- Simple model: Only the extensive margin (in or out) matters.

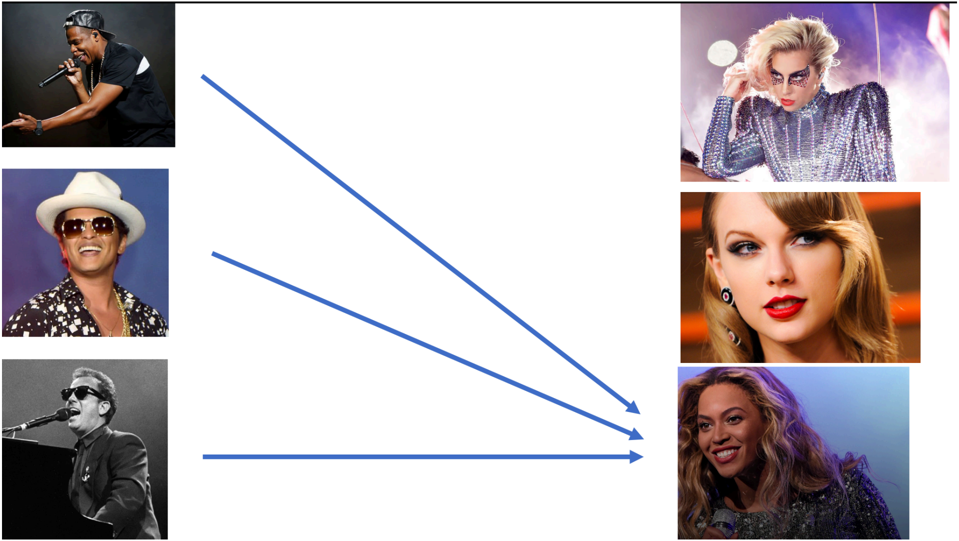
- Pairwise matching models with transferable utility capture in a simple story the economic structures of many settings:
  - assigning tasks to individuals
  - buyers and sellers trading
  - partnerships, and maybe marriages
  - firms hiring workers
- metaphor: two sides of the market are “men” and “women”
- We wish to understand: Who trades with whom? Who pairs with whom? Who marries whom? Who works with whom?

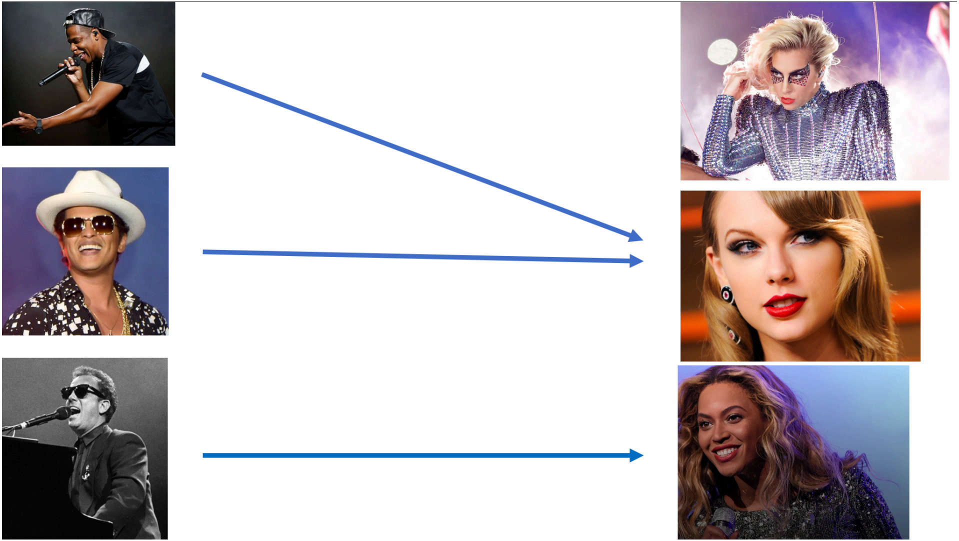
Matching without Transfers: The Girl-Guy Band Contest

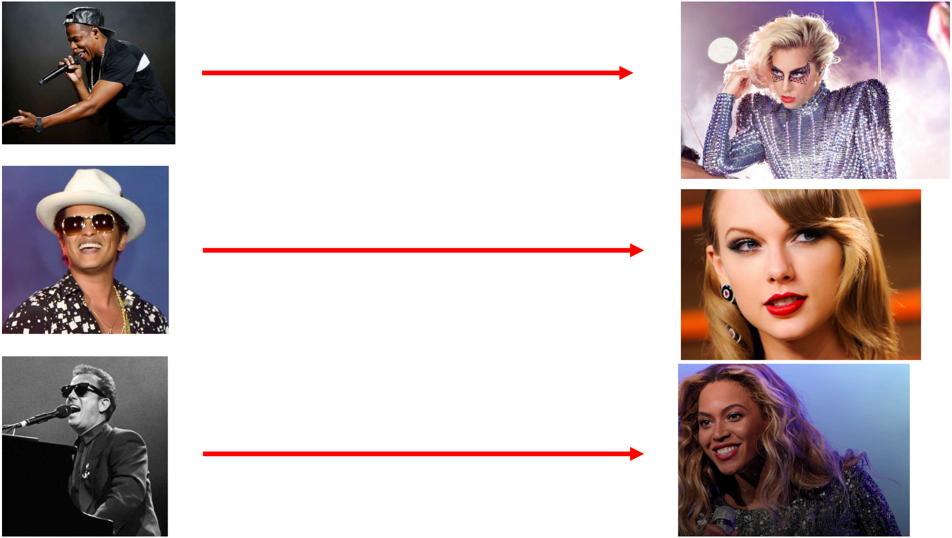
- Contest of Beyonce, Taylor Swift, and Lady Gaga to sing a duet with concert with Billy Joel, Bruno Mars, and Jay-Z
- We first only specify $\boxed{\textbf{ordinal preferences}}$
- Men commonly rank: Beyonce > Taylor Swift > Lady Gaga
- Women commonly rank: Billy Joel > Bruno Mars > Jay-Z

Stable Predictions for Pairwise Matchings

- Matchings must survive new double coincidence of wants
- An assignment is *unstable* if there are men, say Alan and Bob, respectively matched to women Alice and Bea, such that Bob prefers Alice to Bea and Alice prefers Bob to Alan
- Say that the matching of Bob and Alice blocks the matching.
- A matching is *stable* if it is not unstable, i.e. $\nexists$ blocking pair.







Deferred Acceptance Algorithm (DAA)

- Men have preferences over all women and not matching, and women have preferences over all men and not matching

1. All men start unengaged and women start with no suitors.
2. Each unengaged man *proposes* to his most-preferred woman (if any) among those he has not yet proposed to, if he prefers matching to remaining single.
3. Each woman gets *engaged* to the most preferred among all her suitors, including any prior engagements, if she prefers matching with him to remaining single.
4. Rinse and repeat until no more proposals are possible. Engagements become matches.

"I have to go. I'm getting a better call."

Gale-Shapley Theorem

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CONTENTS

Proposition (Gale & Shapley, American Math Monthly, 1962)

(*a*) *Then the DAA stops in finite time.*
(*b*) *Given an equal number of men and women, if matching with someone beats remaining single, then everybody matches.*
(*c*) *The DAA matching is stable, i.e. a stable matching exists.*
(*d*) *Given strict preferences, the DAA yields a unique matching.*

Proof of Gale-Shapley Theorem

- At each iteration, one man proposes to some new woman
- Let Alice and Bob be married, but not to each other.
- **Claim**: *After the DAA, Alice and Bob cannot prefer each other to their match partners.*
  - If Bob prefers Alice to his match partner, then he must have proposed to Alice before his match partner.
  - If Alice accepted, yet ends up not married to him, then she must have dumped him for someone she prefers
    $\Rightarrow$ Alice doesn't prefer Bob to her current partner.
  - If Alice rejected Bob's proposal, then she was already engaged to someone she prefers to Bob. □
- The contradiction proves the theorem!
- The paper's theorem includes many-to-one school matching
- Gale-Shapley (1962) was the 2nd *market design* paper — after Vickrey (1961), introducing second price auctions

Gale-Shapley Theorem

- The band matching example was trivial: When n men and n women had the same preference ranking, it ends in n rounds.
- **Claim**: With n men and n women and arbitrary preferences, there are at most n^2 possible ways men can propose.
  - A each stage, one man proposes to someone to whom he has never proposed before
  - With n men and n women, there are n^2 possible events
  - In fact, the maximum number of DAA steps is $n^2 - 2n + 2$.
    - Exercise: Illustrate this for the cases $n = 2$ and $n = 3$.
    - Proof is in Iota (1978) on canvas
- Al Roth found that the DAA was used to match interns to hospitals. This was a major reason for:

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

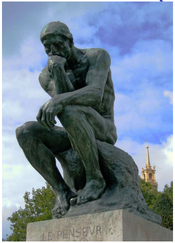
Photo: U. Montan
Alvin E. Roth
Prize share: 1/2

Photo: U. Montan
Lloyd S. Shapley
Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

Note: David Gale died in 2008.

Thinker: The Stable Roommates (i.e. Unisex) Problem



$$
\begin{array}{cc}
 & \begin{array}{cccc} \textit{Ann} & \textit{Beth} & \textit{Cathy} & \textit{Dee} \end{array} \\
\begin{array}{c} \textit{Ann} \\ \textit{Beth} \\ \textit{Cathy} \\ \textit{Dee} \end{array} &
\begin{pmatrix} - & 1 & 2 & 3 \\ 2 & - & 1 & 3 \\ 1 & 2 & - & 3 \\ 1 & 3 & 2 & - \end{pmatrix}
\end{array}
$$

- Show there is no stable allocation. Proof on wikipedia.
- Hint: If a stable allocation exists, someone rooms with Dee.
- Crucially, the DAA does not apply to the unisex model!

Gale and Shapley's Ranking of Stable Matchings

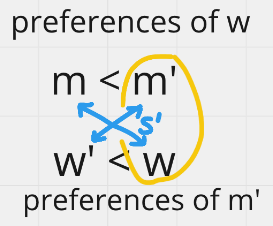
- Assume matching by women x and men y (from XX and XY)
- The set of stable matchings is nonempty.
- x is a **valid partner** of y if they pair in some stable matching.
  - **Male optimal**: each man pairs with best valid partner.
  - **Male pessimal**: each man pairs with worst valid partner.
  - Similarly define **woman-optimal** and **woman-pessimal**.

Proposition (Male Optimality of DAA)

The DAA finds a male-optimal / female-pessimal stable matching.

- Observation: The right to propose is hugely valuable, from bargaining, to agenda setting, to the DAA
- *Just as in the bargaining model, there is an advantage to making the proposal.*

Off Line: Tricky Proof that DAA is Male Optimal



- The proof is by contradiction
- If the DAA matching S is not male optimal, then a valid partner rejects some man, since men propose in order

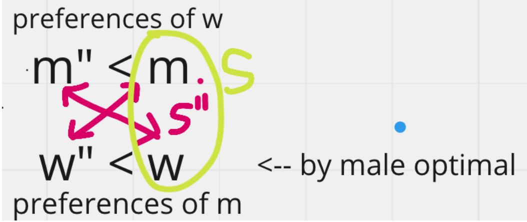
(★) Let m and w be the *first* such rejection in S

- This happens because woman w chose some man $\boxed{m' \succ_w m}$
- (m, w) paired in a stable matching S', since (m, w) is valid
- In stable matching S', let man m' pair with woman w', say
  - Note: m' was not rejected by a valid woman in S before (★)
  - If $w' \succ_{m'} w$ then m' offers to w' first, and must have been rejected if he was available to w, negating "1st" proviso in (★)

  $\Rightarrow$ $\boxed{w \succ_{m'} w'}$

$\Rightarrow$ m' and w form a blocking pair in S'

Off Line: Tricky Proof that DAA is Female Pessimal



- The proof is by contradiction
- Let m and w pair in the DAA matching S, and assume (for a contradiction) that m is not the worst valid partner for w

$\Rightarrow$ $\exists$ a stable matching S'' with w paired to m'', and $\boxed{m \succ_w m''}$

- In matching S'', let man m pair with woman w'', say
- $\boxed{w \succ_m w''}$ by male-optimality

$\Rightarrow$ m and w form a blocking pair in S''

3 Stable Matchings, but DAA Logic Can Only Get Two

	x_1	x_2	x_3
y_1	5,5	6,2	2,6
y_2	2,6	5,5	6,2
y_3	6,2	2,6	5,5

- The default DAA yields the male-optimal and female pessimal matching, where men earn 6 and women 2.
- In the DAA', women do the proposing, rather than men.

$\Rightarrow$ By the above reasoning, DAA' yields the female-optimal and male pessimal matching, where women earn 6 and men 2.

- A third stable matching yields payoffs of 5 for everyone.

Unique Stable Outcomes

- DAA': women do the proposing, rather than men.

Corollary (Uniqueness)

DAA and DAA' yield the same matching if and only if there is a unique stable matching.

- Since DAA and DAA' yield a stable matching, if the stable matching is unique, DAA and DAA' land at same matching
- If DAA and DAA' land at the same matching, then it is both optimal and pessimal for men, and so is unique. □

Offline & Easy: Incentive Issues with DAA

Proposition (Roth, 1982)

DAA is incentive compatible for men and DAA' is incentive compatible for women.

- Proof is omitted, since it is game theory.
- *But women might gain by misreporting their types in DAA.*
- Example:
  - Man A prefers $\boxed{\text{X}}$ to Y to Z, and Man B prefers $\boxed{\text{X}}$ to Z to Y
  - Man C prefers $\boxed{\text{Y}}$ to X to Z
  - Woman X prefers $\boxed{\text{C}}$ to A to B, and Woman Y prefers A to C.
- DAA: Men A & B propose to #1 woman X, and Man C to Y
- X retains A, and B proposes to Z next. Proposals end.
- In the end, X is matched to A
- Machiavellian Deviation by X:
  - X sneakily accepts B's proposal.
  - Then A proposes to Y.
  - Y leaves C for A.
  - Then C proposes to X
  - X ends up matched to top choice C

Cardinal Preferences

- Start with nontransferable payoffs (all in millions of dollars).
- This might be by organizational rule, eg. NCAA rules forbid payoffs to athletes.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	6,21	12,12	**18,3**
Bruno Mars	4,14	**8,8**	12,2
Jay-Z	**2,7**	4,4	6,1

- Men commonly rank: Beyonce $>$ Taylor Swift $>$ Lady Gaga
- Women commonly rank: Billy Joel $>$ Bruno Mars $>$ Jay-Z
- DAA ends in three periods!

Transferable Utility

- Assume cardinal payoffs (or cardinal utility) is money.
- Every man and woman cares only about total money
- This is a special case of quasilinear utility, or utility $U(a, z) = u(a) + z$, where a is a real action and z is money
- Quasi-linear utility precludes income effects on the action
- All fields assume quasilinear utility as a default

Transfers and Bribery

Lady Gaga's Corrupt Thought:

- Gaga schemes to match up with Billy Joel. To do this, she
  - bribes Billy more than his loss of $18 - 6 = 12$ to accept her,
  - pays Beyonce more than her loss of $3 - 1 = 2$, and
  - collects from Jay-Z less than his gain $6 - 2 = 4$ from matching with Billy
- These bribes on net cost as much as $12 + 2 - 4 = 10$. But Lady Gaga gains $21 - 7 = 14$ by matching with Billy Joel.
- We start with this matching

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	6,21	12,12	**18,3**
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Making Matching Immune to Bribery

- The bribery scheme's profitability only depends on total match payoffs

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Jay-Z	$6+21=27$	$12+12=24$	$18+3=21$
Bruno Mars	$4+14=18$	8+8=16	$12+2=14$
Billy Joel	$2+7=9$	$4+4=8$	$6+1=7$

Making Matching Immune to Bribery

- The bribery scheme's profitability only depends on total match payoffs
- Now, the cardinal strength of each party's preference matters.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	27	24	21
Bruno Mars	18	16	14
Jay-Z	9	8	7

Making Matching Immune to Bribery

- A matching is **immune to bribes** if there is no set of matched individuals for whom a profitable re-matching exists.
- An **efficient** matching maximizes the sum of payoffs.

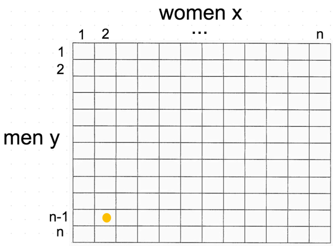
My Theorem *An efficient matching is immune to bribes.*

- Proof: If some bribery scheme is profitable, then rematching those people raises total match output.

♂╲♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	**27**	24	21
Bruno Mars	18	**16**	14
Jay-Z	9	8	**7**

Efficient Matching

- Matching Sudoku: Efficiently match n men to n women.
- $=$ Place exactly one dot in every row and column



- Obviously, an efficient matching exists. But what is it?
- Problem: There are $n! = 1 \times 2 \times \cdots \times n$ possible allocations.
- E.g. there are 10^{158} pairings of 100 men and 100 women. The number of electrons in the universe is estimated at 10^{80}.

Historical Background: "Transportation Problem" (1781)

666. MÉMOIRES DE L'ACADÉMIE ROYALE

MÉMOIRE

SUR LA

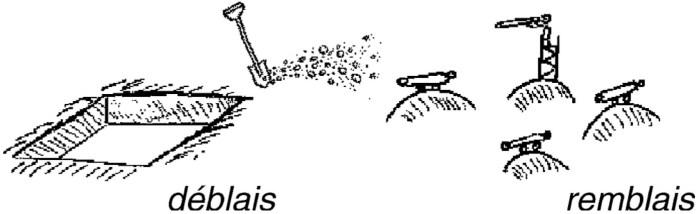
THÉORIE DES DÉBLAIS

ET DES REMBLAIS.

Par M. MONGE.

1781 — Transportation Problem: How Best to Move Dirt

- Transportation problem: a classic resource allocation problem
- The cost $c(x, y)$ of moving dirt from a cut (déblais) x to to a fill (remblais) y depends on the distance, roads, etc.
- Assign unit dirt piles $x_i \in \{x_1, \ldots, x_n\}$ to holes $y_j \in \{y_1, \ldots, y_n\}$ to minimize the sum of transportation costs $c(x_i, y_j)$?

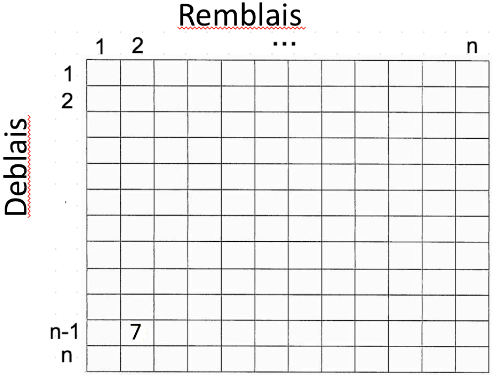


- What is the cheapest way to transport all the earth from every déblais to some other remblais, while omitting no déblais and overfilling no remblais?
- As formulated, this is an impossible combinatorics exercise.

1781 — The Transportation Problem

- Start with an $n \times n$ matrix of costs $[c(x_i, y_j)]$
- E.g: It costs 7 to move the dirt in déblais $n-1$ to remblais 2
- Solve the *minimization* $\sum_{i=1}^{n} c(x_i, y_{i\text{'s partner}})$
  - Maximizing payoffs is the same as minimizing negative payoffs
- The problem is doomed with combinatorial math methods.

$\Rightarrow$ Lesson: *Need to reformulate the story to make it solvable!*



1957: Transportation Problem as the Assignment Problem

- 160 years passes and linear programming is invented in WWII, by many in USA (e.g. Dantzig) and Kantorovich in Russia
- *The TU matching story is so great (i.e. general) it also captures the assignment model (& other economic models!)*

Assignment Problems and the Location of Economic Activities

Tjalling C. Koopmans; Martin Beckmann

Econometrica, Vol. 25, No. 1 (Jan., 1957), 53-76.

$$
\text{Plants}\;
\begin{array}{c|cccc}
 & \multicolumn{4}{c}{\text{Locations}} \\
 & 1 & 2 & 3 & 4 \\
\hline
1 & 25 & \mathbf{20} & 5 & 19 \\
2 & 18 & 3 & \mathbf{0} & 12 \\
3 & \mathbf{22} & 4 & 2 & 12 \\
4 & 16 & 7 & -2 & \mathbf{10}
\end{array}
\qquad
\text{Plants}\;
\begin{array}{c|cccc}
 & \multicolumn{4}{c}{\text{Locations}} \\
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1
\end{array}
$$

3. AN EQUIVALENT LINEAR PROGRAMMING PROBLEM

This problem is obtained by blandly ignoring the indivisibilities of plants, and admitting the assignment of *fractional plants* to locations in our model even though this is supposed to be meaningless from a realistic point of view.

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975

Leonid Vitaliyevich Kantorovich, Tjalling C. Koopmans

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The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975

Leonid Vitaliyevich Kantorovich

Prize share: 1/2

Tjalling C. Koopmans

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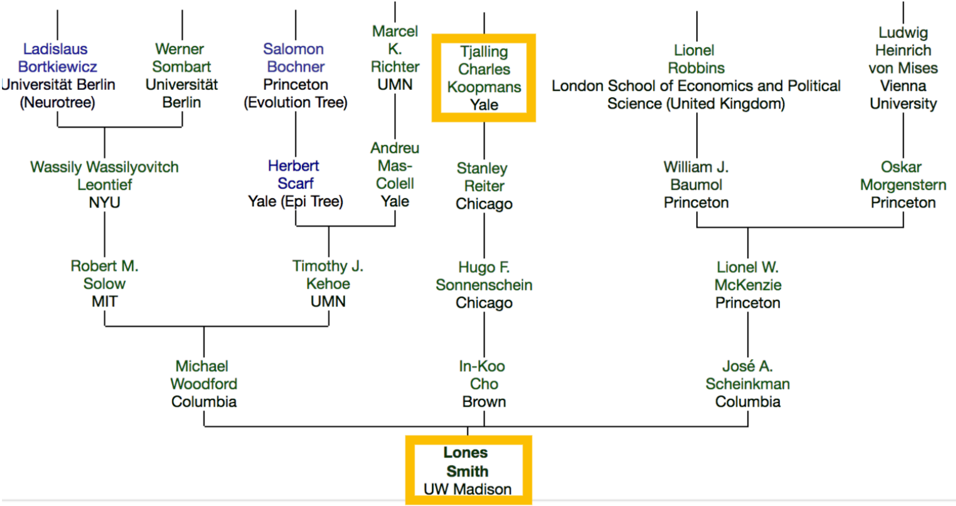

Rear View Mirror on Last Class

I. Ordinal preferences in a matching model of ‘men’ & ‘women’

- DAA leads to stable matching (no blocking pairs)
  - Men are willing to report preferences to a DAA machine.
  - Women can sometimes game these algorithms
- With more than one stable matching, we claimed (no proof):
  - Men all agree ranking stable matchings. So do women.
  - DAA gives the male optimal and female pessimal matching

II. We shifted to cardinal preferences with monetary transfers.

- Our stable allocation might be destabilized by bribes.
- Efficient allocation (max match payoff sum) cannot be bribed.

Koopman's Idea: Convexify the Feasible Matchings Space

- Choices or Actions
- *Finitely* many women x and men y (from XX and XY)
  - $m(x,y) = 1$ if x is matched to man y, and $m(x,y) = 0$ if not.
  - So a woman x remains single if $m(x,y) = 0$ for all $y \in Y$.
- Matching Space $\mathcal{M} = [m(x,y)]$ are all *feasible* matchings
  - $\mathcal{M}$ is *symmetric*: $m(x,y) = m(y,x)$ for all x, y
  - $\mathcal{M}$ is *convex* provided:
    - A fraction $m(x,y) \geq 0$ of woman x matches with man y
    - Or, with a continuum mass of men and women of finitely many types $\{x_i, y_j\}$, a mass $m(x,y)$ of types x and y match.
  - $\mathcal{M}$ is *bounded* (*no overmatching* any man or woman)
    - Finite world: for every x, $m(x,y) = 1$ for at most one y, and for every y, $m(x,y) = 1$ for at most one x.
    - Convex world: $\sum_y m(x_0, y), \sum_x m(x, y_0) \leq 1 \ \forall x_0, y_0$

Socially Efficient Matching with Transferable Utility

- $h(x, y)$ = match payoff of man x and woman y
  - Normalize unmatched payoff to zero: $h(x, \emptyset) = h(\emptyset, y) = 0$
- A **(socially) efficient matching** $[m^*(x, y)]$ maximizes the sum of all match outputs $\sum_x \sum_y m(x, y)h(x, y)$ over $m \in \mathcal{M}$

Proposition

An efficient matching $m \in \mathcal{M}$ exists.

- Proof: By Weierstrass Theorem, the maximum of a continuous function (the sum) on a compact set exists
- Compactness is trivial with finitely many types.
  - With a type continuum, we need weak-* topology. (hard

Decentralizing the Matching Market with Middlemen

- Payoffs: We derive **wages** $v(x)$ & $w(y)$ of women x & men y
- *Middlemen* compete in wages, earning *profits* for (x, y) match:

$$h(x, y) - v(x) - w(y)$$

- Free exit of middlemen $\Rightarrow$ profits ≥ 0 for all actual matches

$$v(x) + w(y) \leq h(x, y) \text{ if } m(x, y) > 0$$

  - This may capture a "free market", possibly with market power
  - I.e. No one is forced to stay in a market
- Free entry of middlemen $\Rightarrow$ profits ≤ 0 for all matches

$$v(x) + w(y) \geq h(x, y) \text{ for any } (x, y)$$

  - i.e. No profitable opportunity goes unexploited!
  - Free markets need not be competitive (e.g. Russian oligarchs)
- A **competitive equilibrium** (m, w, v) satisfies feasibility and:

$$\Rightarrow \quad v(x) + w(y) \begin{cases} \geq h(x, y) & \text{for all women and men } x, y \\ = h(x, y) & \text{if } x, y \text{ are matched.} \qquad (\star\star) \end{cases}$$

$\Rightarrow$ Unmatched x or y earn zero wage: $v(x) = 0$ or $w(y) = 0$ $(\star)$

- This is intuitive now. We will prove it soon.

Coordinated Middlemen?

- Gaga arranged the bribes, but anyone could have!
- There is a free market in match makers
- Middlemen — real or metaphorical — determine prices.
- Without free entry of middlemen, Google can make profits

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FOR IMMEDIATE RELEASE Tuesday, January 24, 2023

Justice Department Sues Google for Monopolizing Digital Advertising Technologies

Through Serial Acquisitions and Anticompetitive Auction Manipulation, Google Subverted Competition in Internet Advertising Technologies

Today, the Justice Department, along with the Attorneys General of California, Colorado, Connecticut, New Jersey, New York, Rhode Island, Tennessee, and Virginia, filed a civil antitrust suit against Google for monopolizing multiple digital advertising technology products in violation of Sections 1 and 2 of the Sherman Act.

Competitive Equilibrium is Efficient

Proposition (First Welfare Theorem of Matching)

A competitive equilibrium (m, v, w) yields an efficient matching m.

- Proof: If a competitive equilibrium (m, v, w) is not efficient

$\Rightarrow$ some feasible matching $\hat{m} \in \mathcal{M}$ has a strictly higher payoff (2):

$$\begin{aligned}
\textstyle\sum_x v(x) + \sum_y w(y) &\geq \textstyle\sum_y \sum_x h(x,y)\hat{m}(x,y) && (1)\\
&> \textstyle\sum_y \sum_x h(x,y)m(x,y). && (2)\\
&= \textstyle\sum_y \sum_x [v(x) + w(y)]m(x,y) && (3)\\
&= \textstyle\sum_x v(x) + \sum_y w(y) && (4)
\end{aligned}$$

- Contradiction!
- Free entry $\Rightarrow$ inequality (1)
  - free entry: For $v(x) + w(y) \geq h(x,y)$ for all (x,y)
  - feasibility: $1 \geq \sum_x \hat{m}(x,y)$ $\forall y$ and $1 \geq \sum_y \hat{m}(x,y)$ $\forall x$
- Free exit (★★) $\Rightarrow$ inequality (3)
- Complementary slackness (later on) $\Rightarrow$ equality (4)

(CS) $v(x) = 0$ if $\sum_y m(x,y) < 1$ and $w(y) = 0$ if $\sum_x m(x,y) < 1$

  - Eg if some men are unmatched, their constraint does not bind

Contrast to a Stable Matching without Transfers

	Y_1	Y_2
X_1	2,0	**0,7**
X_2	**0,7**	2,0

	Y_1	Y_2
X_1	2	**7**
X_2	**7**	2

- At left, are the male and female optimal stable outcomes.
- The male optimal one yields higher total payoffs.
- But stability only reflects ordinal and not cardinal preferences.
- If outside options are zero, wages obey $v_1, v_2, w_1, w_2 \geq 0$ and:

$$\begin{array}{ll} v_1 + w_1 \geq 2 & v_1 + w_2 = 7 \\ v_2 + w_1 = 7 & v_2 + w_2 \geq 2 \end{array}$$

- Crucially, there are many competitive equilibrium wages
- One set of equilibrium wages is $v_1 = 5, v_2 = 0, w_1 = 7, w_2 = 2$

Trading Houses (Shapley and Shubik, 1971)

- A good story allows many re-interpretations of its formulation!
- *Our transferable utility matching model is such a great story it captures trading among buyers (men) and sellers (women)!*
- Men and women are just metaphors!
- $I \geq 1$ sellers (homeowners) and $J \geq 1$ prospective buyers.
- i-th seller values his house at opportunity cost $c_i > 0$
- j-th buyer value of i's house is $\xi_{ij} > 0$.
- Let $\xi_{ij} > c_i$. If seller i sells his house to buyer j for price p_i, then i's payoff is $p_i - c_i$ and j's is $\xi_{ij} - p_i$ (quasilinear utility).
- *The match payoff is the gain from trade* (no sale if $\xi_{ij} < c_i$):

$$h_{ij} = \max\{0, \xi_{ij} - c_i\}$$

Primal Problem: Maximizing Total Gains from Trade

- Let seller i sell share $m_{ij} \geq 0$ of house i to buyer j (time share?)
- Constraints on every $m_{ij} \geq 0$:
  - No house is oversold
  - No buyer buys more than one house.
- The Social Planner solves output maximization primal problem:

$$\max_{(m_{ij})} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} m_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} m_{ij} \leq 1 \qquad \forall i \in \{1, \ldots, I\}$$

$$\text{and} \quad \sum_{i=1}^{I} m_{ij} \leq 1 \qquad \forall j \in \{1, \ldots, J\}$$

Dual Problem

Lemma

The dual problem to output maximization is cost minimization:

$$\min_{v_i, w_j} \sum_{i=1}^{I} v_i + \sum_{j=1}^{J} w_j \quad s.t. \quad v_i + w_j \geq h_{ij} \; \forall i, j \quad and \quad v_i, w_j \geq 0 \; \forall i, j$$

- *A great story is mathematically solvable.*
- We argue that primal and dual problems have the same value
  - ⇒ The efficient matching yields the cheapest way to afford all match output subject to entry and free exit constraints of a competitive equilibrium
  - ⇒ Two ways of measuring output — gross national product and gross national income — coincide at the optimum.
- What are prices?
  - In the competitive market, selfish incentive devices.
  - But in the planner's problem, they are measures of social value

Linear Programming Duality

- **Primal problem**:

$$\max\{pz \mid Az \le q, z \ge 0\}$$

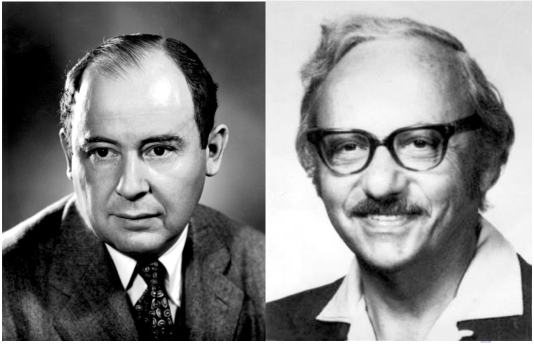
- **Dual problem:**

$$\min\{uq \mid uA \ge p, u \ge 0\}$$

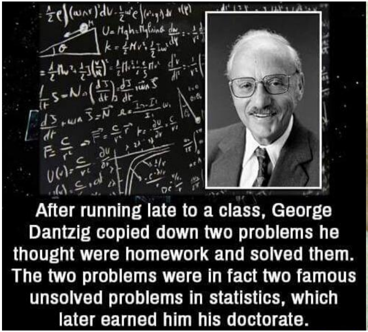
- **Theorem:** These two problems have the same values.
- Primal feasibility $\Rightarrow Az \le q$ and dual feasibility $\Rightarrow p \le uA$.
- **weak duality**: $pz \le uAz \le uq$ for all $u, z \ge 0$
- So *the value of the primal is at most the value of the dual.*
- The reverse (strong) direction is harder to show.

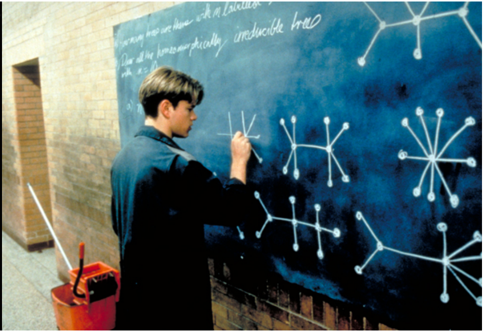
Linear Programming Duality as Deja Vu

- Flashback: von Neumann's Minimax Theorem (Saddle Point)
- George Dantzig, "A Theorem on Linear Inequalities," 1948 — first formal proof of LP duality
  - Air Force *Later Tucker asked me, "Why didn't you publish it?" I replied, "Because it was not my result; it was von Neumann's. All I did was to write up, for internal circulation, my own proof of what von Neumann had outlined to me.*
- von Neumann and Dantzig:



Ideal “PhD Conquer the World” Mindset





Primal and Dual with Two Buyers and Two Sellers

- Example with $I = J = 2$ buyers and sellers,

$$q' = (1, 1, 1, 1)$$
$$h' = (h_{11}, h_{12}, h_{21}, h_{22})$$
$$m' = (m_{11}, m_{12}, m_{21}, m_{22})$$
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Primal Problem: $\max_{m \geq 0} \sum_i \sum_j h_{ij} m_{ij} = \; h' m \quad \text{s.t.} \quad Am \leq q$
- Dual Problem:

$$\min_{w,v \geq 0} \{v_1 + v_2 + w_1 + w_2\} = \min_{v,w \geq 0} (v, w) \cdot q \quad \text{s.t.} \; (v, w) A \geq h$$

Multipliers and Complementary Slackness Conditions

- *Primal*: $\max\{pz | Az \le q, z \ge 0\}$
- *Dual:* $\min\{uq | uA \ge p, u \ge 0\}$
- Imagine a *fictitious zero sum game* with payoff

$$\mathcal{L}(z,u) = pz + uq - uAz$$

- By the 1928 *Minmax Theorem*, this game has *saddle point*:

$$\max_{z\ge 0}\min_{u\ge 0}[pz + uq - uAz] = \min_{u\ge 0}\max_{z\ge 0}[pz + uq - uAz] \qquad (\bigstar)$$

$$\Rightarrow \max_{z\ge 0}\min_{u\ge 0}[(p - uA)z + uq] = \min_{u\ge 0}\max_{z\ge 0}[pz + u(q - Az)]$$

- Let's intuit complementary slackness (CS):
  - A finite saddle point requires $p - uA \le 0 \le q - Az$
  - $\Rightarrow$ maximizer puts 0 weight on $-$ payoffs: $z_\ell = 0$ if $p_\ell - (uA)_\ell < 0$
  - minimizer puts 0 weight on $+$ payoffs: $u_k = 0$ if $q_k - (Az)_k > 0$.
  - Notice that CS $\Rightarrow$ primal value = dual value, given $(\bigstar)$
- Application of complementary slackness in Shapley-Shubik:

$$v_i + w_j \begin{cases} \ge h_{ij} & \text{for all } i, j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases}$$

Multipliers are also Shadow Values!

- *Primal*: $\max\{pz | Az \leq q, z \geq 0\}$
- Social planner's payoff function: $\mathcal{L}(z, u) = pz + u(q - Az)$
- Envelope Theorem $\Rightarrow \frac{\partial}{\partial q}\mathcal{L}(z, u) = u$
  $\Rightarrow$ dq extra constrained resource lifts planner's payoff by $u\,dq$.
- $u =$ shadow value of resource, as it indirectly shows true value

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- $u =$ shadow value of resource, as it indirectly shows true value

Shadow Values in Shapley-Shubik Housing Model

- Application of complementary slackness in Shapley-Shubik:

$$v_i + w_j \begin{cases} \geq h_{ij} & \text{for all } i, j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases}$$

- Intuitive economics of competition yields same inequalities!
- buyer i and seller j trade $\Rightarrow$ gains from trade h_{ij}
- So ε more of *i and j* raises social payoff by εh_{ij}

$\Rightarrow$ All we can say is $v_i + w_j = h_{ij}$

- "It takes two to tango...but who matters more?"

- National political debate: firms vs. workers, buyers vs. sellers
- We cannot separately identify buyers' & sellers' shadow values

1971 — Buyer-Seller Trade: Shapley and Shubik

- Assume three potential home buyers and three sellers

♂\♀	Seller Costs	Buyer Valuations		
		Buyer 1	Buyer 2	Buyer 3
House 1	18	23	26	20
House 2	15	22	24	21
House 3	19	21	22	17

- **Match payoffs are gains from trade, or zero, if negative**

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	$23 - 18 = 5$	$26 - 18 = 8$	$20 - 18 = 2$
Seller 2	$22 - 15 = 7$	$24 - 15 = 9$	$21 - 15 = 6$
Seller 3	$21 - 19 = 2$	$22 - 19 = 3$	$\max(17 - 19, 0) = 0$

1971 — Buyer-Seller Trade: Shapley and Shubik

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House 1	18	23	26	20
House 2	15	22	24	21
House 3	19	21	22	17

- **Match payoffs are gains from trade, or zero, if negative**

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	5	**8**	2
Seller 2	7	9	**6**
Seller 3	**2**	3	0

Solving the Housing Example via the Dual

- Minimize the sum of shadow values $\sum_i v_i + \sum_j w_j$ subject to $v_i \geq 0$ and $w_j \geq 0$ as well as

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 \geq 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 \geq 6 \\ v_3 + w_1 \geq 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- Since the optimum occurs at the red matching, we just solve

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 = 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 = 6 \\ v_3 + w_1 = 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- a solution: $(v_1, v_2, v_3) = (4, 5.5, 0)$ & $(w_1, w_2, w_3) = (2, 4, 0.5)$

$\Rightarrow$ home prices are $p_i = c_i + v_i$, or $p_1 = 22$, $p_2 = 20.5$, $p_3 = 19$

- Example: seller 1 sells his home (cost 18) to buyer 2 (who values it at 26) for a seller surplus $v_1 = 4$ and a buyer surplus $w_2 = 4$: from this, we deduce the price $p_1 = 22$

An Integer Price Solution of the Housing Example

♂\♀	Buyer 1	Buyer 2	Buyer 3	Seller "wage" v_i
Seller 1	5	**8**	2	$v_1 = 4$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 0$
Buyer "wage"	$w_1 = 2$	$w_2 = 4$	$w_3 = 0$	

- We increase the price of home 2 to $p_2 = 21$, increasing the surplus of seller 2 to $v_2 = 6$ and reducing the surplus of buyer 3 to $w_3 = 0$.
- So house prices are now $p_1 = 22, p_2 = 21, p_3 = 19$
- Suggestion: imagine women are middlemen.
  - Why does Buyer 1 choose Seller 3 (match payoff 2) & not 7?
    - Answer: Seller wage $v_2 \geq v_3 + 5$
  - Why does Seller 2 choose Buyer 3 (match payoff 6) & not 9?
    - Answer: Buyer 2 wage $w_2 \geq w_3 + 3$

Offline: Worst Payoffs ("Wages") for Sellers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 3$
Seller 2	7	9	**6**	$v_2 = 5$
Seller 3	**2**	3	0	$v_3 = 0$
Buyers	$w_1 = 2$	$w_2 = 5$	$w_3 = 1$	

- Buyer 1 does not buy house 1 $\Rightarrow v_1 \geq v_3 + 3$
  - Proof: $w_1 + v_1 \geq 5 = 3 + 2 = 3 + w_1 + v_3$ (Buyer 1 buys house 3)
- Buyer 1 does not buy house 2 $\Rightarrow v_2 \geq v_3 + 5$
  - Proof: $w_1 + v_2 \geq 7 = 5 + 2 = 5 + w_1 + v_3$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least seller payoffs $(\underline{v}_1, \underline{v}_2, \underline{v}_3) = (3, 5, 0)$
- Associated maximum buyer payoffs $(\bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 5, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\underline{v}_1 + \bar{w}_2 = 8$, $\underline{v}_2 + \bar{w}_3 = 6$, $\underline{v}_3 + \bar{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

Offline: Worst Payoffs ("Wages") for Buyers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 5$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 1$
Buyers	$w_1 = 1$	$w_2 = 3$	$w_3 = 0$	

- Buyer 1 does not buy house 2 $\Rightarrow w_1 \geq w_3 + 3$
  - Proof: $w_1 + v_2 \geq 7 = 1 + 6 = 1 + w_3 + v_2$ (Buyer 3 buys house 2)
- Buyer 2 does not buy house 2 $\Rightarrow w_2 \geq w_3 + 3$
  - Proof: $w_2 + v_2 \geq 9 = 3 + 6 = 3 + w_3 + v_2$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least buyer payoffs $(\underline{w}_1, \underline{w}_2, \underline{w}_3) = (1, 3, 0)$
- Associated maximum seller payoffs $(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (5, 6, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\bar{v}_1 + \underline{w}_2 = 8$, $\bar{v}_2 + \underline{w}_3 = 6$, $\bar{v}_3 + \underline{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

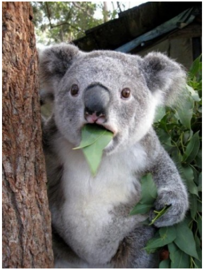
The Welfare Theorems

Welfare Theorems *A competitive equilibrium matching is efficient. Conversely, an efficient matching is a competitive equilibrium, for a suitable set of wages.*

- Proof of ($\Rightarrow$): We already proved this by contradiction
- Proof of ($\Leftarrow$): We use linear programming duality.
  - Maximize output, subject to the linear constraints of not overmatching any man or woman.
  - Call the Lagrange multipliers for these constraints the wages
  - By duality, the maximum total output equals the minimum total wages, subject to all the incentive constraints.
  - These constraints and complementary slackness conditions ensure that Lagrange multipliers are competitive wages
- *The dual problem resolves epic computational complexity issue — we need only find n wages for men and n for women!*

General Type Distributions on Men and Women

- Allow a finite number or a continuum of men and women.



- $M(x)$ gives the mass of women of type $x' \leq x$
  - $\bar{M}$ total mass of women
- $N(y)$ gives the mass of men of type $y' \leq y$
  - $\bar{N}$ total mass of men
- Short and long sides of the market
  - Assume n men and $m < n$ women, or $\bar{M} < \bar{N}$
    - men are on the *long side of the market*
    - women on the *short side of the market*
  - So some men are unmatched

Assortative Matching

- We now ask an allocation question: who matches with whom?
- Pure matching: man $y(x)$ is the partner of woman x
- Assortative matching with finitely many types:
  - *What can be concluded about the predicted matching?*
  - *positive assortative matching (PAM)*: k-th highest man and woman pair for all $k = 1, \ldots, n$
  - *negative assortative matching (NAM)*: woman k with man $51 - k$, for $k = 1, \ldots, 50$, & men $51, \ldots, 100$ unmatched
- Now consider the continuum analogues:
  - PAM if $\bar{M} - M(x) = \bar{N} - N(y(x))$ for all matched women x.
  - NAM if $\bar{M} - M(x) = N(y(x))$ for all matched women x.
- Note: the mass of men and women might even differ
- If $\bar{M} = \bar{N}$, normalize $\bar{M} = \bar{N} = 1$, and think of quantiles:
  - PAM: q-th highest quantile man and woman match
  - NAM: q-th highest quantile man is matched with q-th lowest quantile woman

Assortive Matching?

- Becker (1973), "A Theory of Marriage: Part I"

b) Assortive Mating

We now consider the optimal sorting when M and F differ in a trait, or set of traits, such as intelligence, race, religion, education, wage rate, height, aggressiveness, tendency to nurture, or age. Psychologists and sociologists have frequently discussed whether likes or unlikes mate, and geneticists have occasionally assumed positive or negative assortive mating instead of random mating. But no systematic analysis has developed that predicts for different kinds of traits when likes or unlikes are motivated to mate.[26] Our analysis implies that likes or unlikes mate when that

- I put the @ into Assortive:
  - My 2000 paper, "Assortative Matching and Search"

Key: Competitive Equilibrium Needs Middleman Entry!

- The middleman for concert tickets is a monopolist
- Example: Not Live Nation Entertainment, as it is a monopoly
  - formerly TicketMaster + Live Nation

"Taylor Swift tickets? Be reasonable."

Assortative Matching with Nontransferable Payoffs

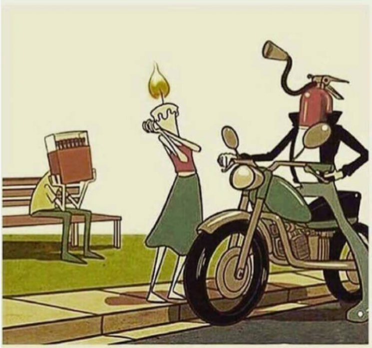
- $f(y|x)$ = payoff of woman x matched with man y,
- $g(x|y)$ = payoff of man y matched with woman x
- f and g are *comonotone* if $\forall y_2 > y_1$ and $x_2 > x_1$,

$$[f(y_2|x) - f(y_1|x)] \cdot [g(x_2|y) - g(x_1|y)] > 0 \;\; \forall x, y$$

- The opposite inequality is *reverse comonotone*
- Let's ignore weak monotonicity, but it has a natural definition
- If f and g are differentiable, then both partial derivatives (in first arguments) have the same sign if comonotone
- **Theorem:** The unique stable matching with NTU is PAM if f and g are comonotone, and NAM if reverse comonotone.

Gentle Proof of NTU Sorting Proposition

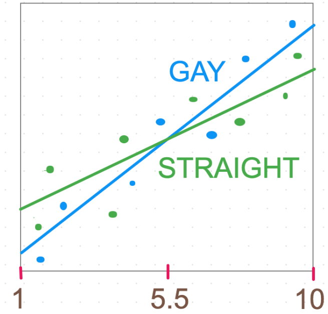
- Assume comonotonicity without PAM in a stable matching
- Then $\exists x' > x$ and $y' > y$ with matches (x, y') and (x', y)
- Claim: either (x', y') or (x, y) is a blocking pair
  1. If $f(y'|x') > f(y|x') \Rightarrow g(x'|y') > g(x|y')$
  2. If $f(y'|x) < f(y|x) \Rightarrow g(x'|y) < g(x|y)$



Positive Sorting is an Empirical Fact

Fun Application (Yale undergrad, 2006): The Dating Market

- Data Source 1: Facebook (Meta?) Dating Market early on
- Data Source 2: Online beauty contest, such as www.rankmyphotos.com



1973 — Becker's Marriage Model

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At left is **positive assortative matching (PAM)**
- Comonotone payoffs: men prefer higher women x and vice versa

$\Rightarrow$ *The stable matching without transfers is PAM.*

- Assume we indexed men or women oppositely.
- Then payoffs are reverse comonotone, and NAM is stable

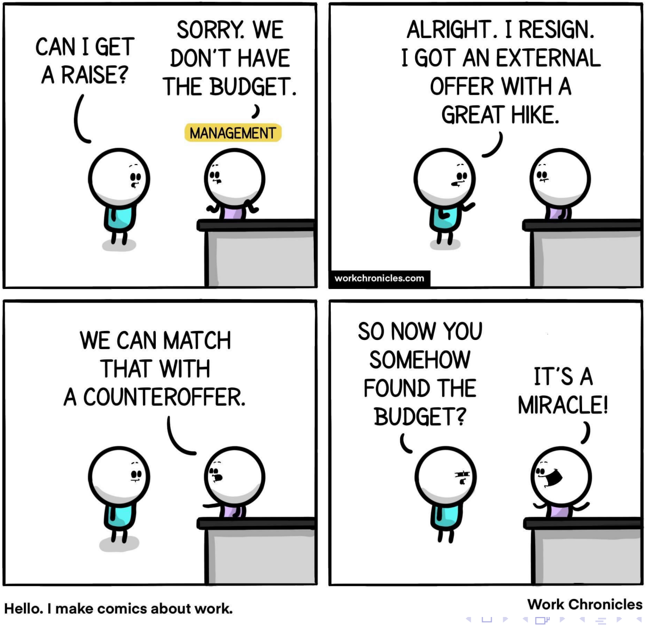
1973 — Becker's Marriage Model

♂\♀	$x=1$	$x=2$	$x=3$
$y=3$	6,21	12,12	**18,3**
$y=2$	4,14	**8,8**	12,2
$y=1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At right, we assume payoffs are transferable (TU)
- Now, **negative assortative matching (NAM)** arises
- Why? Matches all profit from higher men, but the matches that profit most from higher men are those with lower women.
- This forces downward sorting.
- For instance, rematching the two sorted pairs $(1,1)$ and $(2,2)$ as $(1,2)$ and $(2,1)$ changes output by $(18+8)-(16+9)=26-25=1$

Competition



Rear View Mirror on Last Class

- Transferable Utility Matching Model
- Efficiency is now meaningful: maximize the sum of all payoffs
- 1st Welfare Theorem: competitive equilibria are efficient
- 2nd Welfare Theorem: efficiency $\Rightarrow$ competitive equilibrium
  - Computationally, with many men and women, it is easier to find competitive equilibria than compute efficient matchings
  - Proof via linear programming duality theory (Minmax Th'm)
  - Complementary slackness $\Leftrightarrow$ competition logic
    - Anyone can be a middleman: some might find logic easier to follow if you imagine buyers hiring sellers, or vice versa
  - Lagrange multipliers are the wages!!!!!!!!
- Assortative Matching...

Pairwise Efficiency and Efficiency

- Stability with NTU: Can two unmatched people break their matches, to match with each other, & improve their welfare?
  - ⇒ The losses of the dumped partners do not matter
- TU **pairwise efficiency**: Can two matches break, re-match differently, and improve their welfare
  - ⇒ All losses matter: cardinal strength of the preferences matters
- A matching m is **pairwise efficient with TU** if for all matched pairs (x_1, y_1) and (x_2, y_2):

$$h(x_1, y_1) + h(x_2, y_2) - h(x_1, y_2) - h(x_2, y_1) \geq 0$$

- An **efficient** matching maximizes the sum of all match outputs, and so rematching any set of couples cannot help.

Lemma

Any efficient matching $m \in \mathcal{M}$ is pairwise efficient.

- The converse of this lemma is false

Pairwise Efficiency $\not\Rightarrow$ Efficiency

- With NTU, our target is stability: no pairwise blocking.
- But pairwise efficiency does not suffice for TU efficiency:

	y_1	y_2	y_3
x_1	**3**	**3**	0
x_2	0	**3**	**3**
x_3	**2**	0	**3**

- The pairwise efficient green matching has a lower total payoff than the pairwise efficient cyan matching, and is inefficient.
- Q: What bribery scheme would unravel the green matching?

TU — Strategic Substitutes Drives Negative Sorting

♂ \ ♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

Cross Partial Payoff Differences (Synergies)

	12	23
23	$18 + 24 - 27 - 16 = -1$	$16 + 21 - 14 - 24 = -1$
12	$9 + 16 - 18 - 8 = -1$	$8 + 14 - 16 - 7 = -1$

- **Strategic substitutes**:
  - all cross partial differences of match payoffs are negative
  - pairwise efficiency $\Rightarrow$ positive sorting is not locally efficiency
- **Strategic complements**:
  - all cross partial differences of match payoffs are positive
  - pairwise efficiency $\Rightarrow$ negative sorting is not locally efficiency

TU — Strategic Substitutes Drives Negative Sorting

NTU Matching

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

TU Matching

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- **Left**: payoffs are men get $2xy$ and women get $y(10 - 3x)$.
  - Men's payoffs $2xy$ increases in women's type x
  - Women's payoffs $y(10 - 3x)$ increases in men's type y
  - $\Rightarrow$ PAM is the stable allocation without transfers
- **Right**: match payoffs are $2xy + y(10 - 3x) = 10y - xy$.
  - Cross partial derivative is -1
  - $\Rightarrow$ strategic substitutes
  - $\Rightarrow$ NAM

Becker (1973): Assortative Matching with Transfers

- Match payoff $h(x, y)$ is *(strictly) supermodular* [SPM] if
$$h(x', y') + h(x, y) \geq (>)\ h(x', y) + h(x, y') \tag{5}$$
for any women $x' \geq x$ and men $y' \geq y$ (also: *complements*)
- $h(x, y)$ is *(strictly) submodular* if the reverse inequality holds
- For twice differentiable match payoffs, this says $h_{12}(x, y) \geq 0$

Proposition (Becker's Marriage Model)

(a) If $h(x, y)$ is supermodular (SPM), then PAM is efficient. If $h(x, y)$ is strictly SPM, then PAM is uniquely efficient.
(b) If $h(x, y)$ is submodular (SBM), then NAM is efficient. If $h(x, y)$ is strictly SBM, then NAM is uniquely efficient.
(c) If $h(x, y)$ is modular (SPM & SBM), any matching is efficient.

- Proof (by Buz Brock): Assume strictly supermodular (SPM)
- If matching is not PAM, then matching is not pairwise efficient, and so not efficient
- Corollary: If production is modular for a set of agents that match, then any re-matching among them is also efficient.

Example: Matching with and without Transfers

PAM

♂\♀	$x=1$	$x=2$	$x=3$
$y=3$	6,21	12,12	**18,3**
$y=2$	4,14	**8,8**	12,2
$y=1$	**2,7**	4,4	6,1

NAM

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- Women earn $f(y|x) = y(10-3x)$ and men earn $g(x|y) = 2xy$

$\Rightarrow$ $\frac{\partial f(x|y)}{\partial x} = 10 - 3x > 0$ (women prefer higher men)
$\frac{\partial g(y|x)}{\partial y} = 2x > 0$ (men prefer higher women)

$\Rightarrow$ unique stable matching is PAM

$\Rightarrow$ Hence, the DAA delivered PAM

- With transfers, strictly submodular match payoffs $h(x,y) = f(x|y) + g(y|x) = 10y - xy$ since $h_{xy} < 0$
  $\Rightarrow$ unique efficient matching is NAM
  - This is an unusual function that is increasing in x, y and yet with a negative cross partial (since domain is bounded)

How to Compute Competitive Wages with PAM or NAM

- To use calculus, we will assume a continuum of types
  - Example of SPM Match payoffs: $h(x,y) = x^2y$
  - Types: women x and men y uniformly distributed on $[0,1]$
- Since $h_{xy} = 2x > 0$, PAM is the efficient outcome (by Becker)
- Let $w(x)$ and $v(y)$ be the competitive wage functions
- If a middleman matches x and y, paying them their wages, his profits are:

$$\pi(x,y) = x^2y - w(x) - v(y)$$

- Use Topkis' Theorem to prove sorting is a competitive eq'm.
- With free entry by middlemen, competition forces a zero profit max at $y = x$ (competitive equilibrium, by welfare theorem):

$$\begin{aligned} \pi_x &= 0 \Rightarrow \left[2xy = w'(x)\right]_{y=x} \Rightarrow w'(x) = 2x^2 \\ \pi_y &= 0 \Rightarrow \left[x^2 = v'(y)\right]_{x=y} \Rightarrow v'(y) = y^2 \end{aligned}$$

- *Men compete with men, and women compete with women.*
- Aside: This proof applies the "revelation principle" that allows you to solve for bidding strategies in a FPA

Outside Options and the Wages of Men vs. Women

- Evaluating these at the efficient matches, (x, x) and (y, y),

$$
\begin{aligned}
w(x) &= \frac{2}{3}x^3 + \beta \\
v(y) &= \frac{1}{3}y^3 + \delta
\end{aligned}
$$

- By zero profits, $\pi(x, x) = 0 \; \forall x$, and so $\beta + \delta = 0$ because

$$
0 = x^2 \cdot x - w(x) - v(x) = x^3 - \frac{2}{3}x^3 - \frac{1}{3}x^3 - (\beta + \delta)
$$

- If unmatched people earns zero, then $\beta = \delta = 0$
- A *dowry* $\delta > 0$ — a fixed transfer that women pay men — only arises if unmatched women earn a payoff at most $-\delta < 0$
- A *bride price* $\beta > 0$ — a fixed transfer that men pay women — only arises if unmatched men earn a payoff $-\beta < 0$
- If unmatched men and women earn negative payoffs, then a dowry or bride price reflects a social norm (a Nash equilibrium)

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992

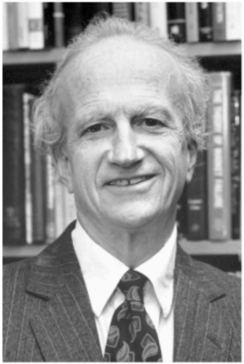


Photo from the Nobel Foundation archive.

Gary S. Becker

Prize share: 1/1

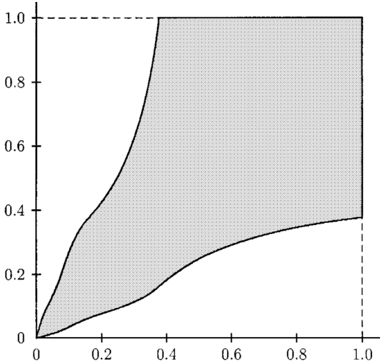
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992 was awarded to Gary S. Becker "for having extended the domain of microeconomic analysis to a wide range of human behaviour and interaction, including nonmarket behaviour."

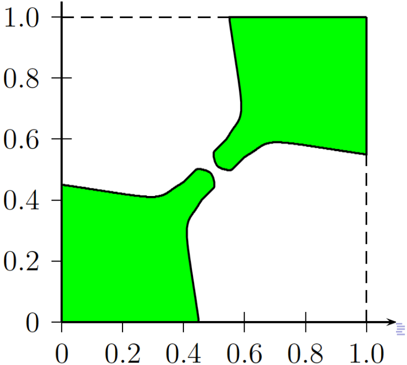
Advanced Theory Topic: Assortative Matching and Search

- We don't see perfect PAM in reality. Find a less wrong model!
- *Economic stories can come from life experience!*
- $\nexists$ no stock exchange for marriage partners, firm-worker pairs
- One must search for partners! I ran school dance coat room!

$\Rightarrow$ Shimer-Smith (2000): Even given SPM, higher types might settle for lower parters since the cost of search is higher.

- With search frictions, PAM requires that $\log h_x(x, y)$ is SPM
- Eg: Matching with $h(x, y) = e^{xy}$ and $h(x, y) = (x + y - 1)^2$:





Advanced Topic: The Comparative Statics of Sorting

- We don't see perfect PAM in reality. Find a less wrong model!
- What if we SPM or SBM fail and thus PAM or NAM fail?
- The transportation problem unsolved $\Rightarrow$ *we cannot say who matches with whom — except in PAM or NAM extreme cases!*
- While we cannot compute the efficient matching, we can still derive the comparative statics of the efficient matching
- **synergy**: any *cross partial difference* of match outputs

$$h(x_2, y_2) - h(x_2, y_1) + h(x_1, y_2) - h(x_1, y_2) \qquad \text{for } x_2 \geq x_1, y_2 \geq y_1$$

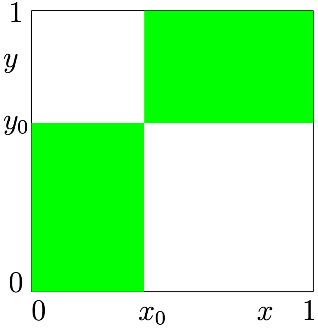
- Special cases: PAM/NAM iff synergy is everywhere $+/-$

The Comparative Statics of Sorting

Axel Anderson Lones Smith

Advanced Topic: Formulating Increasing Sorting

- Assume two (matching) cdfs F and G on the unit square
- F is higher than G in the **positive quadrant dependence (PQD) order** if F puts higher mass than G on $(x, y) \geq (x_0, y_0)$ and $(x, y) \leq (x_0, y_0)$ $\forall (x_0, y_0)$

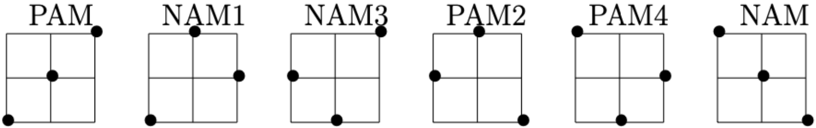


- Anderson-Smith prove: *The OLS regression coefficient of woman's type on man's type increases if the matching increases in this partial order.*
- Example: Recall my claimed Facebook regression

Advanced Topic: Pure Matchings with Three Types

- PQD is a *partial order*, $\succ_{PQD}$.
- PAM is above NAM in the PQD order. In fact:

E.g. PAM $\succ_{PQD}$ {NAM1, NAM3} $\succ_{PQD}$ {PAM2, PAM4} $\succ_{PQD}$ NAM



- NAM1 and NAM3 are incomparable in PQD order!
- Ditto PAM1 and PAM3!

Advanced Topic: Sorting Need not Rise in Synergy

- $\boxed{\textbf{Increasing Sorting Theorem}}$

  Sorting is higher with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

  $$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

- Example: Synergy rises at each stage, but sorting does not

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**14**	18
$y = 2$	5	2	**14**
$y = 1$	**1**	5	9

Matrix of Cross Differences

8	−8
−7	8

Advanced Topic: Sorting Need not Rise in Synergy

- $\boxed{\textbf{Increasing Sorting Theorem}}$

  Sorting is higher with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
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  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.
- Example: Synergy rises at each stage, but sorting does not

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	16	**24**
$y = 2$	**5**	3	16
$y = 1$	1	**5**	9

Matrix of Cross Differences (increases!)

9	-5
-6	9

Advanced Topic: Sorting Need not Rise in Synergy

- $\boxed{\textbf{Increasing Sorting Theorem}}$

  Sorting is higher with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

    $$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

    shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.
- Example: Synergy rises at each stage, but sorting does not

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**20**	30
$y = 2$	5	6	**20**
$y = 1$	**1**	5	9

Matrix of Cross Differences (increases!)

10	-4
-3	10

Advanced Topic: Sorting Need not Rise in Synergy

- **Increasing Sorting Theorem**

  Sorting is higher with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

    $$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

    shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.
- Example: Synergy rises at each stage, but sorting does not

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	22	**36**
$y = 2$	**5**	7	22
$y = 1$	1	**5**	9

Matrix of Cross Differences (increases!)

11	−1
−2	11

Double Auctions

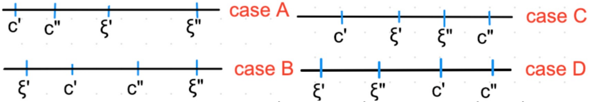
- *Consider a world with homogeneous houses* (Levittown)
- A special case of the housing assignment model with homogeneous houses is the double auction model
- Buyer j's values all goods at ξ_j, so that $\xi_{ij} = \xi_j$ for all i
- So sellers only differ by their opportunity costs and not houses
- Gains from trade: $h(\xi, c) \equiv \max\{0, \xi - c\}$ for a buyer with value ξ and a seller with cost c.
- Efficiency: maximize total trade surplus $\sum_i \sum_j m_{ij} h(\xi_j, c_i)$, where $m_{ij} = 1$ if seller i sells to buyer j, and $m_{ij} = 0$ otherwise.
- Shapley-Shubik: the sum of the shadow values of seller and buyer trading is the match output, $v_i + w_j = h_{ij}$ if $m_{ij} > 0$.
- The price p_i divides this surplus between matched traders
  - producer surplus: $v_i = p_i - c_i$
  - consumer surplus: $w_j = \xi_{ij} - p_i = \xi_j - p_i$
- We next argue that gains from trade $h(\xi, c)$ is submodular.
  - Math intuition: max preserves SBM, and min preserves SPM

Submodular Gains from Trade

Lemma (Gains from Trade)

Gains from trade h is SBM in (ξ, c): Pick any $\xi' \leq \xi''$ and $c' \leq c''$. Then $h(\xi'', c'') + h(\xi', c') \leq h(\xi'', c') + h(\xi', c'')$, with strict inequality iff $\xi' < c' < c'' < \xi''$ or $c' < \xi' < \xi'' < c''$.

Proof:



1. If two trades should occur (case A), then $h(\xi, c)$ is modular.
   $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = \xi'' + \xi' - c' - c''$.
2. If one trade should occur, then $h(\xi, c)$ is strictly submodular.

   $$B. \quad h(\xi'', c'') + h(\xi', c') = \xi'' - c'' < \xi'' - c' = h(\xi', c'') + h(\xi'', c')$$

   $$C. \quad h(\xi'', c'') + h(\xi', c') = \xi' - c' < \xi'' - c' = h(\xi', c'') + h(\xi'', c')$$

3. If no trades should occur (case D), then $h(\xi, c)$ is modular.
   $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = 0$.

- Inequalities are strict if $c' < c''$ and $\xi' < \xi''$, since trade surplus falls when the wrong good is traded.

The Supply and Demand Paradigm

- The market is a set of pairwise trades of buyers and sellers
- The highest value buyers trade with the lowest cost sellers.
- Rank order buyers: $\xi_1 < \cdots < \xi_k < \xi_{k+1} < \cdots < \xi_N$
- Rank order sellers: $c_1 < \cdots < c_k < c_{k+1} < \cdots < c_N$
  - Recall: with finitely many men and women, wages are usually not unique in the marriage model
  - This will imply that the price is usually not unique
  - But it is common across all units traded in a double auction
- Intuitively, the most prized buyers have higher values, *but the most prized sellers have lower costs*
- Since $h(\xi, c)$ is submodular, by Becker's Marriage Theorem, NAM arises: high value buyers trade with low cost sellers.
- Also, since $h(\xi, c)$ is modular matching among agents trading, and among those not trading:
  - *Matching among those trading sellers and buyers is relevant.*
  - *Matching among sellers and buyers not trading is irrelevant.*

Competitive Equilibrium in a Double Auction

Proposition (Double Auctions)

(*a*) *If* $\xi_N < c_1$, *there is no trade. Assume* $c_1 \leq \xi_N$ *henceforth.*
(*b*) *The* k^* *highest value buyers purchase from the* k^* *lowest cost sellers, where* k^* *is the largest* k *with* $c_k \leq \xi_{N+1-k}$.
(*c*) *The* ***law of one price*** *holds, with a common price*

$$p^* \in [\max(c_{k^*}, \xi_{N-k^*}), \min(c_{k^*+1}, \xi_{N+1-k^*})]$$

(*d*) *Any competitive equilibrium is efficient, and so maximizes the sum of gains from trade.*
(*e*) *The final allocation is immune to side bribes.*

- Notice that part (*c*) captures four constraints!
  - The top k value buyers, and bottom k cost sellers want to trade, and the $k+$ 1st highest buyer or lowest seller does not.
- $\boxed{\text{markets clear}}$: supply balances demand
- To understand typical deviations from the law of one price, we can add search or information frictions to the model

Is There One Price? What is it?

- Proof of (c): The social planner equally values buyer j's shadow value $w_j = \xi_j - p_i > 0$ in any optimal trade, namely from low cost sellers i, by the Becker Marriage Theorem

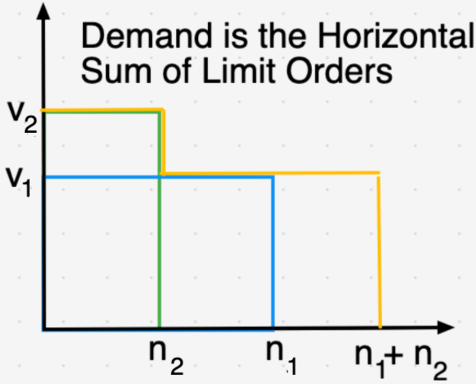
$\Rightarrow$ Seller prices p_i cannot vary with i, assuming they trade

- The price p^* encourages last transaction: $c_{k^*} \leq p^* \leq \xi_{N+1-k^*}$
- The price p^* deters another transaction: $\xi_{N-k^*} \leq p^* \leq c_{k^*+1}$
- Hence, crossing of supply and demand determines quantity:

$$\max(c_{k^*}, \xi_{N-k^*}) \leq p \leq \min(c_{k^*+1}, \xi_{N+1-k^*})$$

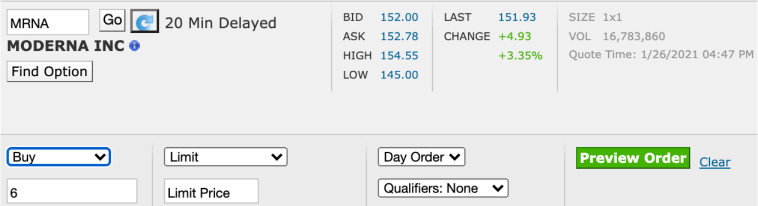
- The competitive price is not pinned down unless the last trade yields no surplus, whereupon the last unit needn't be traded
- A game has a learning dynamic: an impartial *Walrasian auctioneer* finds a competitive equilibrium by raising the price with excess demand and reducing the price with excess supply
  - Our middleman competition is much better in a key way: it is decentralized! That is an advantage of competitive equilibrium
- Opening stock market prices are set to clear the market

Beyond Unit Supply and Demand: Limit Orders



- The same can be done to construct the supply curve.

Overnight Market in Stock Exchanges



- To open/close, many stock exchanges use single price double auction
- The buyer must ask for a limit order (my choice) or a market order (limit order with unspecified price)

Offline: Easy Double Auction Example

- Consider 20 traders, numbered from 1 to 20
- Even traders are buyers, and odd traders are sellers
- Buyer valuations are $\xi_i = 2i$ and sellers costs are $c_j = 3j$.
- Ordering the valuations from high to low:

$$40, 36, 32, 28, 24, 20, 16, 12, 8, 4$$

- Ordering costs from low to high:

$$3, 9, 15, 21, 27, 33, 39, 45, 51, 57$$

- An efficient matching clears the market: the high value buyers and low cost sellers $\Rightarrow k^* = 4$ (but actual pairing irrelevant)
- The price p^* encourages the value 28 buyer and cost 21 seller to trade:

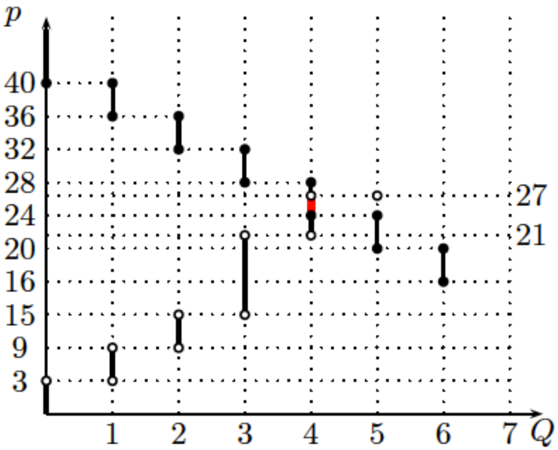
$$21 \leq p^* \leq 28$$

- The price p^* deters the value 24 buyer and cost 27 seller from trading:

$$24 \leq p^* \leq 27$$

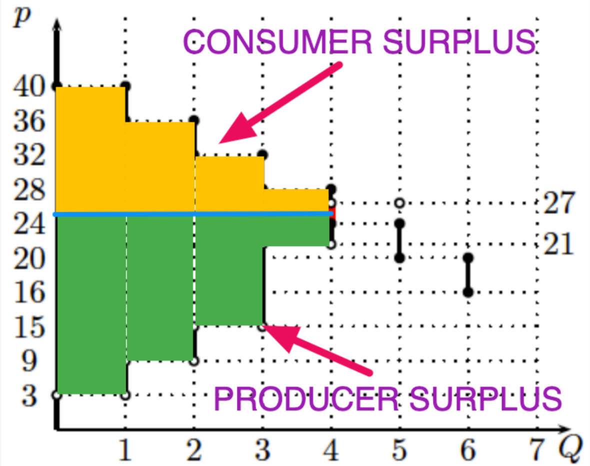
- any price in the interval $[24, 27]$ clears the market

All Positive Gains from Trade are Realized



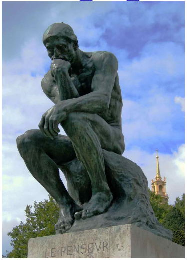
- Okay, I admit my plot is deformed around [15, 16] :)
- All traders earn positive surplus: e.g. at $p^* = 25$, the marginal buyer earns $28 - 25 = 3$ and the marginal seller $25 - 21 = 4$

When are Gains from Trade Larger?



- Heterogeneity is good and the source of all gains from trade.
  - If everyone had identical valuations, then no consumer secures consumer surplus at the market clearing price
  - the more heterogeneous are consumers or producers, the larger the total gains from trade.

Thinker Problem About Merging Markets



- $\exists > 50$ Thinkers (!)
- What happens to the price & quantity if we merge markets?
- Important question as world markets merge via trade!
- *Assume an exchange market for a good in cities A and B. Competitive prices are $p_A < p_B$ and quantities are q_A, q_B. Then the markets merge.*
  1. *How does the new competitive price compare to p_A and p_B?*
  2. *How does the new competitive quantity compare to $q_A + q_B$?*
  3. *Is total trade surplus higher or lower after the merger?*
- *Hint: Find examples where quantity traded rises or falls.*

Paternalism Applications

- Paternalism is imposing your values on another. Examples:
- Volunteer vs. Draft Army (Welfare Theorem Application)
  - A volunteer army maximizes gains from trade: it sets a wage so that the people who most want to serve willingly do so.
  - Milton Friedman's opposition the Draft helped end it in 1973.
    - Old exam Q: how much trade surplus did the draft erase?
- Organ Sale Example: only Iran allows kidney sales
- Scalping Example: Ticket Resale Laws vary (my advisee Axel)
- Regifting Example: Jay Leno's freely gave away Tonight Show tickets to unemployed in Detroit in 2009.
  - People resold tickets on eBay and Leno mocked them.
  - Q: how much trade surplus did resale create?
- Gifting Example: giving gifts usually means value $<$ cost
  - Waldfogel (1993), "The Deadweight Loss of Christmas"
  - Lost surplus was about ten billion dollars per holiday season!

Sorting by Height