An Economic Theory Masterclass

Part IX: General Equilibrium with Uncertainty

Lones Smith

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Review of General Equilibrium

1. HOW do we find equilibrium? Does it exist?
   - The math (Kakutani Fixed point theorem & Nash equilibrium) was invented in the decade before Arrow-Debreu used it
   - The proof used the convexity of preferences and technology
     - And without convexity? without existence? It's not just math!
   - The proof logic is now the basis for numerical simulations
2. WHAT new insights emerge beyond partial equilibrium?
   - If all goods are gross substitutes, there is a unique equilibrium
   - Market prices tend to covary ⇒ so does consumption of people
   - Q: What about prices of complements?

Two Big Ideas: Risk Sharing and Information Revelation

A. Risk Sharing: what markets do for risk averse people

B. Information Revelation: what people do for markets

How Markets Enable Risk Sharing

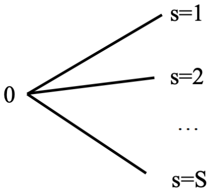
- Robinson Crusoe: shared ownership of firm exists to finance large firms that no one individual could own
- But shared ownership plays another key role: risk-sharing
- Columbus' had a long hunt for funding for his voyage west!
- 1602, the Dutch East India Company officially was the world's first publicly traded company
  - issued shares of the company on Amsterdam Stock Exchange
  - Ships returning from the East Indies had a high chance of loss due to weather, war, or pirates.

  - Instead of investing in one voyage, investors could now purchase shares in multiple companies.

  - The company eventually went bankrupt in 1799.

Arrow-Debreu Securities and Risk Sharing

- Exchange economy with n traders and L goods
- Time-1: A state of the world $s \in S = \{1, \ldots, S\}$ is realized.
  - For simplicity, assume the state s is publicly known.



- Time-0: Only the probability π_s of each state s is known.
  - Label the goods in the Arrow-Debreu model by the state.
- A state-contingent claim or *Arrow security* $x_{\ell s} \in \mathbb{R}^{LS}$ is a contractual claim to a unit of good ℓ in state s.
  - The consumption vector of trader i is $\mathbf{x}^i \in \mathbb{R}^{LS}$.
  - Trade is contractually implemented, in LS forward contracts — binding agreements to buy/sell an underlying asset in the future, at a price set today
  - $p_s =$ price of the state s contingent claim
- *Hereafter, we assume just $L = 1$ good ("money") x in a state.*

Complete Markets

- An Arrow security / contingent claim pays \$1 in just one state
- Complete markets: the securities span the states.
  - Sports Example: If two teams $i = 1, 2$ score X_1 and X_2 points,
    - the *spread* is $X_1 - X_2$
    - the *over/under line* is $X_1 + X_2$.
  - Together, these easily identify the scores X_1 and X_2.
  - If we know the spread and the over-under line, we could identify everything the market knows about the scores X_1, X_2
  - The 2024 Superbowl betting favored SF 49ers over KC Chiefs
    - The spread was 2 points, and over/under line 47.5 points
- incomplete markets. fewer assets than states (realistic)
- *We will assume complete markets, and ignore a vast macro literature on this*

Insurance: The Value of Life in the Two State Model

- Prices reflect probabilities and values in states
- Assume increasing, concave, smooth Bernoulli utility $u(x)$.
- Example: the risk of death (overly spoken of in 2020-21).
  - But gambles that involve a risk of dying allow us to price this.
  - Willingness to accept for a cross town delivery trip, with a chance $\pi > 0$ of deadly accident (costing $\mathcal{L} > 0$) is $p = \$200$.
- Case 1: linear function u (risk neutral) $\Rightarrow$ WLOG $u(x) = x$:

$$w = (1-\pi)(w+p)+\pi(w+p-\mathcal{L}) \iff \pi\mathcal{L} = p \iff \mathcal{L} = p/\pi$$

  - So if $\pi = 0.01\%$, then $\mathcal{L} = \$200/0.0001 = \$2,000,000$
- Case 2: concave u (risk averse, in the sense of Arrow Pratt)

$$\begin{aligned} u(w) &= (1-\pi)u(w+p) + \pi u(w+p-\mathcal{L}) \\ &\leq u((1-\pi)(w+p) + (w+p-\mathcal{L})) \\ \Rightarrow w &\leq (1-\pi)(w+p) + \pi(w+p-\mathcal{L}) \end{aligned}$$

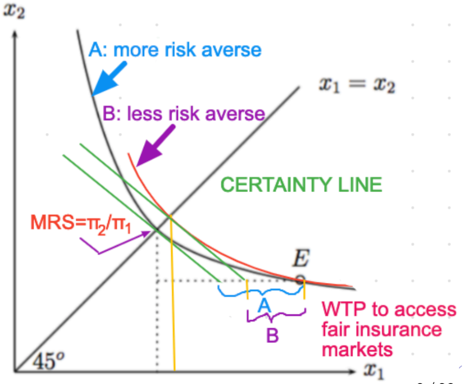
  - Hence, $\pi\mathcal{L} \leq p \iff \mathcal{L} \leq p/\pi$
  - Since individuals are willing to pay $p \geq \pi\mathcal{L}$, insurance companies can make money if they are risk neutral

Offline: 2 State World Risk Aversion Proof (Yaari, 1970)

- Consumption x_1 and x_2 in states 1 & 2 with chances π_1 & π_2
- Expected utility $U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2)$
- Risk aversion $\Rightarrow$ u concave $\Rightarrow$ U concave $\Rightarrow$ U quasiconcave
- A consumption vector x not on certainty line $(x_2 = x_1)$ is risky
- The MRS on full-insurance certainty line is π_1/π_2
- More risk averse $\Leftrightarrow$ willing to pay more for full insurance
- We now relate this economic notion to the concavity of $u(x)$
- Clearly, $MRS_{1,2} = \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$
- Curvature along 45^o diagonal:

$$\left.\frac{dMRS_{1,2}}{dx_1}\right|_{x_1=x_2} = \frac{\pi_1}{\pi_2}\frac{u''(x)}{u'(x)}$$

|Slope| ↓ faster at certainty line
→ indifference curve more curved
→ ↑ Arrow-Pratt coefficient of risk aversion



Insurance: Intensive Margin Choices in the 2 State Model

- The value of life exercise explored an extensive 0-1 margin.
- The optimal insurance question turns on an intensive margin.
- Disaster state wealth has *unit price p* in insurance premiums.

$$\max_{q \geq 0} \pi u(w - \mathcal{L} + q - pq) + (1 - \pi) u(w - pq)$$

- At an interior solution, the FOC is:

$$\pi(1 - p) u'(w - \mathcal{L} + q - pq) - p(1 - \pi) u'(w - pq) = 0$$

- *Actuarially fair insurance* when $p = \pi$, since the premiums paid pq equal expected value of compensation received πq

$$u'(w - \mathcal{L} + q - pq) = u'(w - pq) \quad \Leftrightarrow \quad q^* = \mathcal{L} \qquad \text{(full insurance)}$$

- Typical case is unfair insurance prices: $p > \pi$

$$\text{FOC:} \qquad \frac{u'(w - pq)}{u'(w - \mathcal{L} + q - pq)} = \frac{\pi(1 - p)}{p(1 - \pi)} < 1$$

$$\Rightarrow u'(w - pq) < u'(w - \mathcal{L} + q - pq)$$

- So $q < \mathcal{L}$ if risk averse $\Rightarrow$ *not fully insured.*

The Fundamental Theorem of Risk Bearing (Many States)

- Expected utility $U(x_1, \ldots, x_S) = \sum_{s=1}^{S} \pi_s u(x_s)$
- Assume time-0 market in contingent claims $x_1, \ldots, x_S$

$$\max \sum_{s=1}^{S} \pi_s u(x_s) \quad \text{s.t.} \quad \sum_{s=1}^{S} p_s x_s = \sum_{s=1}^{S} \bar{x}_s$$

- Lagrangian $\mathscr{L} = \sum_{s=1}^{S} \pi_s u(x_s) + \lambda \sum_{s=1}^{S} p_s(\bar{x}_s - x_s)$.
- FOC: $\lambda = \pi_s u'(x_s)/p_s$ for all s
  - $\Rightarrow$ Equalize shadow value of money (bang per buck) across states

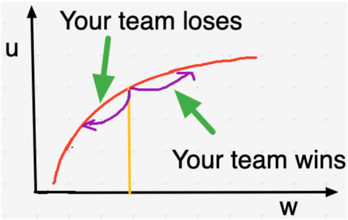
Proposition (Fundamental Theorem of Risk Bearing)

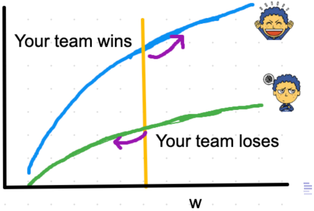
Assuming prices enable an interior solution, we have:

$$\frac{\pi_1 u'(x_1)}{p_1} = \cdots = \frac{\pi_S u'(x_S)}{p_S}$$

Time Permitting: State Dependent Utility?

- Bad state $s = 1$ and good state $s = 2$ (your team loses / wins)
- Assume state-dependent utility functions $u_2(w) > u_1(w)$
- For this intensive margin question, we put an extra dollar where its *expected marginal utility* is highest
- An extra time-0 dollar, used to buy Arrow securities,
  - added to bad state raises expected utility by $\frac{\pi_1}{p_1} u_1'(w)$
  - added to good state raises expected utility by $\frac{\pi_2}{p_2} u_2'(w)$
- With fair prices $p_i = \pi_i$, transfer money to the higher u_i' state.
- State-independent utility $\Rightarrow$ home team win = wealth gain $\Rightarrow$ bet against them to perfectly insure (optimism exception)
- State-dependent utility (home team win lifts marginal utility) $\Rightarrow$ bet for your team, even if utility is more unequal





Risk Sharing: Idiosyncratic Risk

- Assume risk averse traders Iris and Joe, and $S = 2$ states.
- Iris and Joe obey the FOC $\pi_1 u'(x_1)/p_1 = \pi_2 u'(x_2)/p_2 = \lambda$.

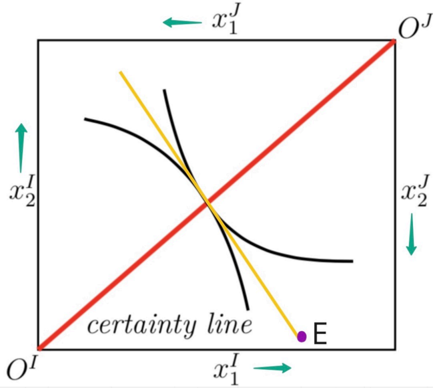
$$x_1 \gtrless x_2 \Leftrightarrow \frac{p_1 \pi_2}{p_2 \pi_1} = \frac{u'(x_1)}{u'(x_2)} \lessgtr 1 \qquad (1)$$

$\Rightarrow$ $x_1^I = x_2^I$ & $x_1^J = x_2^J$, or $x_1^I > x_2^I$ & $x_1^J > x_2^J$, or $x_1^I < x_2^I$ & $x_1^J < x_2^J$.

- Total endowment $\bar{x}_s = \bar{x}_s^I + \bar{x}_s^J$ in state s.
  - purely idiosyncratic risk: $\bar{x}_1 = \bar{x}_2$
  - aggregate risk: $\bar{x}_1 \neq \bar{x}_2$
- Case 1: **Idiosyncratic risk** $\Rightarrow x_1 = x_2$
  - $\Rightarrow$ fair prices: reflect probabilities of states: $p_1/p_2 = \pi_1/\pi_2$
  - $\Rightarrow$ traders fully insure
  - Life insurance premiums reflects death probabilities, and house insurance the chance of a home burning down.
  - Implications: the price of a state-contingent security rises in proportion to the likelihood of the state.
    - Eg. life insurance is really cheap for young buyers, and doubles in price when the death rates double.
    - This allows us to *infer event probabilities from insurance rates*

Risk Sharing: Idiosyncratic Risk

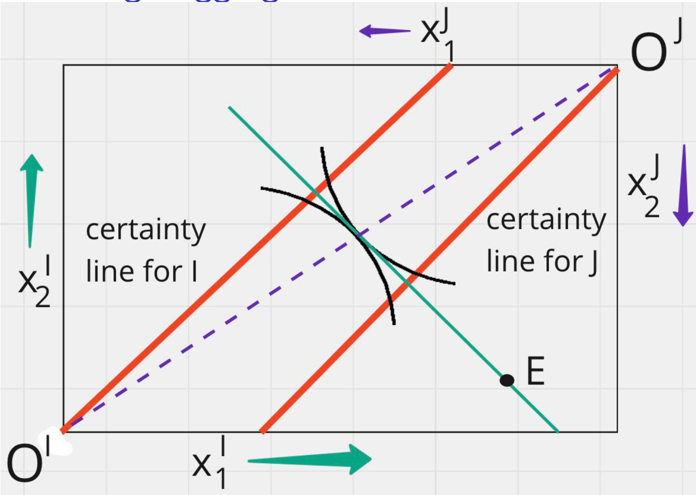
- $U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2) \Rightarrow MRS_{1,2} = \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$
- Along certainty line, with $x_2 = x_1$, we have $MRS_{1,2} = \frac{p_1}{p_2}$
- Puzzle: Which state is less likely below?



Risk Sharing: Aggregate Risk

- Case 2: **Aggregate risk**, with $\bar{x}_1 > \bar{x}_2$ (disaster state is $s = 2$)
  - Fundamental Theorem of Risk Bearing ⇒ traders share risk.
  - $\bar{x}_1 > \bar{x}_2 \Rightarrow x_1^I > x_2^I$ and $x_1^J > x_2^J \Rightarrow p_2/p_1 > \pi_2/\pi_1$
  - Example: logarithmic Bernoulli utility $u^I(x) = u^J(x) = \log x$
    ⇒ utility function over consumption bundles is Cobb Douglas
    - Ordinal utility $U(x_1, x_2) = \pi_1 \log x_1 + \pi_2 \log x_2$
    - We can now compute the earthquake insurance premium
    - The FOC (1) yields $p_2/p_1 = (\bar{x}_1/\bar{x}_2)(\pi_2/\pi_1) > \pi_2/\pi_1$.
    - Calculate the contract curve with log utility $u(x) = \log(x)$.
- Example: earthquake insurance in California is extremely costly, since it only pays out in an overall disastrous state.
  - "force majeure" denies liability for catastrophes

Risk Sharing: Aggregate Risk



Q: Why is contract curve the diagonal with log Bernoulli utility?

- In equilibrium, $\frac{p_2}{p_1} = MRS = \frac{\pi_2 u'(x_2)}{\pi_1 u'(x_1)} > \frac{\pi_2}{\pi_1}$ since $x_2 < x_1$

Q: What is the MRS along each trader's certainty line?

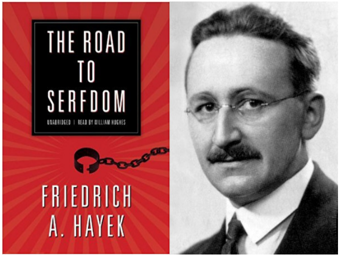
- What happens to prices or risk sharing if Iris' risk aversion ↑?

Information Revelation and Rational Expectations

- planner must know the demage for Pigouvian taxes.
- Prices in Arrow's missing market can figure out that state.

Information Revelation and Rational Expectations

- So far, prices serve as a mechanism to clear markets
- But prices also convey information about supply and demand, if traders are initially asymmetrically informed
- E.g. Idiosyncratic risk: price line slope is probability ratio
  - Austrian economists, non Mises (1920) and Hayek (1935): social planners do not solve the *calculation problem*: aggregate idiosyncratic consumption / production information



- After 1950s, purely verbal/graphical logic did not suffice!
- In a rational expectations equilibrium, agents fully extract information from prices (= Bayesian Nash equilibrium)
- 1970s *rational expectations* work (Radner, Lucas, Sargent,...)

Information Revelation and Rational Expectations

- Can prices "serve two masters": clear markets & convey info?
- Tatonnement process is now delicate:
  - Auctioneer calls out a price
  - Traders make demands
  - Before auctioneer revises his price,
    - traders see demands,
    - learn from them,
    - revise demands, etc.
    - Rinse and repeat

Prices Reveal Information in Prediction Markets

- These let people bet on sporting or presidential etc. events.
- Share price convey the expected probability of events.
- Example: Every individual i has log Bernoulli utility, wealth w_i, and can buy x_i shares at price p ["Joe wins in 2020"]

$$\max_{x_i} \pi_i \log[w_i + x_i(1-p)] + (1-\pi_i)\log[w_i - x_i p]$$

- Individual $i = 1, \ldots, n$'s demand: $x_i^* = w_i \frac{\pi_i - p}{p(1-p)}$.
  - *Traders buy iff more optimistic than the price* $(\pi_i > p)$
- Assume everyone is equally wealthy: $w_i = w$ for all i.
- Clear markets: Market excess demand is $\sum_{i=1}^n x_i^* = 0$, or

$$\sum_{\pi_i > p}(\pi_i - p) = \sum_{\pi_i \le p}(p - \pi_i) \Rightarrow p = \frac{1}{n}\sum_{i=1}^n \pi_i$$

  - No Trade Theorem (Game Theory): $\nexists$ Purely informed trade

$\Rightarrow$ prediction market averages subjective beliefs, not information.

Prediction Market Forecast of President 2024

PredictIt | Markets | Trends | Leaderboards | Account | Dashboard | $0.00 Investment | $0.00 Cash | Deposit

Watchlist | Presidency | Congress | Joe Biden | Donald Tru

Who will win the 2024 US presidential election?

Contract	Latest Yes Price		Best Offer			Best Offer
Donald Trump	48¢	NC	48¢	Buy Yes	Buy No	53¢
Joe Biden	42¢	NC	43¢	Buy Yes	Buy No	58¢
Gavin Newsom	8¢	NC	9¢	Buy Yes	Buy No	92¢
Robert Kennedy Jr.	5¢	NC	6¢	Buy Yes	Buy No	96¢
Nikki Haley	5¢	NC	5¢	Buy Yes	Buy No	96¢
Kamala Harris	4¢	1¢↓	5¢	Buy Yes	Buy No	96¢

Rational Expectations Equilibrium: Nonexistence (Kreps)

- Iris likes x more if $s = 2$: $u^I(x, y) = s \log x + y$ for $s = 1, 2$
- Joe likes x more if $s = 1$: $u^J(x, y) = (3 - s) \log x + y$
- Iris knows s, but Joe thinks $s = 1, 2$ each have 50% chance
- Endowments: $\bar{x} = 2$, and $\bar{y}$ is large. Naturally, $p = p_x / p_y$.
  - Iris's FOC is $x^I(p) = s/p$
  - Joe knows $s \Rightarrow x^J(p) = (3 - s)/p$
- If Joe learns the state from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{3 - s}{p} = \frac{3}{p} = \bar{x} = 2 \Rightarrow p(s) = 1.5$$

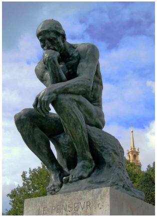
  - This price is the same in $s = 1, 2 \Rightarrow$ conceals Iris's information.
- If Joe learns nothing from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{1.5}{p} = 2 \Rightarrow p(s) = \frac{2}{s + 1.5}$$

  - This price is different in $s = 1, 2 \Rightarrow$ reveals Iris's information.
- $\nexists$ rational expectations equilibrium (REE) in this example.

So Does Rational Expectations Equilibrium Not Exist?

- The problem in the example is that tiny changes in prices suddenly reveal the state, and radically change demand:
  - ⇒ Demand is discontinuous as a function of price.
  - ⇒ Kakutani Fixed Point Theorem does not apply (existence fails)
- Resolution: Assume that some trades do not reflect information but reflect random heterogeneity
- Noisy prices restore continuity
  - ⇒ Small price changes likely reflect noise not fundamentals.
  - Finance typically conceals fundamentals with Gaussian noise
- Thinker Question (MWG):
  Find all REE if $u^I(x, y) = u^J(x, y) = s \log x + y$



Thinker Solution: Revealing REE

- Exercise: Find all REE if $u^I(x,y) = u^J(x,y) = s \log x + y$
- Iris knows s, but Joe thinks $s = 1, 2$ each have 50% chance
- Endowments: $\bar{x} = 2$, and $\bar{y}$ is large.
- Iris maximizes $s \log x + y$ subject to $px^I + y^I = p\bar{x}^I + \bar{y}^I$.
  - FOC is $x^I(p) = s/p$, provided $\bar{y}^I \geq 2p$.
- If Joe learns nothing from the price, then $x^J(p) = (\frac{1}{2} + \frac{1}{2}2)/p$.
  - Clearing the x market,

$$x^I(p) + x^J(p) = \bar{x} \Rightarrow \frac{s}{p} + \frac{3}{2p} = 2 \Rightarrow p(s) = (3 + 2s)/4$$

$\Rightarrow$ price p^* increases in $s \Rightarrow$ reveals Iris's information to Joe.
$\Rightarrow$ $\nexists$ rational expectations equilibrium that conceals the state s.

- If Joe learns the state from the price, then $x^I(p) = x^J(p) = \frac{s}{p}$.

$\Rightarrow$ Endowment $\bar{x} = 2$ is shared equally, and so the price is $p^* = s$.