An Economic Theory Masterclass

Part VIII: General Equilibrium in Production Economies

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Leon Walras

- Leon Walras (1874), *Éléments d'économie politique pure*
  - He first formulates the general equilibrium problem
  - Not translated into English until 1954
  - Econometrica allowed French for decades.

The Walrasian Existence Problem

- Walras formulated the marginal theory of value, based on economic value and not ethical theory of value:
  - *There is here as well no need to take into account the morality or immorality of the need . . . Whether a substance is searched for by a doctor to heal an ill person, or by an assassin to poison his family, this is an important question from other points of view, albeit totally indifferent from ours.*
- He deduced equations for prices and quantities of goods bought and produced, using first order conditions
- Last equation is our Walras Law: all money received is spent
- Suggested *tâtonnement* (French for "trial and error")

Eighty Years Pass, and then Arrow and Debreu (1954)

- "Existence of an Equilibrium for a Competitive Economy"
- *"Walras first formulated the state of the economic system at any point of time as the solution of a system of simultaneous equations . . . Walras did not, however, give any conclusive arguments to show that the equations have a solution"*

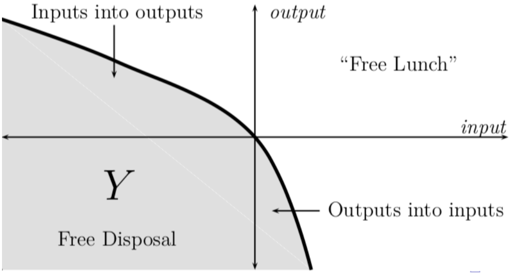
Nash (1951) Inspires Arrow and Debreu (1954)

- "Existence of an Equilibrium for a Competitive Economy"

- Idea: *Professor Nash has formally introduced the notion of an equilibrium point for a game....* .
  - Me: wait, "Professor"???
- *The definition can easily be extended to an abstract economy*
- Goal: introduce an $(m + n + 1)$-player game with
  - m firms maximize profits, and n consumers maximize utility
  - One fictitious Walrasian auctioneer chooses prices to maximize the value of net excess demand $\Rightarrow$ reduce prices of goods in excess supply and raise the prices of goods in excess demand
- They first must extend Nash from a finite game to a continuous action game with quasiconcave payoff functions

Arrow and Debreu's Damn Clever Formulation of Firms

- A firm transforms inputs into outputs
  - A firm is a subset $Y \subset \mathbb{R}^L$, given $L \geq 2$ goods.
  - $y_k \in Y$ is outputs if $y_k > 0$ and input if $y_k < 0$
  - $\Rightarrow$ firm profits are $p \cdot y$, the dot product of prices and quantities.
- *Closed convex technology*
  - *no free lunch* $\Rightarrow Y \cap \mathbb{R}^L_+ = \{0\} \Rightarrow 0 \in Y$ (do nothing)
  - *free disposal* $\Rightarrow Y \supset \mathbb{R}^L_-$
  - $Y \subseteq \mathbb{R}^\ell$ is closed and convex
    - Technologies have diminishing returns
    - Dynamic free lunch is impossible (e.g. 1 rubber unit $\rightarrow$ 2 tires, and 1 tire $\rightarrow$ 2 rubber units)



Formal Model of the *Competitive* Capitalist Economy

- Each consumer i owns a share $\theta_{ij} \geq 0$ of profits of firm j
- A **competitive equilibrium** of a private ownership economy

$$\left(\{Y^j\}_{j=1}^m, \{X^i, u^i, \bar{\mathbf{x}}^i, \{\theta_{ij}\}_{j=1}^m\}_{i=1}^n\right)$$

is an allocation $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{nL} \times \mathbb{R}^{mL}$ and a price $\mathbf{p} \in \mathbb{R}^L$ so that

  - $\forall j$: $\mathbf{y}^j \in Y^j$ maximizes profits, i.e. $\mathbf{p} \cdot \hat{\mathbf{y}}^j \leq \mathbf{p} \cdot \mathbf{y}^j \quad \forall \hat{\mathbf{y}}^j \in Y^j$
  - $\forall i$: $\mathbf{x}^i \in X^i$ maximizes utility u^i in the budget set:

$$B^i(\mathbf{p}) = \{\mathbf{x}^i \in X^i : \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \bar{\mathbf{x}}^i + \sum_{j=1}^m \theta_{ij} \mathbf{p} \cdot \mathbf{y}^j\}$$

  - Markets clear, namely the excess demand vector is nonpositive:

$$\mathbf{z} = D(p) - \bar{x} - S(p) \equiv \sum_{i=1}^n \mathbf{x}^i - \sum_{i=1}^n \bar{\mathbf{x}}^i - \sum_{j=1}^m \mathbf{y}^j \leq 0$$

and if $z_k < 0$, then $p_k = 0$

Existence Theorem

Theorem (Arrow and Debreu, 1954)

Assume every consumer $i = 1, \ldots, n$ has a continuous, nonsatiated and strictly quasiconcave utility u_i, endowment $\bar{\mathbf{x}}^i \in \mathbb{R}^L_+$, and dividend shares (θ_{ij}). Assume firms $j = 1, \ldots, m$ have closed and convex production technologies. A competitive equilibrium exists.

- This generalizes Nash's existence to games from mixtures over finitely many actions to quasiconcave and continuous payoff functions on a compact convex space
  - Quasiconcavity $\Rightarrow$ pure strategy Nash equilibrium exists
  - Glicksberg '52 extended Nash to linear topological spaces

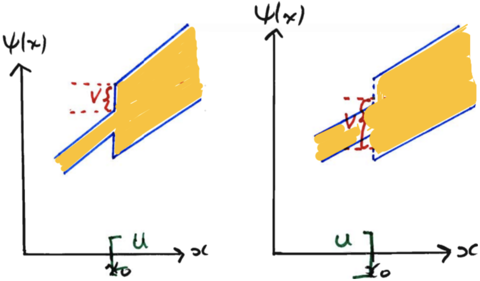
Corollary (Existence in General Exchange Economies)

Assume every consumer $i = 1, \ldots, n$ has a continuous, nonsatiated and strictly quasiconcave utility u_i, and an endowment $\bar{\mathbf{x}}^i \in \mathbb{R}^L_+$. A competitive equilibrium exists.

- Prove: Existence for the exchange economy follows from considering the special case with no firms ($m = 0$)

Closed Graph Property

- Let $\psi : X \rightrightarrows Y$ be a correspondence, so that $\psi(x) \subset Y$ is a set
- Closed graph property (upper hemicontinuity, UHC) if $\{(x, \psi(x))\}$ contains its limit points: has closed graph in $X \times Y$
  - precludes "implosions" of a correspondence function $\psi(x)$
  - Left plot below is upper hemicontinuous, and right plot is not
  - If ψ is a function, then upper hemicontinuity $=$ continuity
  - Lower hemicontinuity (LHC) precludes "explosions" of ψ
    - Rght plot below is lower hemicontinuous, and left plot is not
    - Game theory refinements (e.g. Intuitive Criterion) claim LHC
  - Continuous correspondence $=$ upper $+$ lower hemicontinuous



UHC

LHC

Review: Kakutani Fixed Point Theorem

Theorem (Kakutani Fixed Point Theorem, 1944)

Let ϕ be a correspondence on non-empty, compact, convex $S \subset \mathbb{R}^n$

- *with a closed graph*
- $\phi(x) \neq \varnothing$ *for all* $x \in S$.
- *convex-valued for all* $x \in S$

Then ϕ has a "fixed point" $x \in \phi(x)$

- "Sir, tell us about the Kakutani FPT." Him: "What's that?"

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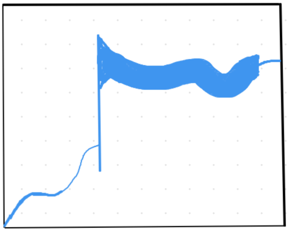
- Kakutani used the von Neumann Approximation lemma to draw a continuous function very close to any closed graph.



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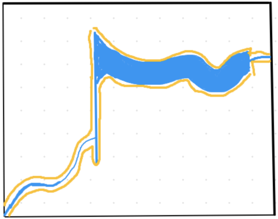
- Loosely, tighten a tube-sock around the closed graph:



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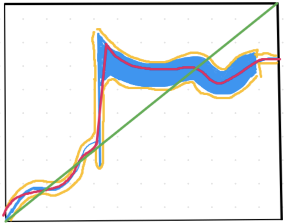
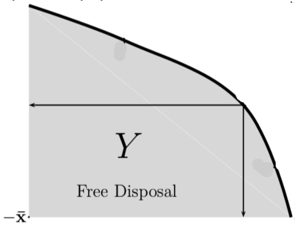
- Each such function has a fixed point, by Brouwer. Take limits.



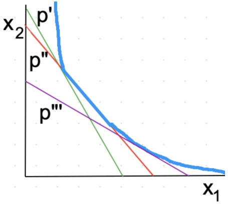
Are Action Domains Compact and Convex?

- A competitive equilibrium is a triple (x, y, p) such that:
  - Given p, consumers and firms choose x, y.
  - Given x, y, auctioneer chooses price p (& it clears the market).
- Auctioneer has compact convex action space & finite payoffs
  - Use compact price domain $P = \{p \in \mathbb{R}^L_+ | p_1 + \cdots + p_L = 1\}$.
  - This uses the degree of freedom in Walras Law differently
- Firms and consumers have compact convex action spaces
  - Markets clear $\Rightarrow$ excess demand $\mathbf{z} = D(p) - S(p) - \bar{\mathbf{x}} \leq 0$.
  - $\Rightarrow$ $S(p) = D(p) - \mathbf{z} - \bar{\mathbf{x}} \geq -\bar{\mathbf{x}}$, since $D(p) - \mathbf{z} = S(p) + \bar{\mathbf{x}} \geq 0$.
  - $\Rightarrow$ Since Y_j is convex, and $Y_j \cap \mathbb{R}^L_+ = \{0\}$, it is bounded above
  - $\Rightarrow$ So every firm's optimization is on a compact domain Y_j.
  - Likewise, $D(p) \leq S(p) + \bar{\mathbf{x}}$ is then uniformly bounded above



Arrow-Debreu (1954) Proof Sketch Theorem

- Correspondence $\phi(x_0, y_0, p_0)$ is all best replies in $X \times Y \times P$:
  - all bundles x maximizing utility, given p_0
  - all profit maximizing inputs and outputs y, given p_0
  - all prices p maximizing value of net excess demand, given x_0, y_0
- Continuous u and compact domain $\Rightarrow \phi(x, y, p) \neq \varnothing$
- Theorem of the Maximum $\Rightarrow \phi(x, y, p)$ has a closed graph
- Convex preferences and technologies $\Rightarrow \phi(x, y, p)$ convex

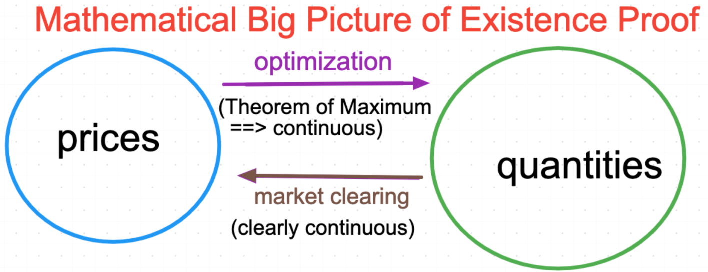


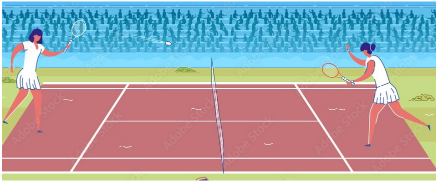
$\Rightarrow$ correspondence $\phi \neq \varnothing$ has a closed graph and convex-valued

- *By Kakutani's Fixed Point Theorem,* $\exists (x, y, p) \in \phi(x, y, p)$
  - Namely, each consumer is optimizing his utility at x, each firm maximizes its profits at y, and markets clear at price p

Insights for Iterative Computer Equilibrium Computation

- Arrow-Debreu proof: $X \times Y \times P \to X \times Y \times P$ and Kakutani
- Computer: A "tennis game" captures the two maps $X \times Y \to P$ and $P \to X \times Y$ (convergence requires prayer)



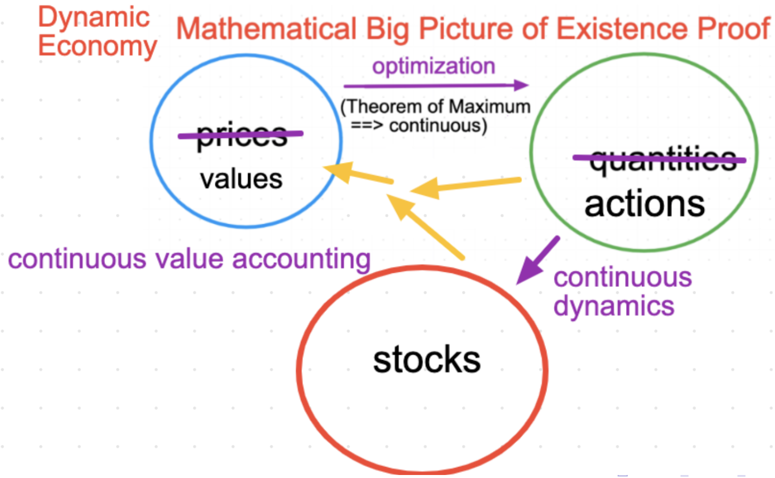


- Walrasian *tâtonnement* usually gives convergence, but there is no general theorem on when this works!

Equilibrium Computation for Dynamical Economic Systems

(Foretaste of my Advanced Theory Search MiniCourse)

- Computer simulations often inspire my theorems.



McKenzie's Slightly Parallel Existence Result

- Lionel McKenzie (1954) also used Kakutani to prove existence
  - But his was a trade model, and he did not model consumers
  - He did not have the elegant description of firms.
  - Crucially, he did not approach it as a game, but instead as a Kakutani fixed point theorem application
- He did not cite Arrow-Debreu (1954), nor did they cite him!!
- The editor Strotz wrote him in 1953:
  - *"I have given up. Letters have gone to both referees requesting the return of your manuscript to this office right away. I hope to God I can have better luck with the next people. I don't know whether this is a matter of concern to you, but let me assure you that it is my intention not to publish the paper by Arrow and Debreu (which has also been submitted) before the publication of your paper (if both are found acceptable). I think this would only be fair to you.'*
  - Strotz wrote the classic 1955 paper on time consistent preferences. He was President of Northwestern 1970–84.
- Mackenzie founded the Rochester economics department

Socially Efficiency for General Equilibrium with Production

- Given feasible consumption $X \subset \mathbb{R}^{n\ell}$ production: $Y \subset \mathbb{R}^{m\ell}$
- An allocation $(x, y) \in X \times Y$ of a private ownership economy is *socially efficient* if $\nexists(\hat{x}, \hat{y}) \in X \times Y$ such that
  - every consumer i is weakly better off, $u^i(\hat{x}) \geq u^i(x)$
  - some consumer k is strictly better off, $u^k(\hat{x}) > u^k(x)$
  - the allocation $(\hat{x}, \hat{y})$ is feasible (so "markets clear"):

$$\sum_{i=1}^{n} \hat{x}^i - \sum_{i=1}^{n} \bar{\mathbf{x}}^i - \sum_{j=1}^{m} \hat{y}^j \leq 0$$

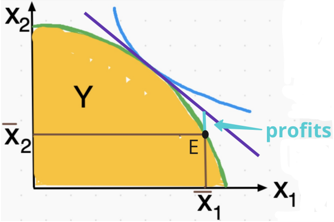
- **Firm profits do not matter here, and do not appear in social welfare functions!**
- Economics welfare analysis is 100% focused on people!
  - We do not treat corporations as people!

1st and 2nd Welfare Theorems with Production

Theorem (Efficiency $\Leftrightarrow$ Competition)

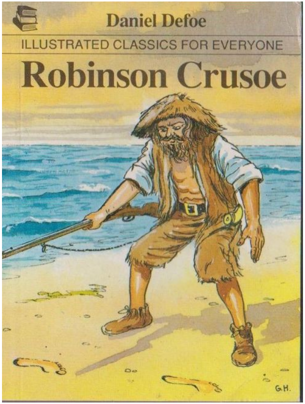
First: *If* $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ *is a competitive equilibrium and preferences are not locally satiated, then* $(\mathbf{x}, \mathbf{y})$ *is a socially efficient allocation.*
Second: *Assume monotonic and convex preferences, and closed convex technologies. If* $(\mathbf{x}, \mathbf{y})$ *is socially efficient, then* $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ *is a competitive equilibrium, for some prices* $\mathbf{p} \in P$*, endowments* $\bar{\mathbf{x}}$*, and ownership shares* θ*.*

- Intuition: Choose the endowment vector $\bar{\mathbf{x}}$ as the Y origin, since it corresponds to the zero production exchange economy
- Open: Modify earlier 2nd welfare theorem proof for production
- Classic Separating Hyperplane logic in a picture



Robinson Crusoe Economies

- $M = 1$ firms, $N = 1$ consumers
- Karl Marx made this metaphor famous in *Das Kapital*

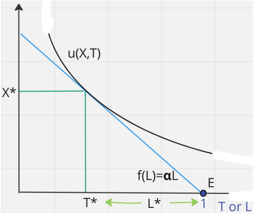


Example: Constant Return to Scale Technology

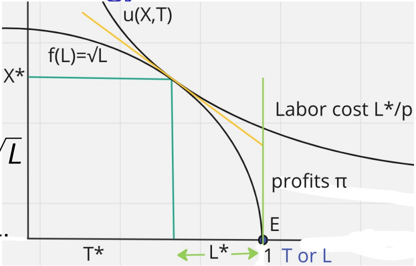
- $L = 2$ goods, produced by $M = 1$ firm, for $N = 1$ consumer
  - Technology: fish or fowl $f(L) = \alpha L$, where L is labor.
  - Preferences: $u(X, T) = X^{\alpha} T^{1-\alpha}$, where T is leisure.
  - Endowment: one unit of time $1 = \bar{T} = L + T$
  - Need not specify firm ownership shares: it earns no profits
- Solution: Let T be numéraire, and p the relative price of X
- Crusoe Inc. maximizes $pf(L) - L$ iff $p\alpha = 1$.
- As endowment income is $\bar{T} = 1$, Cobb Douglas demands are:

$$X = \alpha/p = \alpha^2 \quad \text{and} \quad T = 1 - \alpha$$

- Clear the labor market $\Rightarrow$ Robinson works $L = \alpha$ hours
- Crusoe, Inc. hires $L = \alpha \Rightarrow$ produces $X = \alpha^2$
- Finally, notice that the X market clears (Walras Law)



Example: Diminishing Returns Technology and Profits



- Diminishing returns $\Rightarrow$ profits
- Technology: fish $f(L) = \sqrt{L}$
- Planner: max $u(X, T)$ s.t. $X = \sqrt{L}$
- Competitive Equilibrium (easier)
  - Crusoe Inc. maximizes $p\sqrt{L} - L$.
    - The FOC is $L = p^2/4$
    - $\Rightarrow$ Production is $X = p/2$
    - Profits $pX - L$ are as depicted:

$$\pi = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4} > 0$$

  - Robinson's income is his endowment value and profits: $1 + \pi$.
    - Leisure demand (using $L = p^2/4$) is

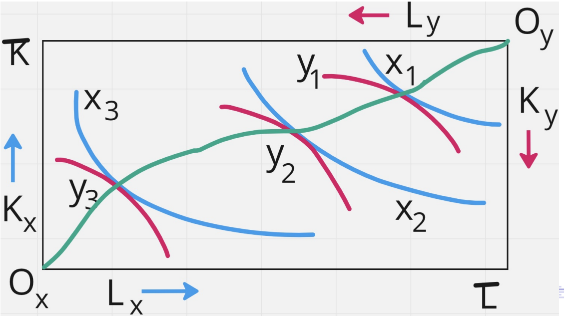
$$T = (1 - \alpha)(1 + p^2/4)$$

    - $T + L = 1 \Rightarrow L^* = \alpha/(2 - \alpha)$ and $T^* = 2(1 - \alpha)/(2 - \alpha)$.
    - $\Rightarrow$ Supply $X^* = \sqrt{\alpha/(2 - \alpha)}$
    - $\Rightarrow$ Competitive price $p = 2X^* = 2\sqrt{\alpha/(2 - \alpha)}$

Multiple Factors of Production or Goods in Competition

- $\nexists$ general solution recipe.
- When in doubt: Find the socially efficient allocation!
- Use the welfare theorems to find wages and prices

1. Efficient consumption: Edgeworth Box contract curve
2. Efficient Product Mix: $MRS = MRT$ (taught in public goods)
   - Max utility possibility on production possibility frontier
3. Efficient production: Find the efficient allocation of factors
   - Every factor is paid the value of its marginal product
     - wage ratio $w_i/w_j = f_i(x)/f_j(x)$ for every industry
     - Every industry has the same the value of its marginal product



ChatGPT Question

- See fun 2023 ChatGPT Prelim Question on canvas
    - Will labor suffer losses from AI?