American Economic Review 2024, 114(3): 709–751
https://doi.org/10.1257/aer.20210890

The Comparative Statics of Sorting†

By Axel Anderson and Lones Smith*

We create a general and tractable theory of increasing sorting in pairwise matching models with monetary transfers. The positive quadrant dependence *partial order subsumes Becker* (*1973*) *as the extreme cases with most and least sorting and implies increasing regression coefficients. Our theory turns on* synergy*—the cross-partial difference or derivative of match production. This reflects basic economic forces: diminishing returns*, *technological convexity*, *insurance*, *and learning dynamics. We prove sorting increases if match synergy globally increases*, *and is cross-sectionally monotone or single crossing. We use our results to derive sorting predictions in major economics sorting papers and in new applications.* (*JEL* C78, D21, D82, D86, J12)

This paper considers optimal pairwise matching, as in Becker's (1973) "marriage" model. Becker uses this metaphor for the economics of actual marriages and allegorical ones like employment, partnerships, optimal assignment, pairwise trade, and other matches with monetary transfers. In this reduced-form model, each side of the market has a scalar type, and payoffs solely depend on the matched individuals' types. Becker showed that *positive assortative matching* (PAM) emerges when partner types are complementary (or more formally, the match payoff function is supermodular): so the highest "man" pairs with the highest "woman," the next-highest man with the next-highest woman, and so on. Also, when match types are substitutes (submodular payoffs), *negative assortative matching* (NAM) arises—highest man with lowest woman, etc.

Little is known about matching models with neither supermodular nor submodular payoffs. This paper targets this gap with a tractable general theory on how match payoff function changes impact equilibrium sorting patterns. To do so, we first identify a simple economically meaningful partial order that captures increasing sorting: *positive quadrant dependence* (PQD). We ask when the output-maximizing matching under one production function entails more sorting in the PQD order than under another production function with "higher" synergy. We derive this comparative static conclusion under many notions of "higher" and assuming a cross-sectional synergy restriction.

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†Go to https://doi.org/10.1257/aer.20210890 to visit the article page for additional materials and author disclosure statements.

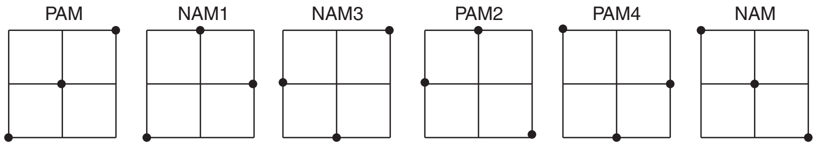

Figure 1. Pure Matchings with Three Types

Note: The possibilities are negative and positive assortative matching (NAM and PAM), negative sorting in quadrants 1 and 3 (NAM1 and NAM3), and positive sorting in quadrants 2 and 4 (PAM2 and PAM4).

Since agents are described by scalar types, a matching describes which types from each side are paired together and thus is a cdf on $\mathbb{R}^2$. One matching cdf is higher than another in the PQD order if it has more mass weakly below any pair of types (x, y) (i.e., in every southwest quadrant). For example, consider the PQD partial order over the six possible pure matchings among three men $(1, 2, 3)$ and three women $(1, 2, 3)$ in Figure 1. NAM pairs the highest types with the lowest types, while PAM pairs the two highest types, the two middle types, and the two lowest types. NAM1 also pairs the two lowest types but pairs middle types with highest types. PAM is strictly higher than NAM1 in the PQD order since it has more matches weakly below any pair (x, y) and strictly more matches weakly below $(2, 2)$. But notice that NAM1 and NAM3 are not PQD ranked, as NAM1 pairs $(1, 1)$, while NAM3 has more matches weakly below $(2, 2)$. The PQD order for all six matchings is

$$\text{PAM} \succ_{PQD} [\text{NAM1}, \text{NAM3}] \succ_{PQD} [\text{PAM2}, \text{PAM4}] \succ_{PQD} \text{NAM}. \tag{1}$$

Lemma 1 argues that increases in the PQD order imply all of (i) the average distance between matched types falls, (ii) the correlation of matched types increases, and therefore, (iii) the regression coefficient of women on their partners' types increases. In other words, increases in the PQD order imply that commonly used measure of sorting rise. By contrast, we show that no coherent sorting theory can emerge premised on increasing covariance, correlation, or falling average distance between match partners.

We next introduce a partial order on match production functions that connects submodularity and supermodularity. Our building block is a local complementarity measure: *Synergy* is the cross-partial difference of production with finitely many types and the cross-partial derivative with continuous types. Synergy is everywhere positive for supermodular functions and everywhere negative for submodular functions. To highlight its central role, we show how to express total match output as a constant plus an average of all match synergies weighted by the matching distribution. This means that any matching characterization *must* turn on synergy. For instance, Becker (1973) deduces positive sorting with all synergies positive and negative sorting with all synergies negative. We subsume intermediate cases, where synergy changes sign.

Since globally positive synergy implies assortative matching, is sorting greater with more synergistic production? A three-type example refutes this conjecture—

the optimal matching oscillates between the two non-PQD comparable matchings NAM1 and NAM3 as synergy rises in Figure 3. While increasing synergy is not enough for increasing sorting, Proposition 1 finds that sorting cannot fall in the PQD order when synergy globally weakly rises. This exhausts the strength of monotone comparative statics logic and allows unranked oscillations, like NAM1 to NAM3, as synergy rises.

To secure increasing sorting, we need stronger assumptions. We add in cross-sectional restrictions on synergy. Our easiest to state such result is Proposition 2—sorting increases if synergy weakly increases for all pairs and if synergy is cross-sectionally monotone, i.e., monotone across pairs of types before and after the shift in production. But these monotonicity assumptions may be too demanding since synergy is not monotone in many matching applications.

Our most general sorting result, Proposition 3, replaces monotonicity conditions in Proposition 2 with sign change provisos. The new assumption across matching markets is that total synergy aggregated on unions of rectangular partner sets changes sign only from negative to positive. The new cross-sectional premise is that the total synergy on rectangular sets changes sign just once as it shifts toward higher types.

Next, Proposition 4 replaces the cross-sectional premise of Proposition 3 with an assumption on marginal rectangular synergy. Finally, to subsume continuum types matching papers, Proposition 5 formulates an increasing sorting result solely in terms of local synergy. It posits that synergy changes sign only from negative to positive, with the same sign change cross-sectionally. But this is not enough, as single crossing is not preserved under addition. We therefore also assume that synergy is the product of an increasing and log-supermodular function. This ensures that positive synergy rises proportionately more than absolute negative synergy.

Finally, the logical arc of the paper is that Proposition 3 implies Proposition 4 implies Proposition 5 implies Proposition 2. We prove Proposition 3 for finitely many types. The proof in Appendix C.B by induction on the number of types is a key contribution of the paper. Notably, it never solves for an optimum. Rather, it chases down failures of the comparative static to the possible shift from the n-type version of NAM3 to NAM.

For our final general results, we deduce comparative statics for distributional shifts, such as an increase in the mass of high types of women. We show that a distributional shift can be reinterpreted as a change in the match payoff function and then apply our previous results to show that first-order shifts in type distributions increase sorting when synergy is cross-sectionally increasing (Corollary 1).

Economic Applications of Our Theory.—Our theory is targeted at applications. We show how our conditions on synergy can be readily derived in many standard economic problems, where our theory makes immediate predictions. We show that

(i) The typical economic force of *diminishing returns* lowers synergy and so sorting.

(ii) Match synergy is greater for *"weakest link" technologies* and lesser for *"strongest link"* technologies—where the lesser/higher type matters more, respectively.

(iii) In the principal-agent matching model of Serfes (2005), NAM obtains—more risk-averse agents with safer projects—when the disutility of effort is below a lower bound, while PAM obtains when disutility crosses an upper threshold. Our theory shows that sorting rises between these two thresholds, provided types (risk aversion and project variance) are not too far apart.

(iv) Our theory also speaks to dynamic matching with evolving types. In a model of *mentor-protégé workplace learning*, matching with a better mentor improves the protégé's future type. This strongest link technology lowers match synergy.[1]

Our model properly is a *transportation problem*, whose literature dates back over two centuries (see Villani 2008). Notably, it is not solved, except in special cases like Becker's. But we provide comparative statics predictions without ever deriving the optimal solution. We also build on a math literature on the PQD order. Lehmann (1966) introduced the PQD order and showed that several common correlation measures are weakly positive for any matching that is PQD higher than uniform random matching. Cambanis, Simons, and Stout (1976) found that total output weakly rises when the matching shifts up in the PQD order whenever synergy is everywhere nonnegative. Our Proposition 1 is a corollary of this result. Techen (1980) showed that nonnegative synergy is necessary for total output to rise for any upward shift in the PQD order.

Longer proofs and new monotone comparative statics results are in the Appendixes.

I. Becker's Marriage Model and Planner's Result

Our model is standardly adapted from Becker (1973) and the pairwise matching literature with two groups (men and women, firms and workers, buyers and sellers) or one (partnership model). To subsume both finite and continuum type models, we posit a unit mass of "women" and "men" with respective *types* $x, y \in [0,1]$ and cdfs G and H. We assume absolutely continuous type distributions G and H, and for the finite type model, G and H are discrete measures with equal weights on female types $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ and male types $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$ for $n \geq 2$. In the finite types case, we relabel women and men as $i, j \in \{1, 2, \ldots, n\}$, respectively.

[1] Bayesian updating need not inherit supermodularity in Anderson and Smith (2010). Supermodularity is often not preserved in our work with evolving human capital (Anderson and Smith 2012).

We assume a C^2 production function $\phi > 0$, so that types x and y jointly produce $\phi(x,y)$. In the finite type model, the output for match (i,j) is $f_{ij} \equiv \phi(x_i,y_j) \in \mathbb{R}$. Production is *supermodular* or *submodular* (SPM or SBM) if for all $x' < x''$ and $y' < y''$,

$$\phi(x',y') + \phi(x'',y'') \geq (\leq) \; \phi(x',y'') + \phi(x'',y'). \tag{2}$$

Strict supermodularity (respectively, strict SBM) asserts globally strict inequality in (2).

Since output is positive, everyone matches—even if allowed not to. A *matching* is a bivariate cdf $M \in \mathcal{M}(G,H)$ on $[0,1]^2$ with marginals G and H. A finite matching is a nonnegative matrix $[m_{ij}]$, with cdf $M_{i_0j_0} = \sum_{1\leq i\leq i_0, 1\leq j\leq j_0} m_{ij}$ and unit marginals $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0j}$ for all women i_0 and men j_0. In a *pure matching*, $[m_{ij}]$ is a matrix of 0s and 1s, with everyone matched to a unique partner.

There are two perfect sorting flavors. In *positive assortative matching*, any woman type of x at quantile $G(x)$ pairs with a man of type y at the same quantile $H(y)$, and thus, the match cdf is $M(x,y) = \min\{G(x),H(y)\}$. In *negative assortative matching*, complementary quantiles match, and so $M(x,y) = \max\{G(x)+H(y)-1,0\}$. Matched types are *uncorrelated* given uniform matching, and so $M(x,y) = G(x)H(y)$.

The *partnership* (or unisex) model is a special case where types x and y share a common distribution, $G = H$, and the production function ϕ is symmetric ($\phi(x,y) = \phi(y,x)$). In this case, PAM is simply matching with the same type, $y = x$.

A social planner maximizes total match output, namely, $\sum_{i=1}^n \sum_{j=1}^n f_{ij}(\theta) m_{ij}$ with finite types, or more generally, $\int_{[0,1]^2} \phi(x,y\,|\,\theta) M(dx,dy)$, where we index output $\phi(x,y\,|\,\theta)$ by a (*often suppressed*) *state* $\theta \in \Theta$, a partially ordered set (poset). The optimal matchings $\mathcal{M}^*(\theta)$ solves

$$\mathcal{M}^*(\theta) \equiv \underset{M\in\mathcal{M}(G,H)}{\arg\max} \int_{[0,1]^2} \phi(x,y\,|\,\theta) M(dx,dy). \tag{3}$$

Gretsky, Ostroy, and Zame (1992) prove existence and show that $\mathcal{M}^*$ is the core of the matching game among women x and men y, or workers x and capital y. They also show that solutions can be decentralized as a competitive equilibrium.[2] So our theory applies to equilibrium sorting in such markets.

Problem (3) has been solved in just three general cases: all feasible matchings are optimal with additive production, while Becker solved for SBM and SPM production.

BECKER'S SORTING RESULT: *Given SPM* (*SBM*) *production ϕ, PAM* (*NAM*) *is an optimal matching. Given strict SPM* (*SBM*), *these pairings are uniquely optimal.*

[2] Villani (2008, p. 61) states that existence "has probably been known from time immemorial," and his Theorem 4.1 provides existence for very general type spaces. Koopmans and Beckmann (1957) decentralize the finite type solution as a competitive equilibrium. Legros and Newman (2007) show that some nontransferable utility models can be mapped into the transferable utility paradigm.

For an intuition, assume finitely many types and SPM (2). A maximum of problem (3) obviously exists. To see uniqueness, note that if ever women $x' < x''$ and men $y' < y''$ are negatively sorted into matches (x', y'') and (x'', y'), then total output is raised by rematching them as (x', y') and (x'', y''). A proof for any number of types is in Section II.

Without SBM or SPM, solving the general social planner's problem (3) is a hard open question. We bypass this and ask how the optimal set $\mathcal{M}^*(\theta)$ changes in θ. We derive its comparative statics in θ when output $\phi(x, y \mid \theta)$ is neither SPM nor SBM. Hereafter, a *time series* property suggestively refers to changes in the state θ,[3,4] and a *cross-sectional property* to production changes over the type space. We then apply our finding in several matching models across economics, without SPM or SBM output.

Throughout the paper, we present finite type and continuum type results together, as synergy is a common theme. We draw both intuition and our overall inductive proof logic from the finite type case and derive the continuum type results by taking limits.

II. Sorting Measurement and Synergy

This section introduces the building blocks of our theory. First, we define and discuss the partial order that we use to measure sorting. We then define a property of payoff functions called synergy and show that optimal matching only depends on synergy.

A. *The Positive Quadrant Dependence Order*

The PQD order is a binary partial order on bivariate probability distributions $M, M' \in \mathcal{M}(G, H)$. Matching measure M' is *PQD higher than* M, or $M' \succeq_{PQD} M$, if $M'(x, y) \geq M(x, y)$ for all types x, y. So M' puts more weight than M on all lower (southwest) orthants. As M and M' share marginals, M' puts more weight than M on all upper (northeast) orthants too (Figure 2).

As noted in the introduction, PQD only partially orders the six possible pure matchings on three types. In terms of Becker's bounds, match cdf's are sandwiched above NAM and below PAM:

$$\max\{G(x) + H(y) - 1, 0\} \leq M(x, y) \leq \min\{G(x), H(y)\}. \tag{4}$$

The second inequality says that the mass of matched men and women in $[0, x] \times [0, y]$ is at most the total mass of men or women. The first inequality—rewritten as $1 - M(x, y) \leq \min\{1 - G(x) + 1 - H(y), 1\}$, says the mass of matches not in $[0, x] \times [0, y]$ is at most the total mass of women above x plus the mass of men above y.

[3] The term "time series" is used to distinguish variation *across* matching markets from changes across types within a market. The state could also represent geographic differentiation in matching markets.

[4] Equivalently, our theory compares sorting for two production functions ϕ_1 and ϕ_2 (i.e., $\theta_1 < \theta_2$).

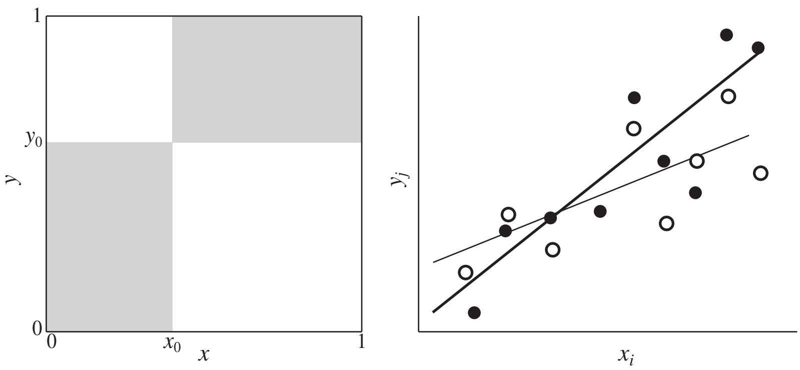

FIGURE 2. PQD ORDER

Notes: Left panel: PQD increases for cdfs on $[0,1]^2$ raise the probability mass on all lower-left rectangles (corners $(0,0)$ and (x_0, y_0)), and so on all upper-right rectangles (corners (x_0, y_0) and $(1,1)$). Right panel: The best fit regression line is steeper (thick black line and ● versus thin black line and ○) after a PQD increase (Lemma 1(c)).

The PQD sorting measure implies typical economically relevant measures for measured traits $u(x)$ and $v(y)$ of women x and men y, increasing in x and y:

LEMMA 1: *Fix nondecreasing functions u and v. Given a PQD order upward shift,*

(*a*) *the average distance* $E\left[\left|u(X) - v(Y)\right|^{\gamma}\right]$ *for matched types weakly falls*, *if* $\gamma \geq 1$;

(*b*) *the covariance* $E_M\left[u(X)v(Y)\right] - E\left[u(X)\right]E\left[v(Y)\right]$ *across matched pairs weakly rises*;

(*c*) *the linear regression coefficient of* $v(y)$ *on* $u(x)$ *across matched pairs weakly rises.*

PQD is an *ordinal sorting ranking*, like PAM—not dependent on type scaling. So if educational sorting PQD rises, then this holds regardless of whether it is measured in highest degree, schooling years, etc. But for non-PQD comparable matching changes, the sorting conclusion can reverse if the choice of cardinal measure changes. This highlights why we use the stronger ordinal PQD sorting order.

To see this, assume three types, and consider a non-PQD comparable NAM1 to NAM3 change. If $x \in \{1,2,3\}$ and $y \in \{0.5, 1.8, 3\}$, then the covariance between matched types and average distance between partners both fall; i.e., sorting falls if measured by type correlation but rises if measured by average distance between matched types. But if $y \in \{0.5, 2.5, 5\}$, match type correlation rises, and average distance between matched types falls. Both sorting measures fall if $y \in \{0.5, 2.5, 3\}$, and both rise if $y \in \{0.5, 2.5, 3\}$. So *any sign pattern is consistent with a NAM1 to NAM3 shift*.

If we convert to *quantile space*, then the covariance and the average distance ranking coincide for (NAM1, NAM3) and (PAM2, PAM4). But equivalence fails with four types. For example, let M' be the four type matching $\{(1,4),(2,2),(3,3),(4,1)\}$ and M'' be the PQD incomparable matching $\{(1,3),(2,4),(3,1),(4,2)\}$. Then covariance-based sorting statistics deem M'' more sorted (e.g., a higher correlation coefficient) than M', while M' is more sorted than M'' by the average distance between partners.

B. *Synergy*

We now introduce a local measure of Becker's supermodularity assumption. In finite type models, we suggestively call the cross-partial difference of output *synergy*:

$$s_{ij}(\theta) \equiv f_{i+1j+1}(\theta) + f_{ij}(\theta) - f_{i+1j}(\theta) - f_{ij+1}(\theta).$$

Synergy is the net change in output from positively sorting pairs (i,j) and $(i+1,j+1)$ versus negatively sorting as $(i,j+1)$ and $(i+1,j)$. Equivalently, it is the difference between the gain in output that woman $i+1$ gets when matching with the next-higher man, $f_{i+1j+1} - f_{i+1j}$, and this same change for the next-lower woman i, $f_{ij+1} - f_{ij}$.

The central importance of synergy is revealed by expressing match output as a weighted sum of match synergies. Appendix A proves the following identity by double summation of match output by parts:[5]

$$(5) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij} m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left[f_{nj+1} - f_{nj}\right] j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij} M_{ij}.$$

In other words, any two production functions with identical synergies share the optimal matching. For instance, if production is linear, then synergy vanishes, and all match distributions yield the same output.

Becker's Result follows immediately from the bounds (4) and the summation by parts formula (5). For example, output is SPM when all $s_{ij} \geq 0$, and so by (5), output is highest when the cdf $M(x,y)$ is maximal: PAM dominates all other matchings. Similarly, if output is SBM, then all $s_{ij} \leq 0$, and thus, output is highest when the match cdf $M(x,y)$ is minimal, namely, for NAM. More generally, the PQD and SPM orders coincide in $\mathbb{R}^2$; i.e., *increases in the PQD order increase* (*reduce*) *the total output for any SPM* (*SBM*) *function* ϕ:[6]

$$(6) \qquad M' \succeq_{PQD} M \Leftrightarrow \int \phi(x,y) M'(dx,dy) \geq \int \phi(x,y) M(dx,dy), \quad \forall \phi \text{ SPM}.$$

III. What Happens When Synergy Rises?

Since Becker shows that globally negative synergy leads to NAM, and globally positive synergy leads to PAM, one might surmise that sorting increases if synergy

[5] Lemma 3 in online Appendix D.2 derives the analog for types on a continuum.

[6] Lehmann (1966) introduces the PQD order, and Cambanis, Simons, and Stout (1976) prove that the SPM order implies the PQD ranking in $\mathbb{R}^2$. Techen (1980) proves the converse.

Match payoffs

	x_1	x_2	x_3
y_3	9	**14**	18
y_2	5	2	**14**
y_1	**1**	5	9

→

	x_1	x_2	x_3
y_3	9	16	**24**
y_2	**5**	3	16
y_1	1	**5**	9

→

	x_1	x_2	x_3
y_3	9	**20**	30
y_2	5	6	**20**
y_1	**1**	5	9

→

	x_1	x_2	x_3
y_3	9	22	**36**
y_2	**5**	7	22
y_1	1	**5**	9

Cross-partial differences of match payoffs

	x_1x_2	x_2x_3
y_2y_3	8	–8
y_1y_2	–7	8

→

	x_1x_2	x_2x_3
y_2y_3	9	–5
y_1y_2	–6	9

→

	x_1x_2	x_2x_3
y_2y_3	10	–4
y_1y_2	–3	10

→

	x_1x_2	x_2x_3
y_2y_3	11	–1
y_1y_2	–2	11

Figure 3. Sorting Need Not Rise in Synergy

Notes: Top: The unique efficient matching (bold) alternates between NAM1 and NAM3. Bottom: Match synergies (cross-payoff differences) strictly increase as we move right, but sorting does not PQD rise. Sorting by two common cardinal measures can move contrarily. If $x \in \{1,2,3\}$ and $y \in \{0.5, 1.8, 3\}$, *NAM1 to NAM3 shifts reduce both covariance* and *average distance between partners.*

increases everywhere. This natural conjecture fails: in Figure 3, synergy strictly increases at each step, and yet the uniquely optimal matching oscillates between the non PQD-comparable NAM1 and NAM3. What goes wrong?

The synergy sign is all that matters for determining whether NAM or PAM is optimal for any pair of couples, but the magnitude of synergy impacts global sorting patterns. For example, one can verify that NAM1 yields a higher payoff than NAM3 if and only if synergy is larger in the lower-left rectangle, s_{11}, than in the upper-right rectangle, s_{22}. This makes sense of the sorting monotonicity failure in Figure 3: synergy strictly increases in θ, but the difference $s_{11}(\theta) - s_{22}(\theta)$ changes sign for every increase in θ. Consequently, the optimal matching oscillates between NAM1 and NAM3.

Technically, our objective function is single crossing in (M, θ) by (5). But standard monotone comparative statics results do not apply because the domain of matching cdf's is not a lattice with the PQD order (Müller and Scarsini 2006). Indeed, NAM1 and NAM3 in (1) are both pure upper bounds for PAM2 and PAM4, but neither is least. More strongly, there is no mixed least upper bound for PAM2 and PAM4.

While the optimal matching oscillates in Figure 3, it never falls in the PQD order.We show in online Appendix D that this is the comparative statics conclusion for our case with a single crossing condition but not on a lattice domain. Specifically, for our matching context, say that *sorting is nowhere decreasing* in θ if the matching never falls in the PQD order. So for all $\theta'' \succeq \theta'$, if $M' \in \mathcal{M}^*(\theta')$ and $M'' \in \mathcal{M}^*(\theta')$ are ranked $M' \succeq_{PQD} M''$, then we have $M'' \in \mathcal{M}^*(\theta')$ and $M' \in \mathcal{M}^*(\theta'')$.

PROPOSITION 1: *Sorting is nowhere decreasing in θ if synergy is nondecreasing in θ.*[7]

[7] For completeness, online Appendix D.1 generalizes Proposition 1, deriving a more general theory of comparative statics on posets. We thank a referee for the proof of the following special case of this general theory. He derived it as a corollary of Cambanis, Simons, and Stout (1976).

PROOF:

By match payoff formulation (5), the payoff gain moving from matching M'' to matching M' is $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M''_{ij})$. Since $M' \succeq_{PQD} M''$ (namely, $M' \geq M''$), if $\theta'' \succeq \theta'$, then *the Planner's objective function obeys increasing differences in* (M,θ):

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta'')(M'_{ij} - M''_{ij}) \geq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M''_{ij}).$$

Assume that M' is optimal at θ' and M'' at θ''. Then,

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M''_{ij}) \geq 0 \geq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta'')(M'_{ij} - M''_{ij}).$$

But then equality holds everywhere: hence, M' is optimal at θ'' and M'' at θ'.∎

IV. Increasing Sorting

We now provide conditions that guarantee that matching is increasing in the PQD order. To preclude the increasing sorting failures as in Figure 3, we cross-sectionally restrict how synergy evolves across types.

A. *Strictly Monotone Synergy in Types*

First consider the simplest case: *synergy is* (*strictly*) *monotone in types* if synergy is either nondecreasing (increasing) or nonincreasing (decreasing) in (x,y); i.e., synergy is monotone to the "north and east" or "south and west" in the type space.[8]

PROPOSITION 2: *Let synergy be nondecreasing in θ. If M'' and M' are respectively optimal for $\theta'' \succ \theta'$, then $M'' \succeq_{PQD} M'$ in* (a) *generic finite type models for synergy monotone in types and* (b) *continuum type models for synergy strictly monotone in types.*

To illustrate this first sorting result, consider the production function $\phi = \alpha xy + \beta(xy)^2$. If $\alpha\beta < 0$, then Becker's Sorting Result does not apply. But since synergy $\phi_{12} = \alpha + 2\beta xy$ strictly increases in (α,β), sorting rises in both parameters, by Proposition 2.

Assuming that synergy is monotone in types rules out either NAM1 or NAM3 in three-type models. But with more types, this cross-sectional assumption still allows for rich matching patterns. For example, assume the synergy function $\phi_{12}(x,y) = \alpha - \beta\min\{x,y\}$ (recall that synergy fully determines the optimal matching by (5)). Synergy is monotone in types—nondecreasing or nonincreasing as $\beta \lesseqgtr 0$. Synergy is increasing in α and decreasing in β. Thus, by Proposition 2, sorting increases in α and falls in β. Figure 4 illustrates this

[8] This cross-sectional assumption is not so strong that it eliminates the partialness of the PQD order. For instance, PAM2 and PAM4 can both emerge as optimal matchings when synergy is strictly monotone in types (Figure 5, left panel).

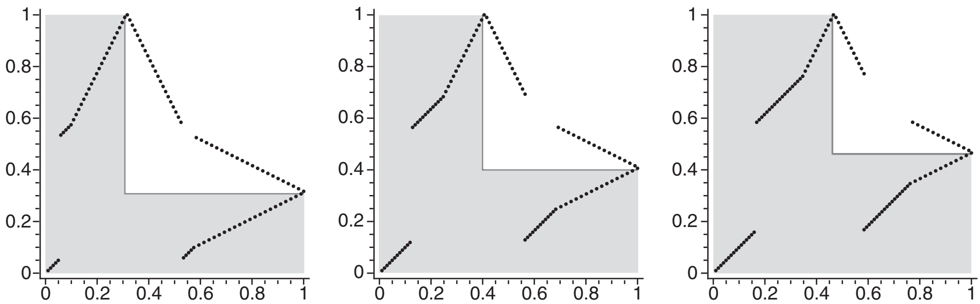

FIGURE 4. MATCHING EXAMPLE FOR PROPOSITION 2

Notes: We numerically depict the matching support for the synergy function $\alpha - \beta\min\{x_i, x_j\}$. *All matching plots depict optimally matched pairs (dots) for a uniform distribution on a finite* 100×100 *matching array.* In each graph, synergy is positive (negative) on the shaded (unshaded) regions. Left to right, plots assume $(\alpha, \beta) = (0.4, 1.3), (0.4, 1)$, and $(0.6, 1.3)$.

comparative static with 100 equally spaced types on each side of the market. Notice that the matching alternates between locally positive and locally negative sorting for fixed α and β. Furthermore, these finite type plots also *suggest* that the optimal matching will not be pure (one-to-one) with continuum types. In fact, none of our continuum type sorting results require purity.

B. *One-Crossing Rectangular Synergy in Types*

The conditions in Proposition 2 are quick to check but do not hold in many applications. We now prove a sorting result with a weaker premise, which we apply to several applications in Section VI. Let $(T, \succeq)$ be a partially ordered set. A function $\Upsilon : T \mapsto \mathbb{R}$ is *upcrossing in t*[9] if $\Upsilon(t) \geq (>) \ 0$ implies $\Upsilon(t') \geq (>) \ 0$ for all $t' \succeq t$, *downcrossing in t* if $-\Upsilon$ is upcrossing, and *one-crossing in t* if it is upcrossing *or* downcrossing. Strict versions of these conditions require that weak inequalities imply strict inequalities. For example, Υ is *strictly upcrossing* if $\Upsilon(t) \geq 0$ implies $\Upsilon(t') > 0$, for all $t' \succ t$.

The *rectangle* $r \equiv (i_1, j_1, i_2, j_2) \in \mathbb{N}^4$ has diagonally opposite corners given by two women $i_1 < i_2$ and men $j_1 < j_2$. *Rectangular synergy* $S(r \mid \theta) : \mathbb{N}^4 \to \mathbb{R}$ sums synergies $s_{ij}(\theta)$ inside the rectangle r:

$$(7) \qquad \mathcal{S}(r \mid \theta) \equiv \sum_{i=i_1}^{i_2-1} \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) = f_{i_1 j_1}(\theta) + f_{i_2 j_2}(\theta) - f_{i_1 j_2}(\theta) - f_{i_2 j_1}(\theta).$$

This is the gain on rectangle r to positively sorting (creating couples $(i_1, j_1) < (i_2, j_2)$) versus negatively sorting (creating couples (i_1, j_2) and (i_2, j_1)).

For a type continuum, rectangular synergy is the integral of synergy over a rectangle; namely, $S(R \mid \theta) \equiv \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y \mid \theta) dx dy$ for any $R = (x_1, y_1, x_2, y_2)$. Our next

[9] The "single crossing property" usually implies a two-dimensional functional domain. To avoid this confusion, and clarify the direction, we instead use the suggestive terms *upcrossing* and *downcrossing*.

result requires *summed rectangular synergy*—namely, the sum $\sum_k \mathcal{S}(r_k \mid \theta)$ on a finite set of disjoint rectangles $\{r_k\}$ with finite types, or $\sum_k \mathcal{S}(R_k \mid \theta)$ on finite disjoint set $\{R_k\}$ with continuum types.[10]

Since rectangular synergy is the net gain to positively rematching the negatively sorted pair of couples (i_1,j_2) and (i_2,j_1), summed rectangular synergy is the net gain to a sequence of such positive couple swaps. When summed rectangular synergy is upcrossing in θ, any such sequence of positive swaps increases aggregate output at all $\theta' \succeq \theta$, whenever this sequence increases aggregate output at θ. This ordinal assumption weakens the time series assumption in Proposition 2 since summed rectangular synergy is upcrossing in θ if synergy is nondecreasing in θ.

Our first ordinal cross-sectional assumption uses the northeast partial order on rectangles: $r \succeq_{NE} r'$, if diagonally opposite corners of r are weakly higher than r'. Rectangular synergy is *one-crossing* in types if $\mathcal{S}(r \mid \theta)$ is upcrossing or downcrossing in r, for all θ. This assumption demands that the sign of the change in output from any positive swap can only change once as we increase types. For example, if rectangular synergy is upcrossing in types and positively rematching the negatively sorted pair of couples (i_1,j_2) and (i_2,j_1) increases output, then any positive swap involving couples with higher type indices must also increase output. This is an ordinal weakening of the cross-sectional assumption in Proposition 2 since rectangular synergy is one-crossing in types when synergy is monotone in types.

PROPOSITION 3 (Increasing Sorting): *Assume summed rectangular synergy is upcrossing in θ and rectangular synergy is one-crossing in types. If M'' and M' are uniquely optimal for respectively $\theta'' \succ \theta'$, then $M'' \succeq_{PQD} M'$.*

Proposition 3 is our most general result. Since its time series premise is weaker than monotone synergy, we cannot deduce it from Proposition 1.[11] Proposition 3 applies to output functions with a unique optimal matching, but optimal matchings are generically unique in finite type models by Koopmans and Beckmann (1957). We prove uniqueness for continuum type models with a stronger cross-sectional proviso in Section IVD.

C. *Logic of the Proof of the Increasing Sorting Theorem*

To help build intuition, we show how our cross-sectional and time series assumptions jointly rule out a PAM2 to PAM4 shift as θ rises. Toward a contradiction, assume PAM2 uniquely optimal at θ' and PAM4 uniquely optimal at $\theta'' \succ \theta'$, as illustrated in Figure 5 (right). Local optimality implies the synergy signs given in Step A. Then, since synergy is upcrossing in θ, $s_{12}(\theta') > 0$ implies $s_{12}(\theta'') > 0$ as indicated in Step B. Now notice that PAM4 involves negatively sorting couples $(1,3)$ and $(3,2)$; and thus, the synergy sum across the top row obeys $s_{12}(\theta'') + s_{22}(\theta'') < 0$. But then, since $s_{12}(\theta'') > 0$ (Step B), we conclude in Step C that $s_{22}(\theta'') < 0$. Likewise, PAM4 negatively sorts pairs $(1,3)$ and $(2,1)$, implying

[10] The proof only needs this assumption for sums of rectangles sharing a common northeast corner.

[11] In fact, the time series assumption in Proposition 3 is weaker than the robustly necessary condition for nowhere decreasing sorting, as seen in Theorem 4 in online Appendix D.2.

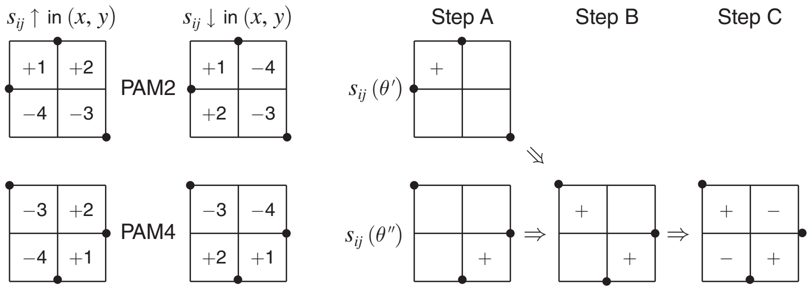

Figure 5. The Role of Our Cross-Sectional Synergy Assumption

Notes: At left, we show that even strictly monotone synergy in types still allows PAM2 and PAM4, and so PQD is still a partial order on allowable matchings. At right is a schematic illustrating our logic precluding a PAM2 to PAM4 *change* when synergy is also upcrossing in θ.

the synergy sum in the first column satisfies $s_{11}(\theta'') + s_{12}(\theta'') < 0$. But then, since $s_{12}(\theta'') > 0$ (Step B), we can also sign $s_{11}(\theta'') < 0$. Notice that the sign pattern in Step C violates synergy one-crossing in types. Altogether, PAM2 optimal at θ' and PAM4 optimal at θ'' is impossible. Symmetric logic rules out PAM4 optimal at θ' and PAM2 optimal at θ''.

The preceding logic rules out one non-PQD comparable shift. We now trace the logic of our induction proof in Appendix C.B for three-type models with rectangular synergy upcrossing in types. Assume M' and M'' are uniquely optimal for $\theta'' \succ \theta'$. As shown in Appendix C.B, uniqueness implies purity for finite type models: M' and M'' are pure.

- **Step (i):** *Sorting rises in θ in two-type models if rectangular synergy upcrosses in θ.*

- **Step (ii):** *If rectangular synergy upcrosses in types, then NAM1 is impossible.* Indeed, rectangular synergy upcrossing in types precludes $s_{11} + s_{12} > 0 > s_{22}$ (Figure 6), as required if NAM1 is uniquely optimal. Notice that this step rules out the monotone sorting counterexample in Figure 3. We use the fact that this holds for any 3×3 subset of $n \times n$ types throughout our proof in the Appendix.

- **Step (iii):** *Partners of woman 1 and man 1 each rise by one if the matching does not weakly rise.* This corresponds to Step 3 in Appendix C.B. Indeed, shifting from θ' to θ'':

Case 1 of Step (iii): *The partner of woman 1 cannot rise by 2.* Since there are only three types, the only way the partner of woman 1 can rise by 2 is if woman 1 is matched to man 1 at θ' and man 3 at θ'', which implies that man 3 is paired with a woman $i > 1$ at θ', while woman i is matched to a man $j < 3$ at θ''. Now, remove matched couples $(i, 3)$ at θ' and (i, j) at θ'' and consider the induced matching among the remaining two women and men. By Fact 2 in Appendix C.B, synergy will be

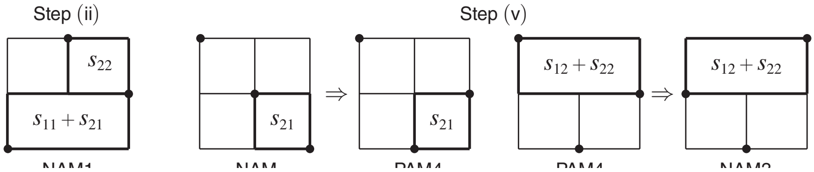

Figure 6. Illustrations for Three-Type Version of Proposition 3 Proof

Notes: Step (ii) shows that NAM1 (left) for any 3×3 subset of types is impossible when synergy is upcrossing in types. Step (v) uses the fact that mapping from NAM to PAM4 changes the payoff by s_{21}, while mapping from PAM4 to NAM3 changes the payoff by $s_{12} + s_{22}$.

upcrossing in θ in this two-type model since we have removed the same woman and a weakly higher man at θ'. So the matching in the induced two-type model must be PQD higher at θ'' than θ', by Step (i). But by assumption, woman 1 pairs with man 1 at θ', and woman 1 pairs with (the new) man 2 at θ''; i.e., the induced two-type model is PAM at θ and NAM at θ''.

Case 2 of Step (iii): *The partner of woman 1 strictly rises.* Assume instead that her partner weakly falls from k to j. As in Case 1, synergy must be upcrossing in θ in the induced two-type model, if we remove couple $(1,k)$ at θ' and couple $(1,j)$ at θ''. Thus, the induced two-type matching is PQD higher at state θ'' than θ' by Step (i). But adding couple $(1,k)$ and couple $(1,j)$ to the optimal two-type matchings under θ' and θ'' preserves the PQD ordering by $k \geq j$ and Fact 5 in Appendix C.B. So if the matching fails to weakly rise in the PQD order, then woman 1's partner cannot weakly fall.

Combining Cases 1 and 2 of Step (iii), woman 1's partner increases by 1. Symmetric arguments establish that man 1's partner also increases by 1.

- **Step (iv):** *If matching does not weakly PQD rise, then it falls from NAM3 to NAM.* By Step (iii), woman 1 cannot pair with man 3, nor man 1 with woman 3, at θ'. E.g., in the first case, by Step (iii), woman 1 matches with nonexistent man 4 under θ''.

But woman 1 and man 1 cannot match at θ'. For if so, there are only two possible matchings for M': either types 2 and 3 positively sort, and so $M' = $ PAM, or they do not, whence $M' = $ NAM1. Since Step (ii) precludes NAM1, assume $M' = $ PAM. As the lowest two types match at θ', by Step (iii), woman 1 pairs with man 2 and man 1 with woman 2 at θ''. All told, the lowest two types positively sort at θ' and negatively sort at θ''—violating rectangular synergy upcrossing in θ.

Now consider the remaining case: woman 1 pairs with man 2, and man 1 with woman 2, at θ'. Having matched the two lowest men and women, woman 3 must match with man 3. Altogether, M' is NAM3—namely, couples $\{(1,2),(2,1),(3,3)\}$. By Step (iii), woman 1 matches with man 3, and man 1 with woman 3 at θ''. But then, the remaining man 2 and woman 2 match; i.e., M'' is NAM: $\{(1,3),(2,2),(3,1)\}$.

	x_1	x_2	x_3
y_3	6	6	**11**
y_2	**4**	6	6
y_1	0	**4**	6

$\rightarrow$

	x_1	x_2	x_3
y_3	**7**	6	11
y_2	4	**6**	6
y_1	0	4	**7**

	x_1x_2	x_2x_3
y_2y_3	−2	5
y_1y_2	−2	−2

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	−3	5
y_1y_2	−2	−3

Figure 7. Falling Matching with Rectangular Synergy Upcrossing in Types and θ

Notes: The unique efficient matching falls from NAM3 to NAM as θ' shifts up to θ''. The sorting premium S is upcrossing in rectangles r for each θ, and the signs of $S(r\,|\,\theta')$ and $S(r\,|\,\theta'')$ coincide for all r; thus, S is upcrossing from θ' to θ''. But Proposition 3 does not apply, as total synergy falls from 1 to -1 for the set that only excludes s_{11}.

Step (iv) captures Steps 4–7 in the n-type proof, although the logic is significantly more involved with many types. The next item distills Step 8 in the n-type proof:

- **Step (v):** *The matching cannot fall from NAM3 to NAM.* As in Figure 6, one can switch from NAM to NAM3, by first moving to PAM4, then to NAM3. The first shift rematches couples $(2,2)$ and $(3,1)$, into $(2,1)$ and $(3,2)$, changing output by synergy s_{21}. The second switch to NAM3 rematches couples $(1,3)$ and $(3,2)$ into $(1,2)$ and $(3,3)$, changing output by the synergy sum $s_{21}+s_{22}$. Combining these two swaps, we see that the NAM3 payoff exceeds the NAM payoff by synergy sum $s_{12}+s_{21}+s_{22}$. Since NAM3 is uniquely optimal for θ', and NAM uniquely optimal for θ'', we have

$$s_{12}(\theta'')+s_{21}(\theta'')+s_{22}(\theta'')\;<\;0\;<\;s_{12}(\theta')+s_{21}(\theta')+s_{22}(\theta').$$

  This contradicts summed rectangular synergy upcrossing in θ.

Steps (iv) and (v) together imply that the matching weakly rises from θ' to θ''.

Only in Step (v) did we use *summed* rectangular synergy upcrossing in θ. Absent this assumption, sorting can fall in θ. For example, in Figure 7, rectangular synergy is upcrossing in types and θ, and yet the uniquely optimal matching falls from NAM3 to NAM as θ rises. We generalize Steps (iv) and (v) in Appendix C.B, with an n-type generalization of NAM3; namely, couple (n,n) matched, and lower types matched according to NAM.

D. *One-Crossing Marginal Rectangular Synergy in Types*

We now provide a stronger, but easier to check, cross-sectional assumption to deliver increasing sorting.

Specifically, the *x-marginal rectangular synergy* $\Delta_i(i\,|\,j_1,j_2)$ is the sum of synergy between woman i and men in the interval $[j_1,j_2-1]$, and the *y-marginal rectangular synergy* $\Delta_j(j\,|\,i_1,i_2)$ is the sum of synergy between man j and women in the interval $[i_1,i_2-1]$; i.e.,

$$\Delta_i(i\,|\,j_1,j_2,\theta)\;\equiv\;\sum_{j=j_1}^{j_2-1}s_{ij}(\theta)\quad\text{and}\quad\Delta_j(j\,|\,i_1,i_2,\theta)\;\equiv\;\sum_{i=i_1}^{i_2-1}s_{ij}(\theta). \tag{8}$$

Equivalently, the x-marginal rectangular synergy is the difference between the gain in output that woman $i+1$ gets when matching with a higher-index man, $f_{i+1j_2} - f_{i+1j_1}$, and this same change for the next lower woman $i, f_{ij_2} - f_{ij_1}$.

Marginal rectangular synergy is upcrossing in the finite types case if the left sum in (8) is upcrossing in i *and* the right sum is upcrossing in j. In the continuum types case, we require the integrals $\Delta_x(x\,|\,y_1,y_2,\theta) \equiv \int_{y_1}^{y_2} \phi_{12}(x,y\,|\,\theta)dy$ upcrossing in x for all $y_2 > y_1$ and $\Delta_y(y\,|\,x_1,x_2,\theta) \equiv \int_{x_1}^{x_2} \phi_{12}(x,y\,|\,\theta)dx$ upcrossing in y for all $x_2 > x_1$. Finally, marginal rectangular synergy is one-crossing if it is either upcrossing or downcrossing.

Notably, one-crossing marginal rectangular synergy is an ordinal implication of monotone synergy. To see this, notice that synergy ϕ_{12} is nondecreasing in x if and only if $\phi_1(x,y_2\,|\,\theta) - \phi_1(x,y_1\,|\,\theta)$ is nondecreasing in x for all $y_2 > y_1$, i.e., if x-marginal rectangular synergy $\Delta_x(x\,|\,y_1,y_2,\theta)$ is nondecreasing in x.

PROPOSITION 4: *Assume summed rectangular synergy is upcrossing in θ. If M'' and M' are optimal for respectively $\theta'' \succ \theta'$, then $M'' \succeq_{PQD} M'$ in (a) generic finite type models if marginal rectangular synergy is one-crossing and (b) continuum type models if marginal rectangular synergy is strictly one-crossing.*

The proof in Appendix C.D shows that these conditions imply those of Proposition 3. These propositions share the same time series assumption. The cross-sectional assumption in Proposition 4 implies Proposition 3's cross-sectional assumption. To verify this, recall that a function $f : \mathbb{R}^k \mapsto \mathbb{R}$ is *log-supermodular* (LSPM) if $f \geq 0$ and $\forall a,b \in \mathbb{R}^k$

$$f\big(\max\{a,b\}\big)f\big(\min\{a,b\}\big) \geq f(a)f(b). \tag{9}$$

Now, rewrite rectangular synergy as

$$\begin{aligned} \mathcal{S}(x_1,x_2,y_1,y_2\,|\,\theta) &= \int_{x_1}^{x_2} \Delta_x(x\,|\,y_1,y_2,\theta)dx \\ &= \int_0^1 \Delta_x(x\,|\,y_1,y_2,\theta)\mathbf{1}\big\{x \in [x_1,x_2]\big\}dx. \end{aligned} \tag{10}$$

We show in Appendix C.D that the indicator function $\mathbf{1}\{x \in [x_1,x_2]\}$ is LSPM in (x, x_1,x_2). Thus, by the classic result of Karlin and Rubin (1956) on upcrossing preservation in integrals, $\mathcal{S}$ is upcrossing in (x_1,x_2) whenever Δ_x is upcrossing in x. Likewise, $\mathcal{S}$ is upcrossing in (y_1,y_2) whenever y-marginal rectangular synergy is upcrossing in y. Loosely, log-supermodularity of a kernel is the key way to ensure that upper portions of the domain are proportionately weighted more and thus upcrossing is preserved.

To apply Proposition 3, we also need the optimal matching to be unique. This is generically true for finite type models. Fortuitously, strictly one-crossing marginal rectangular synergy implies a known sufficient condition in the optimal transport literature for uniqueness in our continuum types model.

E. *Purely Local Assumptions on Synergy*

In this section we give a theory of increasing sorting based on synergy alone, rather than summed synergy. A possible conjecture is that sorting is increasing in θ whenever synergy is upcrossing in θ and one-crossing in types. But in Figure 7 sorting *falls* in θ, despite the fact that synergy is both upcrossing in θ and in types. The reason for this failure is that *summed rectangular* synergy is not upcrossing in θ since it falls from 1 to -1 for the set that only excludes s_{11}.

This example illustrates the fact that sums of upcrossing functions need not be upcrossing. We need additional assumptions to ensure that *summed* synergy inherits the upcrossing assumptions required by our earlier theory. Appendix C.E presents our most general increasing sorting result based on synergy assumptions alone. Here, we pursue a robust special case that generalizes Proposition 2, which we apply to a class of applications in Section VIA. Specifically, assume that synergy has a product structure, $s_{ij}(\theta) = \zeta(x_i, y_j \mid \theta)\kappa(x_i, y_j \mid \theta)$ in the discrete case and $\phi_{12}(x, y \mid \theta) = \zeta(x, y \mid \theta)\kappa(x, y \mid \theta)$ for continuum types with κ nonnegative. We say ζ is (strictly) monotone in types if it is either nondecreasing (increasing) or nonincreasing (decreasing) in (x, y).

PROPOSITION 5: *Assume synergy is the product $\zeta\theta$, where ζ is monotone in types and nondecreasing in θ, and κ is LSPM. If M'' and M' are optimal for respectively $\theta'' \succ \theta'$, then $M'' \succeq_{PQD} M'$ in (a) generic finite type models and (b) continuum type models if ζ is also strictly monotone in types and $\kappa > 0$.*

By setting $\kappa \equiv 1$, this result trivially generalizes Proposition 2.

To prove Proposition 5, we show it implies Proposition 4's premise. For example, we show that marginal rectangular synergy is strictly upcrossing when ζ is strictly increasing in (x, y) and $\kappa > 0$ is LSPM. Consider y-marginal rectangular synergy $\Delta_y(y) = \int_{x_1}^{x_2} \zeta(x, y \mid \theta)\kappa(x, y \mid \theta)dx$, suppressing arguments (x_1, x_2, θ). Intuitively, κ LSPM ensures that the integral weights the positive parts of the increasing function ζ proportionately more than the negative parts, as θ increases. The general theory in Appendix C.E dispenses with the product structure but retains this key implication of LSPM.

V. Increasing Sorting and Type Distribution Shifts

Our analysis thus far focused on differences in production functions. We now ask, what if we fix the production function and vary the type distributions? It turns out that our results readily apply because changes in the distribution can be reinterpreted as changes in the production function. In particular, we can deduce sorting predictions for changes in the type distributions $G(\cdot \mid \theta)$ and $H(\cdot \mid \theta)$ by analyzing sorting by quantiles (rather than types). We say that *X types shift up* $(down)$ in θ if $G(\cdot \mid \theta)$ stochastically increases (decreases) in θ; i.e., $G(\cdot \mid \theta') \leq G(\cdot \mid \theta)$ if $\theta' \succeq \theta$. Similarly, *Y types shift up* $(down)$ in θ if $H(\cdot \mid \theta)$ stochastically increases (decreases) in θ.

We need to adapt our notion of sorting since PQD in Section II only ranks matching distributions with the *same* marginals G and H. Instead, we consider sorting in quantile space. First, label every type by its *quantile* in the distribution, so $p \equiv G\big(G^{-1}(p\,|\,\theta)\,|\,\theta\big)$ and $q \equiv H\big(H^{-1}(q\,|\,\theta)\,|\,\theta\big)$. The bivariate *copula* defines the sorting by quantiles $C(p,q) = M\big(G^{-1}(p\,|\,\theta), H^{-1}(q\,|\,\theta)\big)$. Say that *quantile sorting* is higher at M'' than M' when the associated copulas are ranked $C'' \succeq_{PQD} C'$; i.e., C'' has more mass than C' in all lower and upper orthants in (p,q) space. This order generalizes the PQD order. For if M'' and M' share the same marginals, then $C'' \succeq_{PQD} C'$ if and only if $M'' \succeq_{PQD} M'$. And since all copulas have uniform marginals by definition, we can compare two copulas in the PQD order even if the associated matching distributions do not share marginals.

By Lemma 1, greater quantile sorting reduces the average geometric distance between matched quantiles and raises the covariance across matched quantile pairs, and the coefficient in linear regression of male on female match partner quantiles.

COROLLARY 1: *Assume types shift up (down) in θ. If C'' and C' are optimal copulas, respectively for $\theta'' \succ \theta'$, then $C'' \succeq_{PQD} C'$*

(*a*) *generically with finite types, if synergy is nondecreasing (nonincreasing) in types;*

(*b*) *given G and H absolutely continuous, if synergy is increasing (decreasing) in types.*

For some insight into the proof in Appendix C.F, consider the *quantile production function* $\varphi(p,q\,|\,\theta) \equiv \phi\big(G^{-1}(p\,|\,\theta), H^{-1}(q\,|\,\theta)\big)$ with *quantile synergy*:

$$\varphi_{12}(p,q\,|\,\theta) \equiv \frac{\phi_{12}\big(G^{-1}(p\,|\,\theta), H^{-1}(q\,|\,\theta)\big)}{g\big(G^{-1}(p\,|\,\theta)\big)h\big(H^{-1}(q\,|\,\theta)\big)}. \tag{11}$$

For concreteness, assume synergy ϕ is increasing in types and that θ stochastically shifts up types. Then $\phi_{12}\big(G^{-1}(p\,|\,\theta), H^{-1}(q\,|\,\theta)\big)$ is increasing in quantiles p,q and θ. But we cannot conclude that quantile synergy is increasing in q and θ since (11) includes g and h, which need not be monotone in q or θ. Nonetheless, quantile synergy is upcrossing in types and θ. We verify in Appendix C.F that the premise of Corollary 1 implies that of Proposition 4. Figure 8 depicts this result for quadratic production.

VI. Economic Applications

A. *Diminishing Returns*

Assume that matched pairs produce an intermediate output within a firm. In this case, the overall match synergy will depend on synergies in intermediate output production, and the returns to intermediate outputs. In this section we fix the intermediate output production function and focus on the returns to intermediate outputs. As we will see, *diminishing returns reduces match synergies, and increasing returns*

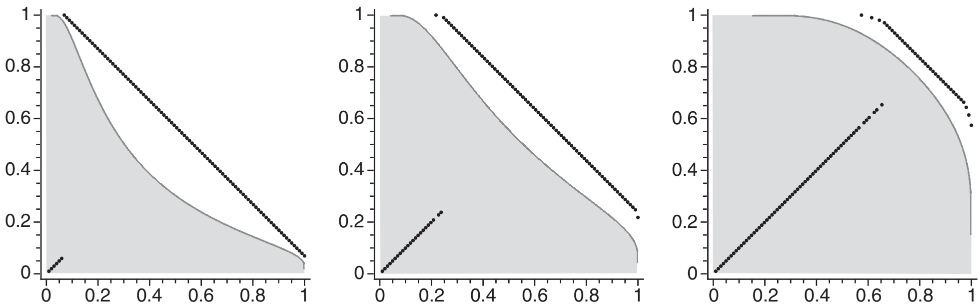

FIGURE 8. DISTRIBUTION SHIFT EXAMPLE

Notes: We plot optimally matched quantile pairs (dots) for quadratic production $xy - (xy)^2$ and exponential distributions on types $G(x\,|\,\theta) = 1 - e^{-x/\theta}$ and $H(y\,|\,\theta) = 1 - e^{-y/\theta}$, for $\theta = 1, 2/3, 1/3$ at left, middle, and right. By Corollary 1, *quantile sorting increases as θ falls* since synergy falls in types.

amplifies them. We then explore how sorting changes as the returns to intermediate outputs change.

Specifically, assume that a type x worker on a type y machine has an increasing intermediate *output* $q(x,y)$. Assume the monetary value of q is given by the increasing *revenue function* ψ. The match payoff is then $\phi(x,y\,|\,\theta) = \psi(q(x,y)\,|\,\theta)$, and synergy

$$\phi_{12} = \psi'(q\,|\,\theta)q_1 q_2\left[\frac{q_{12}}{q_1 q_2} + \frac{\psi''(q\,|\,\theta)}{\psi'(q\,|\,\theta)}\right] \tag{12}$$

rises in complementarity q_{12} and falls in the Arrow-Pratt risk aversion measure $-\psi''/\psi'$.

By Becker's Sorting Theorem, if ψ is convex and q is SPM, then perfect sorting arises, whereas if ψ is concave and q is SBM, then perfect negative sorting arises. But perhaps the most natural case is ψ concave (diminishing returns to q) and q SPM (complementarity in intermediate output production). For concreteness, consider the special case $q(x,y) = xy$. Then by (12), synergy is negative if the "relative risk aversion" $-q\psi''(q\,|\,\theta)/\psi'(q\,|\,\theta)$ exceeds one. If relative risk aversion is falling in q, then we have negative synergies at low types and positive synergies at high types, and so sorting failures occur for low types. The opposite synergy signs arise for rising risk aversion.

Figure 9 depicts the first result. Online Appendix E shows that if relative risk aversion falls in q, but rises in a parameter θ, then synergy is the product of a function that is increasing in x, y, and $t = 1 - \theta$ and a positive function that is LSPM in (x,y,t). Thus, by Proposition 5, *sorting falls as the risk aversion parameter θ rises.*

As a quick application, we compare sorting in the manufacturing and service sectors of the economy. Assume $q(x,y) = xy$ is the *effective labor* of matched workers (x,y) and $\psi(q\,|\,\kappa) = (q^\eta + \kappa^\eta)^{1/\eta}$, where κ is the exogenous capital requirement of the tasks performed by workers in the industry. When $\eta < 1$, effective labor and

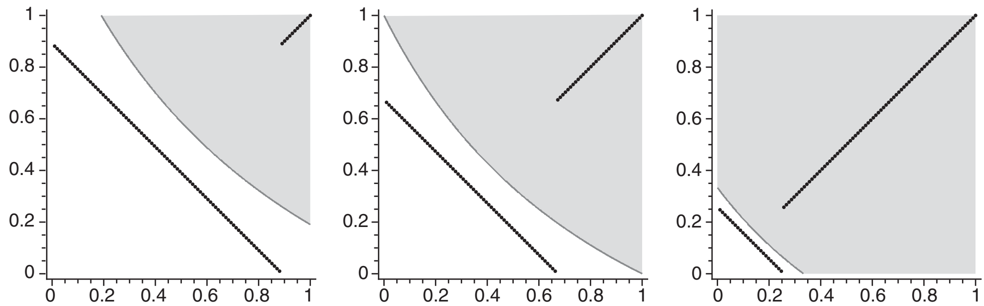

Figure 9. Increasing Sorting with Diminishing Returns

Notes: These graphs depict optimally matched pairs (dots) with $\phi(x,y) = \psi(q(x,y)\,|\,\theta)$ for $q(x,y) = xy$ and $\psi(q\,|\,\theta) = (xy-1)^{1-\theta}$. In all cases synergy is upcrossing in types, which follows from relative risk aversion $-q\psi''/\psi'$ falling in q. Sorting rises from left to right as the risk aversion parameter θ falls from $\theta = 0.58, 0.5, 0.25$. In order to ensure that ϕ increases in types, we assume types are uniform on $[1,2]$ and depict matches by quantiles.

capital are complements, and also ψ is concave. In this case, relative risk aversion $-q\psi''(q\,|\,\kappa)/\psi'(q\,|\,\kappa) = (1-\eta)\kappa^{\eta}/(\kappa^{\eta}+q^{\eta})$ falls in q and rises in κ; and so sorting falls in capital intensity κ. Hence, sorting is higher in the service than manufacturing sector.

B. *From Weakest to Strongest Link Technologies*

We now consider a complementary thought experiment: fixing the revenue function $\psi(q)$ and varying the intermediate output function $q(x,y)$. The CES technology $q(x,y) = (x^{-\rho}+y^{-\rho})^{-1/\rho}$ is a helpful tractable class for this exercise. It is SPM when $\rho \geq -1$, and otherwise SBM. Thus, by Becker's Sorting Result, the optimal sorting is PAM for $\rho \geq -1$ and NAM for $\rho \leq -1$, when ψ is linear. To avoid this knife-edge result and explore how sorting varies in the CES parameter ρ, we again assume diminishing returns to output q. To keep things simple, assume increasing quadratic payoffs $\psi(q) = \alpha q - \beta q^2$, so that $\alpha, \beta > 0$ and $\alpha > 2\beta q(1,1)$, where all types $(x,y) \in [0,1]^2$. Then output is $\phi(x,y) = \alpha q(x,y) - \beta q(x,y)^2$, and its synergy is continuous in ρ, and synergy tends to $-2\beta < 0$ as $\rho \downarrow -1$. By online Appendix E, its synergy is also upcrossing in ρ and strictly positive for ρ sufficiently large; also, there exist $\bar{\rho} > \underline{\rho} > -1$ such that production is SBM (yielding NAM) for all $\rho < \underline{\rho}$ and SPM (giving PAM) for $\rho > \bar{\rho}$. We then use Proposition 4 to prove that *sorting is increasing in ρ, for all $\rho \in [0,\bar{\rho}]$.*

For additional economic insight, notice that whenever ψ is increasing, the $\rho \to \infty$ limit yields an SPM function $\psi(\min\{x,y\})$ and $\rho \to -\infty$ yields the SBM function $\psi(\max\{x,y\})$. Intuitively, for any increasing $\psi(q)$, we get PAM for high ρ, i.e., when q is close to the "weakest link" technology, $\min\{x,y\}$. Equally shared tasks, like jointly lifting a couch, have this flavor: output is more responsive to the lower type. But when $q(x,y)$ is close to the "strongest link" technology $\max\{x,y\}$, we get NAM. Here, output is more responsive to the higher type, such as for mutually insured matched pairs. Altogether, *match synergies are higher with weak link technologies and lower with strong link technologies.*

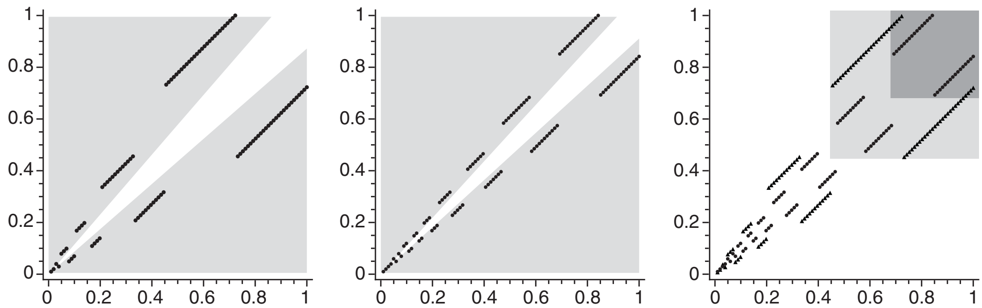

FIGURE 10. KREMER-MASKIN SYNERGIES AND MATCHING

Notes: These graphs depict optimal matchings for production (13) with $\varrho = -20$ and a uniform distribution on 100 types. In the left graph $\theta = 0.4$ and rises to $\theta = 0.45$ in the middle. Synergy is positive on the shaded region and is not one-crossing in types. *So our sorting monotonicity theory is silent here.* Indeed, the matching for $\theta = 0.45$ has more (circle) couples in the dark rectangle in the right graph, while the matching for $\theta = 0.4$ has more (triangle) couples in the light rectangle. Online Appendix E proves sorting is nowhere decreasing in θ.

Kremer and Maskin (1996) explore a famous strong link technology that arises with role assignment. Agents can be assigned either to the manager or deputy roles. Fixing $\theta \in [0,1/2)$, their output is $x^\theta y^{1-\theta}$ if x is the manager and y the deputy. As a unisex model, match output is then the maximum of two SPM functions $\max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\}$, which is neither SPM nor SBM (noting that maximization preserves SBM but not SPM).

To apply our theory, we introduce a smooth production function

$$\phi(x,y\,|\,\theta,\rho) \;=\; x^\theta y^\theta \big(x^{-\rho} + y^{-\rho}\big)^{\frac{2\theta-1}{\rho}} \tag{13}$$

that converges to the Kremer-Maskin production function as $\varrho \to -\infty$. The x,y cross partial of the smooth function $\phi(x,y\,|\,\theta,\rho)$ in (13) is $+,\,-,\,+$ as types increase (Figure 10). Thus, the essential assumption of Proposition 3 that rectangular synergy is one-crossing in types fails, nor is sorting monotone in either θ or ρ (Figure 10 illustrates the nonmonotonicity in θ).

Furthermore, synergy is not monotone in θ or ρ for the "smooth" production function (13), nor is finite synergy monotone in θ for the limit case $\phi(x,y\,|\,\theta) = \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\}$. So Proposition 1 does not imply nowhere decreasing sorting. But we show in online Appendix E that synergy (13) obeys a weaker one-crossing assumption in Theorem 4 (which generalizes Proposition 1) and that sorting cannot fall in (θ,ρ).

C. *Moral Hazard with Endogenous Contracts*

Serfes (2005) explores pairwise matching among principals and agents. He assumes project output is the sum of the agent's unobservable effort e and a mean zero Gaussian error. The risk-neutral principal's project variance y is their type; this varies in $[\underline{y},\bar{y}]$. Agents have constant absolute risk aversion utility function $1 - e^{x(w-\theta e^2)}$, given wage w, effort e, and a monetary cost of effort θe^2. Agents share the same *disutility of effort parameter* $\theta > 0$ but differ in their types—namely, the risk aversion coefficient x in $[\underline{x},\bar{x}]$. After a principal and agent match, the principal makes a take-it-or-leave-it contract offer, specifying the agent's wage as a function of realized output. Serfes (2005) derives (in his equation (2)) the equilibrium expected output of an (x,y) match:

$$\phi(x,y\,|\,\theta) = \frac{1}{2\theta(1+\theta xy)} \Rightarrow \phi_{12}(x,y\,|\,\theta) = \frac{\theta xy - 1}{2(1+\theta xy)^3}. \tag{14}$$

Serfes observes that synergy is globally negative for $\theta\bar{x}\bar{y} < 1$ and globally positive for $\theta\underline{x}\,\underline{y} > 1$. Thus, by Becker's Sorting Result, NAM obtains for $\theta < (\bar{x}\bar{y})^{-1}$ and PAM obtains for $\theta > (\underline{x}\,\underline{y})^{-1}$. This result reflects two countervailing forces for sorting. First, if all contracts were the same, then efficient insurance across principal-agent pairs favors NAM: less risk-averse agents work on higher variance projects. But the slope of the equilibrium wage contract is $(1+\theta xy)^{-1}$; and thus, the incentives to provide effort are SPM for high types. The sign of synergy (14) implies that the insurance effect dominates for low types and the incentive effect dominates for high types.[12]

Serfes (2005) is silent when $\theta\bar{x}\bar{y} > 1 \geq \theta\underline{x}\,\underline{y}$: our theory partly fills this gap. We claim that *Sorting is increasing in the disutility of effort parameter θ when types are not too far apart*, namely, when $\bar{x}\bar{y} \leq 2\underline{x}\,\underline{y}$ (‡). To see this, assume $\theta' > \theta$. If $\theta\bar{x}\bar{y} < 1$, then synergy (14) is globally negative at θ, and so NAM is uniquely optimal. If $\theta'\underline{x}\,\underline{y} > 1$,then synergy is globally positive at θ', and so PAM is uniquely optimal. In both cases, sorting is weakly higher at θ' than θ. Now assume $\theta'\underline{x}\,\underline{y} \leq 1 < \theta\bar{x}\bar{y}$. Then $\theta'\bar{x}\bar{y} \leq 2\theta'\underline{x}\,\underline{y} \leq 2$ by (‡) and $\theta'\underline{x}\,\underline{y} \leq 1$. Thus, $\theta xy < \theta'\bar{x}\bar{y} \leq 2$ for all (x,y), and so synergy in (14) is increasing in θxy—for $(t-1)/(1+t)^3$ is increasing for $t \in (0,2]$. Altogether, sorting increases in θ by Proposition 2, as in Figure 11. Since synergy increases in types when PAM is suboptimal, quantile sorting increases when types shift up (i.e., when projects become more variable or agents become more risk averse), by Corollary 1.

D. *Mentor-Protégé Learning Dynamics*

Dynamic matching with evolving types can be understood through the lens of match synergies. Let's assume a two-period model with pairwise matching in periods one and two. Let $\phi^0(x,y)$ be the increasing and SPM match output of types x and y.

We capture learning dynamics by the increasing *transition function* τ. Specifically, after producing output in period one, types x and y evolve to new types $x' = \tau(x,y)$ and $y' = \tau(y,x)$ in period two. For matching between workers within a firm, τ describes learning from coworkers. In a neighborhood sorting application, τ may reflect peer influences on children. Or in a procreation context, couple (x,y) produces offspring of type $\tau(x,y)$. In this latter case, $\tau(x,y) = \max\{x,y\}$ and $\tau(x,\ y) = \min\{x,y\}$ formalize the respective extremes of dominant and recessive

[12] Ackerberg and Botticini (2002) investigate matching between landowners (principals) and tenants (agents) in fifteenth-century Tuscany. Matched crop-tenant pairs exhibit positive covariance in crop types (project variance y) and tenant wealth (risk aversion x). But since match sorting is imperfect (not PAM), our theory provides a framework for analyzing changes in crop-tenant matching across markets.

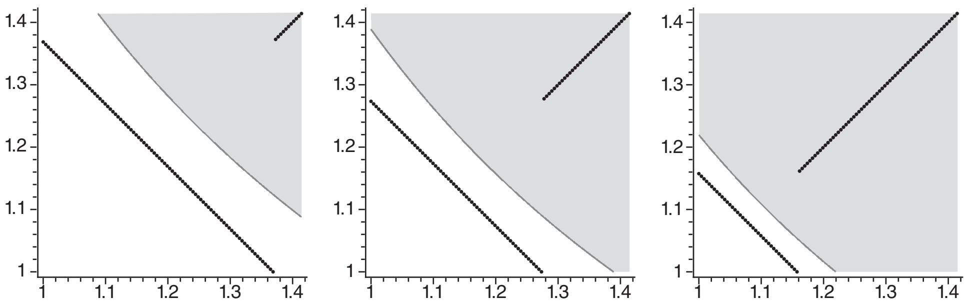

FIGURE 11. INCREASING SORTING IN THE PRINCIPAL-AGENT MODEL

Notes: These graphs depict optimal matched pairs (dots) for a uniform distribution on 100 types of principals and agents. Sorting rises from left to right as θ rises on $\{0.65, 0.72, 0.82\}$.

type transmission—namely, one or both high-achieving parents suffice for high-achieving children.

Matching must be statically optimal in period two, and thus, PAM occurs.[13] For instance, in the partnership model, the social planner has period one payoff:

$$\phi(x,y) = (1-\delta)\phi^0(x,y) + \tfrac{\delta}{2}\big[\phi^0\big(\tau(x,y),\tau(x,y)\big) + \phi^0\big(\tau(y,x),\tau(y,x)\big)\big]$$

given discount factor δ. So synergy ϕ_{12} is a $(1-\delta,\delta)$ weighted average of static synergy $\phi^0_{12} > 0$ and *dynamic synergy*—namely, if τ is twice differentiable, the first term is

$$\big[\phi^0\big(\tau(x,y),\tau(x,y)\big)\big]_{12} = \big(\phi^0_{11} + 2\phi^0_{12} + \phi^0_{22}\big)\tau_1\tau_2 + \big(\phi^0_1 + \phi^0_2\big)\tau_{12}. \tag{15}$$

Since τ is increasing, the first term in (15) is positive when $\phi^0(x,x)$ is convex but negative when $\phi^0(x,x)$ is concave. That is, convexity pushes toward positive synergy and concavity toward negative synergy, as in Section VIA.

But in this evolving type world, negative synergy may also reflect a submodular transition function τ. This arises in learning environments, where the lower type learns from the higher, as a protégé from a mentor. In particular, given the normalization $\tau(x,x) = x$, strictly SBM τ implies

$$\tau(x,y) + \tau(y,x) > \tau(x,x) + \tau(y,y) \Leftrightarrow \tau(x,y) - x > y - \tau(y,x).$$

So when unequal types match, the higher partner pulls up the lower more than the latter pulls him down—as in a workplace when skilled coworkers pass on insights.

[13] Anderson and Smith (2010) consider an infinite horizon with stochastic type transitions. In a special case of the model where types are the common knowledge chance that an agent is high (versus low) productivity, they show that synergy is negative for (x,y) close to $(0,0)$ or $(1,1)$ with sufficient patience. Thus, PAM cannot be optimal given sufficient patience.

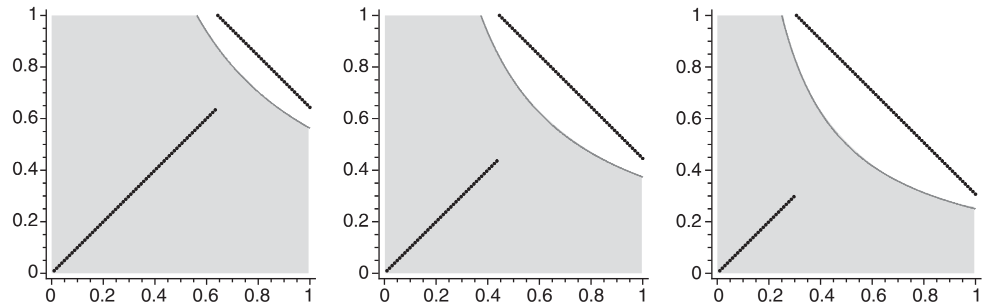

Figure 12. Increasing Sorting with Peer Learning

Notes: These graphs depict optimally matched pairs with static output $\phi^0(x,y) = \sqrt{xy}$ and transitions $\tau = x + 0.7(y-x) + 0.5(x^2 - xy)$ and a uniform distribution on 100 types. Sorting falls as the discount factor rises from $\delta = 0.4$ (left) to $\delta = 0.45$ (middle) to $\delta = 0.5$ (right).

In particular, Herkenhoff et al. (forthcoming) find negative dynamic synergy in a learning setting.[14] Our model affords comparative statics in the discount factor. Since synergy is increasing in $1-\delta$, the *time series premise* of each of our increasing sorting results is met. But stronger assumptions are required for the *cross-sectional* assumptions. The most transparent case is when static synergy and dynamic synergy (15) are both monotone in types in the same direction. Then sorting falls in δ, by Proposition 2. Figure 12 shows this comparative static in a parametric example.

VII. Conclusion

Becker's finding that complementarity (or supermodularity) yields positive sorting launched the immense literature on pairwise matching. While perfect sorting does not emerge in many economic settings, an impassable wall of mathematical complexity has stopped any general predictive matching theory. This paper develops a theory linking changes in the pairwise production function to changes in the PQD stochastic sorting order, without solving for an optimal matching.

Showing that total match output is a weighted average of synergy, we center our theory on this local complementarity notion. Our easiest result is that sorting increases when synergy increases, provided that synergy is monotone in types. We then weaken the assumptions on how synergy rises and prove more general comparative statics.

We apply our theory to several applications in the matching literature, deriving new predictions. We hope this offers a tractable foundation for future theoretical and empirical analysis of matching. A subtle and valuable direction for future work is a multidimensional extension of our theory (Lindenlaub 2017).

We assumed an equal mass of men and women, like Becker. If types are imagined as quality, this is without loss of generality: lowest men are queued out if men are

[14] They estimate a matching model with search frictions and find SPM static production but negative dynamic synergy. Synergy is positive for low types and negative for high types.

in surplus. Extending our increasing sorting results to a horizontal model of types is an open question.

We considered the planner's sorting exercise and are silent on transfers. Future research could characterize the behavior of wage changes as sorting increases.

Appendix A. Match Output Reformulation: Derivation of (5)

PROOF:

Summing $\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}m_{ij}$ by parts in j and then i yields an expression for total match output in terms of synergy:

$$\begin{aligned}
\sum_{i=1}^{n}\left(\sum_{j=1}^{n} f_{ij}m_{ij}\right) &= \sum_{i=1}^{n}\left[f_{in}\sum_{j=1}^{n} m_{ij} - \sum_{j=1}^{n-1}\left(f_{i,j+1}-f_{ij}\right)\sum_{k=1}^{j} m_{ik}\right] \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\sum_{i=1}^{n}\left(f_{i,j+1}-f_{ij}\right)\sum_{k=1}^{j} m_{ik} \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left[\left(f_{n,j+1}-f_{n,j}\right)\sum_{\ell=1}^{n}\sum_{k=1}^{j} m_{\ell k} - \sum_{i=1}^{n-1} s_{ij}\sum_{\ell=1}^{i}\sum_{k=1}^{j} m_{\ell k}\right] \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left[\left(f_{n,j+1}-f_{nj}\right)j - \sum_{i=1}^{n-1} s_{ij}M_{ij}\right]. \blacksquare
\end{aligned}$$

Appendix B. Integral Preservation of Upcrossing Properties

A. *Integral Preservation of Upcrossing Functions on Lattices*

Given a real or integer lattice $Z \subseteq \mathbb{R}^N$ and poset $(\mathcal{T},\succeq)$, the function $\sigma : Z\times\mathcal{T} \to \mathbb{R}$ is proportionately upcrossing[15] if $\forall z,z' \in Z$ and $t' \succeq t$.

$$\sigma^-(z\wedge z',t)\sigma^+(z\vee z',t') \geq \sigma^-(z,t')\sigma^+(z',t). \tag{B1}$$

THEOREM 1: *Let $\sigma(z,t)$ be proportionately upcrossing. Then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is weakly upcrossing in t and upcrossing in t if $\sigma(z,t)$ is also upcrossing in t.*

This result is stronger than needed,[16] as it applies to general lattices; we just need it for $\mathbb{R}^2$. It generalizes an information economics result by Karlin and Rubin (1956): *If $\sigma_0(z)$ is upcrossing in $z\in\mathbb{R}$, and* $\log(\sigma_1)$ *is SPM, then* $\int\sigma_0(z)\sigma_1(z,t)d\lambda(z)$ *is upcrossing*. Our result subsumes theirs when $n = 1$ and $\sigma = \sigma_0\sigma_1$ is proportional upcrossing.

[15] Proportionately upcrossing implies *weakly upcrossing*; namely, $\sigma(z,t) > 0$ implies $\sigma(z',t') \geq 0$ for all $(z',t') \succeq (z,t)$. To see this, fix $t = t'$ and suppress t. If $z' \succeq z$, inequality (B1) is an identity. If $z \succ z'$, inequality (B1) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

[16] This result is related to Theorem 2 in Quah and Strulovici (2012). They do not assume (B1). Rather, they assume σ is upcrossing in (z,θ), and a time series condition: signed ratio monotonicity. Our results are independent but overlap more closely for our smoothly LSPM condition in Appendix B.B.

PROOF:

Karlin and Rinott (1980) prove the following: *If functions* $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ *obey* $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ *for* $z \in Z \subseteq \mathbb{R}^N$, *then for all positive measures* λ,[17]

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z). \tag{B2}$$

Now, if $t' \succeq t$, then (B1) reduces to $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for the functions

$$\xi_1(z) \equiv \sigma^+(z,t), \quad \xi_2(z) \equiv \sigma^-(z,t'), \quad \xi_3(z) \equiv \sigma^+(z,t'), \quad \xi_4(z) \equiv \sigma^-(z,t).$$

Thus, by (B2),

$$\int \sigma^+(z,t')d\lambda(z) \int \sigma^-(z,t)d\lambda(z) \geq \int \sigma^+(z,t)d\lambda(z) \int \sigma^-(z,t')d\lambda(z). \tag{B3}$$

This precludes $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$ and $\int \sigma^+(z,t')d\lambda(z) < \int \sigma^-(z,t')d\lambda(z)$, simultaneously. And thus, $\Sigma(t) > 0$ implies $\Sigma(t') \geq 0$, proving weakly upcrossing.

We now argue Σ upcrossing. First, assume $\Sigma(t) > 0$. Then, $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$. By (B3), either $\int \sigma^+(z,t')d\lambda(z) > \int \sigma^-(z,t')d\lambda(z)$ or $\int \sigma^+(z,t')d\lambda(z) = \int \sigma^-(z,t')d\lambda(z) = 0$. But the latter is impossible since $\int \sigma^+(z,t')d\lambda(z) = 0$ implies $\int \sigma^+(z,t)d\lambda(z) = 0$, as $\sigma(z,t)$ is upcrossing in t—contradicting $\Sigma(t) > 0$. So $\Sigma(t') > 0$.

Next, posit $\Sigma(t) = 0$, then $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z)$. By (B3), either $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, and so $\Sigma(t') \geq 0$, or we have $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z) = 0$, whereupon $\int \sigma^-(z,t')d\lambda(z) = 0$— as $\sigma(z,t)$ is upcrossing in t, and so $\sigma^-(z,t)$ is downcrossing. Thus, $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, or $\Sigma(t') \geq 0$. ∎

B. *Proportionately Upcrossing and log-Supermodularity*

Let $\theta \in \mathbb{R}$, $z \in \mathbb{R}^N$, and abbreviate $w = (z,\theta) \in \mathbb{R}^{N+1}$. The function $\sigma : \mathbb{R}^{N+1} \mapsto \mathbb{R}$ is *smoothly log-supermodular* (*LSPM*) if all of its pairwise derivatives obey $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$.

THEOREM 2: *If* $\sigma(z,\theta)$ *is upcrossing and smoothly LSPM*, *then* σ *obeys* (B1).

PROOF:

Assume $\hat{w} \geq w$, sharing the i coordinate $w_i = \hat{w}_i$, with $\sigma(\bar{x}, w_{-i}) < 0 < \sigma(\hat{w})$ for some $\bar{x} > w_i$. Then we claim that

$$\sigma_i(x, w_{-i})\sigma(x, \hat{w}_{-i}) \geq \sigma_i(x, \hat{w}_{-i})\sigma(x, w_{-i}), \quad \forall x \in [w_i, \bar{x}]. \tag{B4}$$

[17] The proof for the integer lattice requires that λ be a counting measure. Also true: if λ does not place all mass on zeros of σ, then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is upcrossing in t.

Since σ is upcrossing, $\sigma(x, w_{-i}) < 0 < \sigma(x, \hat{w}_{-i})$ for all $x \in [w_i, \bar{x}]$. If (B4) fails, then for some $x' \in [w_i, \bar{x}]$,

$$\frac{\sigma_i(x', w_{-i})}{\sigma(x', w_{-i})} > \frac{\sigma_i(x', \hat{w}_{-i})}{\sigma(x', \hat{w}_{-i})}.$$

This contradicts smoothly LSPM, as $(\sigma_i/\sigma)_j \geq 0$ for all $\sigma \neq 0$ and $i \neq j$. So (B4) holds. Given $\sigma(x, \hat{w}_{-i}) \neq 0$, the ratio $\sigma(x, w_{-i})/\sigma(x, \hat{w}_{-i})$ is nondecreasing in x on $[w_i, \bar{x}]$, so that

$$\text{(B5)} \qquad \frac{\sigma(w)}{\sigma(\hat{w})} \leq \frac{\sigma(\bar{x}, w_{-i})}{\sigma(\bar{x}, \hat{w}_{-i})}.$$

By assumption, $\theta' \geq \theta$ (now a real). So if $(z, \theta') \leq (z \wedge z', \theta)$, we have $z \leq z'$ and $\theta' = \theta$, in which case (B1) is an identity. If not $(z, \theta') \leq (z \wedge z', \theta)$, then let $i_1 < \cdots < i_K$ be the indices with $(z, \theta')_{i_k} > (z \wedge z', \theta)_{i_k}$ for $k = 1, \ldots, K$. Let's change $w^0 \equiv (z \wedge z', \theta)$ into $w^K \equiv (z, \theta')$ in K steps, $w^0, \ldots, w^K$, one coordinate at a time, and likewise $\hat{w}^0 \equiv (z', \theta)$ into $\hat{w}^K \equiv (z \vee z', \theta')$, changing coordinates in the same order. Notice that $w_{i_k}^{k-1} = \hat{w}_{i_k}^{k-1} = (z', \theta)_{i_k} < (z, \theta')_{i_k}$ and $\hat{w}^k \geq w^k$ for all k.

Now, inequality (B1) holds if its RHS vanishes. Assume instead the RHS of (B1) is positive for some $\theta' \geq \theta$, so that $\sigma(z, \theta') < 0 < \sigma(z', \theta)$; and so, replacing $\hat{w}^0 = (z', \theta)$ and $w^K = (z, \theta')$, we get $\sigma(w^K) < 0 < \sigma(\hat{w}^0)$. But then since the sequences $\{w^k\}$ and $\{\hat{w}^k\}$ are increasing and σ is upcrossing, we have $\sigma(w^k) < 0 < \sigma(\hat{w}^{k-1})$ for all k. Altogether, we may repeatedly apply inequality (B5) to get

$$\frac{\sigma(z \wedge z', \theta)}{\sigma(z', \theta)} \equiv \frac{\sigma(w^0)}{\sigma(\hat{w}^0)} \leq \frac{\sigma(w^k)}{\sigma(\hat{w}^k)} \leq \cdots \leq \frac{\sigma(w^K)}{\sigma(\hat{w}^K)} \equiv \frac{\sigma(z, \theta')}{\sigma(z \vee z', \theta')}.$$

So given $\sigma(z \wedge z', \theta), \sigma(z, \theta') < 0 < \sigma(z', \theta), \sigma(z \vee z', \theta')$, inequality (B1) follows from

$$\frac{\sigma^-(z \wedge z', \theta)}{\sigma^+(z', \theta)} \geq \frac{\sigma^-(z, \theta')}{\sigma^+(z \vee z', \theta')}. \blacksquare$$

Appendix C. Omitted Proofs

A. *Proof of Lemma 1*

Part (a): By inequality (6), it suffices that $|u(x) - v(y)|^\gamma$ is SBM for all $\gamma \geq 1$. Since $-\psi(u - v)$ is SPM for all convex ψ, by Lemma 2.6.2-(b) in Topkis (1998), we have $-|u - v|^\gamma$ SPM for all $\gamma \geq 1$. So $|u(x) - v(y)|^\gamma$ is SBM for all increasing u and v.

Part (**b**)**:** Since the marginal distributions on X and Y are the same for all $M \in \mathcal{M}(G,H)$, and $u(x)v(y)$ is supermodular for all increasing u and v, the covariance $E_M[XY] - E[X]E[Y]$ between matched types increases in the PQD order, by (6).

Part (**c**)**:** The coefficient $c_1 = \text{cov}\big(u(X)v(Y)\big)/\text{var}\big(v(X)\big)$ in the univariate match partner regression $v(y) = c_0 + c_1 u(x)$ increases in the PQD order, by part (b). ∎

B. *Proof of Proposition 3: Increasing Sorting for Finite Types*

LEMMA 2: *An optimal matching is generically unique and pure for finite types.*

PROOF:

The optimal matching is generically unique, by Koopmans and Beckmann (1957). A nonpure matching M is a mixture $M = \sum_{\ell=1}^{L} \lambda_\ell M_k$ over $L \leq n+1$ pure matchings $M_1, \ldots, M_n$, with $\lambda_\ell > 0$ and $\sum_\ell \lambda_\ell = 1$.[18] As the objective function (3) is linear, if the nonpure matching M is optimal, so is each pure matching M_ℓ, contradicting uniqueness. ∎

For a big picture, we show that matching models in some domain $\hat{\mathcal{D}}_n$ obey our sorting conclusion for all n. Our induction argues the stronger claim that it holds on a larger recursively convenient domain $\mathcal{D}_n^* \supset \hat{\mathcal{D}}_n$. Our proof building blocks are

(a) Consider the generic case with unique optimal pure matchings μ, described by men partners $(\mu_1, \ldots, \mu_n)$ of women or women partners $\omega = (\omega_1, \ldots, \omega_n)$ of men.

(b) To emphasize the dependence on the number of types n, write rectangular synergy as $S^n(r\,|\,\theta)$ and the *summed rectangular synergy* as $\mathbb{S}^n(K\,|\,\theta) \equiv \sum_k S^n(r_k\,|\,\theta)$ for any finite set of nonoverlapping rectangles $K \equiv \{r_k\}$.

(c) We consider the *summed rectangular synergy dyad* $\big(\mathbb{S}^n(K\,|\,\theta'), \mathbb{S}^n(K\,|\,\theta'')\big)$ for generic $\theta'' \succeq \theta'$. Let domain $\mathcal{D}_n$ be the space of summed rectangular synergy dyads $\big(\mathbb{S}^n(K\,|\,\theta'), \mathbb{S}^n(K\,|\,\theta'')\big)$ that are each upcrossing in K on rectangles $\mathcal{R}$ and upcrossing in θ on $\{\theta', \theta''\}$ for any $K \in \mathcal{R}$. The domain $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n$ further insists that they be upcrossing in θ for finite sets of nonoverlapping rectangles K. Proposition 3 assumes that summed rectangular synergy dyads are in $\hat{\mathcal{D}}_n$ for all n.

(d) *Removing couple* (i,j) from an n-type market *induces rectangular synergy* S_{ij}^{n-1} among the remaining $n-1$ types, satisfying the natural formula

$$\text{(C1)} \qquad S_{ij}^{n-1}(r\,|\,\theta) \equiv S^n\big(r + \mathcal{I}_{ij}(r)\,|\,\theta\big),$$

$$\text{for } \mathcal{I}_{ij}(r) \equiv \big(\mathbf{1}\{r_1 \geq i\}, \mathbf{1}\{r_2 \geq j\}, \mathbf{1}\{r_3 \geq i\}, \mathbf{1}\{r_4 \geq j\}\big),$$

[18] This follows from Carathéodory's Theorem. It says that a nonempty convex compact subset $\mathcal{X} \subset \mathbb{R}^n$ is a weighted average of extreme points of $\mathcal{X}$. The extreme points here are the pure matchings.

where $\mathcal{I}_{ij}(r)$ increments by one the index of the women $i' \geq i$ and men $j' \geq j$, *where the type indices refer to the original model whenever removing types* henceforth.

(e) To avoid ambiguity when changing the number n of types, we denote by (i_n, j_n) the i-th highest woman and the j-th highest man. Now, consider the sequence of models with $\kappa = n+k, n+k-1, \ldots, n$ types induced by removing couple (i'_κ, j'_κ) at θ' and (i''_κ, j''_κ) at θ'' from the κ type model. We say the sequence of couples has *higher partners at θ' than θ''* if $(i'_\kappa, j'_\kappa) \geq (i''_\kappa, j''_\kappa)$ and $i'_\kappa = i''_\kappa$ or $j'_\kappa = j''_\kappa$.

(f) Domain $\mathcal{D}^*_n$ is the set of summed rectangular synergy dyads $(\mathbb{S}^n(K|\theta'), \mathbb{S}^n(K|\theta''))$ induced by sequentially removing k optimally matched couples with higher partners at θ' than θ'' from dyads $(\mathbb{S}^{n+k}(K|\theta'), \mathbb{S}^{n+k}(K|\theta''))$ $\in \hat{\mathcal{D}}_{n+k}$, for some $k \in \{0, 1, \ldots\}$.

Key Properties of Our Domains and Pure Matchings.—

Fact 1: *Fix a summed rectangular synergy dyad in $\mathcal{D}^*_{n+1}$. Removing couple (i', j') at θ' and (i'', j'') at θ'' induces such a dyad in $\mathcal{D}^*_n$ if $(i', j') \geq (i'', j'')$ and $i' = i''$ or $j' = j''$.*

Fact 2: *Given a summed rectangular synergy dyad in $\mathcal{D}_{n+1}$, removing couple (i', j') at θ' and (i'', j'') at θ'' induces a summed rectangular synergy dyad in $\mathcal{D}_n$ if $\langle i' = i''$ and $j' \geq j'' \rangle$ or $\langle j' = j''$ and $i' \geq i'' \rangle$.*

PROOF:

We prove this for $i' = i''$ and $j' \geq j''$. For any θ, rectangular synergy $S^n_{ij}(r|\theta)$ is upcrossing in r, needing fewer inequalities. To see that summed rectangular synergy is upcrossing in θ on rectangular sets in $\mathbb{Z}^2_{n-1}$, assume $S^n_{ij'}(r|\theta') \geq (>) \ 0$ for some r. Then

$$\begin{aligned} S^{n+1}(r + \mathcal{I}_{ij'}(r)|\theta') \geq (>) \ 0 &\Rightarrow S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta') \geq (>) \ 0 \\ &\Rightarrow S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta'') \geq (>) \ 0 \\ &\Rightarrow S^n_{ij'}(r|\theta'') \geq (>) \ 0, \end{aligned}$$

respectively, as (i) $S^{n+1}(r|\theta)$ is upcrossing for rectangles r, nonincreasing $\mathcal{I}_{ij}$ in j, and $j'' \leq j'$, and (ii) $S^{n+1}(r|\theta)$ is upcrossing in θ for rectangles r, and (iii) by (C1). ∎

Fact 3: *The domains are nested: $\hat{\mathcal{D}}_n \subseteq \mathcal{D}^*_n \subseteq \mathcal{D}_n$.*

PROOF:

Trivially, $\hat{\mathcal{D}}_n \subseteq \mathcal{D}^*_n$ since we may set $k = 0$ in the definition of $\mathcal{D}^*_n$.

To get $\mathcal{D}_n^* \subseteq \mathcal{D}_n$, pick $\left(\mathbb{S}^n(K|\theta'),\mathbb{S}^n(K|\theta'')\right) \in \mathcal{D}_n^*$. This dyad is induced by removing k optimally matched couples with higher partners at θ' than θ'' from a dyad $\left(\mathbb{S}^{n+k}(K|\theta'),\mathbb{S}^{n+k}(K|\theta'')\right) \in \hat{\mathcal{D}}_{n+k} \subseteq \mathcal{D}_{n+k}$, where $k \geq 0$. For $\ell = 1,\ldots,k$, induce dyads $\left(\mathbb{S}^{n+k-\ell}(K|\theta'),\mathbb{S}^{n+k-\ell}(K|\theta'')\right)$, sequentially removing optimally matched couples. So $\left(\mathbb{S}^{n+k-\ell}(K|\theta'),\mathbb{S}^{n+k-\ell}(K|\theta'')\right) \in \mathcal{D}_{n+k-\ell}$ for $\ell = 1,\ldots,k$, as removed couples are ordered, as Fact 2 needs. So $\left(\mathbb{S}^n(K|\theta'),\mathbb{S}^n(K|\theta'')\right) \in \mathcal{D}_n$. ∎

Fact 4: *If $M \neq \hat{M}$ are pure n-type matchings, $\hat{\mu}_i > \mu_i$ at some i and $\hat{\omega}_j > \omega_j$ at some j.*

PROOF:

Since $M \neq \hat{M}$, there is a highest-type man j matched with woman $\hat{\omega}_j > \omega_j$. Logically then, woman $i = \hat{\omega}_j$ is matched to a lower man under M; i.e., $j = \hat{\mu}_i > \mu_i$. ∎

Adding a couple (i_0,j_0) to a matching μ creates a new matching $\hat{\mu}$ with indices of women $i \geq i_0$ and men $j \geq j_0$ renamed $i+1$ and $j+1$, respectively. Equivalently, this means inserting a row i and column j into the matching matrix m—with all 0s except 1 at position (i,j)—and shifting later rows and columns up one.

Fact 5: *Adding respective couples $(1,\hat{m}) \leq (1,m)$, or $(\hat{w},1) \leq (w,1)$, to the n-type matchings $\hat{\mu} \succeq_{PQD} \mu$ preserves the PQD order for the resulting $n+1$ type matchings.*

PROOF:

We just consider adding couples $(1,\hat{m}) \leq (1,m)$, as the analysis for $(\hat{w},1) \leq (w,1)$ is similar. For pure matchings μ, let $C^{\mu}(i_0,j_0)$ count matches by women $i \leq i_0$ with men $j \leq j_0$, and so call $C^{\mu}(0,j) = C^{\mu}(i,0) = 0$. So $\hat{\mu} \succeq_{PQD} \mu$ if and only if $C^{\hat{\mu}} \geq C^{\mu}$.

By adding a couple $(1,m)$, the new count is

$$\mathcal{C}^{\mu}_m(i,j) \equiv C^{\mu}\left(i-1,j-\mathbf{1}\{j \geq m\}\right)+\mathbf{1}\{j \geq m\},$$

$$\text{for all} \quad i,j \in \{1,2,\ldots,n+1\}.$$

To prove the fact, we must show that if $\hat{\mu} \succeq_{PQD} \mu$, then $\mathcal{C}^{\hat{\mu}}_{\hat{m}} \geq \mathcal{C}^{\mu}_m$ for all $\hat{m} \leq m$.

By assumption, $\hat{\mu} \succeq_{PQD} \mu$, and thus, $C^{\hat{\mu}} \geq C^{\mu}$. So since $\hat{m} \leq m$,

$$\mathcal{C}^{\hat{\mu}}_{\hat{m}}(i,j) - \mathcal{C}^{\mu}_m(i,j)$$

$$= \begin{cases} C^{\hat{\mu}}(i-1,j) - C^{\mu}(i-1,j) \geq 0, & \text{if } j < \hat{m}; \\ C^{\hat{\mu}}(i-1,j-1)+1-C^{\mu}(i-1,j) \geq 0, & \text{if } \hat{m} \leq j < m; \\ C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) \geq 0, & \text{if } j \geq m. \end{cases}$$

To understand the middle line, note that this match count can be written as

$$C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) - \left[C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) - 1\right].$$

As $C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) \leq 1$, this is at least $C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) \geq 0$. ∎

The Induction Proof: Detailed Steps: Let M_n' and M_n'' be uniquely optimal n-type matchings at θ' and θ''. Proposition 3 assumes summed rectangular synergy dyads in $\hat{\mathcal{D}}_n$. Until Step 8, we work on the larger domain $\mathcal{D}_n^*$.

PREMISE $\mathcal{P}_n$: *Summed rectangular synergy dyad is in* $\mathcal{D}_n^* \Rightarrow M_n'' \succeq_{PQD} M_n'$.

Step 1: *Base Case* $\mathcal{P}_2$: *Summed rectangular synergy dyad is in* $\mathcal{D}_2^* \Rightarrow M_2'' \succeq_{PQD} M_2'$.

PROOF:

If not, then NAM is uniquely optimal at θ'' and PAM at θ'. Since $\mathcal{D}_2^* \subseteq \mathcal{D}_2$ by Fact 3, rectangular synergy is upcrossing in θ. This precludes negative rectangular synergy at θ'' (NAM) and positive rectangular synergy at θ' (PAM). ∎

- A *pair* refers to two *couples*, such as (i_1,j_1) and (i_2,j_2).
- A pair is a *PAM pair* if $(i_1,j_1) < (i_2,j_2)$, and a *NAM pair* if $i_1 < i_2$ and $j_1 > j_2$.

Step 2: *If the summed rectangular synergy dyad is in* $\mathcal{D}_{n+1}^*$, *then neither* M_{n+1}' *nor* M_{n+1}'' *includes a subset of types that match according to NAM1.*

PROOF:

We prove the stronger conclusion that neither M_{n+1}' nor M_{n+1}'' includes a matched NAM pair above a matched PAM pair. Indeed, by Fact 3, $\mathcal{D}_{n+1}^* \subseteq \mathcal{D}_{n+1}$. So $S^{n+1}(r\,|\,\theta)$ is upcrossing in rectangles r for θ' and θ''. Also, PAM (NAM) is optimal for a pair if and only if $S^{n+1}(r\,|\,\theta) \geq (\leq)\ 0$ on rectangle r. As the optimal matching is unique, $S^{n+1}(r\,|\,\theta) \neq 0$ for all optimally matched pairs. ∎

Steps 3–8 impose premises $\mathcal{P}_2, \ldots, \mathcal{P}_n$. We then supposed by contradiction that $\mathcal{P}_{n+1}$ is not satisfied. Equivalently, we suppose by contradiction,

(‡‡).—*In a model with summed rectangular synergy dyads in* $\mathcal{D}_{n+1}^*$, *the generically uniquely optimal matchings at* $\theta'' \succ \theta'$ *are not ranked* $\mu'' \succeq_{PQD} \mu'$ $(\omega'' \succeq_{PQD} \omega')$.[19]

Our cross-sectional assumption rules out NAM1 for any three-type subset of agents. Steps 3–7 show this restriction along with the inductive hypothesis, and (‡‡) implies that the optimal matching for θ'' must be NAM for some subset of types $\{1,2,\ldots,m\}$ and a multitype generalization of NAM3 under θ' for this same

[19] We cannot apply Theorem 4 to rule out $\mu' \succeq_{PQD} \mu''$ since the time series premise of Theorem 4 is stronger than the time series assumption in Proposition 3.

subset of types that we call NAM*; namely, (m, m) matched and the remaining types $\{1, 2, \ldots, m-1\}$ matched according to NAM. Step 8 then applies the cross-sectional and time series properties of the space $\mathcal{D}^*_{n+1}$ to rule out such NAM to NAM* transitions as θ rises.

Step 3: *At states θ' and θ'', the matchings obey $\mu''_1 = \mu'_1 + 1 \geq 2$ and $\omega''_1 = \omega'_1 + 1 \geq 2$.*

We establish the first relationship. Symmetric steps would prove the second.

PROOF OF $\mu''_1 > \mu'_1$:

If not, then $\mu''_1 \leq \mu'_1$. In this case, remove couple $(1, \mu'_1)$ at θ' and couple $(1, \mu''_1)$ at θ''. The remaining matching is PQD higher at θ'', by Induction Premise $\mathcal{P}_n$ and Fact 1. By Fact 5, if we add back the optimally matched pairs $(1, \mu'_1)$ and $(1, \mu''_1)$, then the PQD ranking still holds with $n+1$ types, given $\mu''_1 \leq \mu'_1$, namely $\mu'' \succeq_{PQD} \mu'$. This contradiction to $(\ddagger\ddagger)$ proves that $\mu''_1 > \mu'_1$. ∎

PROOF OF $\mu''_1 < \mu'_1 + 2$:

If not, then $\mu''_1 \geq \mu'_1 + 2$. By Fact 4, choose a woman $i > 1$ with $\mu''_i < \mu'_i$. Remove couples (i, μ'_i) at θ' and (i, μ''_i) at θ''. Since $\mu''_i < \mu'_i$, the resulting matching is PQD higher at θ'' than θ', by Fact 1 and Premise $\mathcal{P}_n$. In the resulting model, woman 1 is not matched to a higher man at θ'' than θ'. This is impossible if $\mu''_1 \geq \mu'_1 + 2$, as $\mu''_1 - \mu'_1$ falls by at most 1 when removing man μ_i at θ' and μ''_i at θ''. ∎

Step 4: *The couple (ω''_1, μ''_1) is matched at θ', namely, $\mu'_{\omega''_1} = \mu''_1$ and $\omega'_{\mu''_1} = \omega''_1$.*

In words: the man matched to the lowest woman under θ'' and the woman matched to the lowest man under θ'' must match together under θ'.

PROOF OF $\mu'_{\omega''_1} \geq \mu''_1$ AND $\omega'_{\mu''_1} \geq \omega''_1$:

We prove the first inequality. If not, then $\mu'_{\omega''_1} < \mu''_1$. As man $\mu'_1 = \mu''_1 - 1$ is matched at θ' by Step 3, $\mu'_{\omega''_1} < \mu''_1 - 1 = \mu'_1$. Removing couple $(\omega''_1, \mu'_{\omega''_1})$ at θ' and $(\omega''_1, 1)$ at θ'' induces an n-type matching that is PQD higher at θ'' by $\mathcal{P}_n$ and Fact 1. Since man $\mu'_{\omega''_1}$ removed at θ' and man 1 removed at θ'' are below $\mu'_1 = \mu''_1 - 1$, the match count at $(1, \mu'_1 - 1)$ is unchanged at θ'' and θ'. By Step 3, this count is higher at θ' than θ'', contradicting the n-type matching PQD higher at θ''. ∎

PROOF OF $\mu'_{\omega''_1} = \mu''_1$ AND $\omega'_{\mu''_1} = \omega''_1$:

Just one strict inequality is impossible, as it overmatches some type: $\omega'_{\mu''_1} > \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$ or $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} > \mu''_1$. Next, assume two strict inequalities. As $\mu'_{\omega''_1} > \mu''_1$, the θ' matching includes the PAM pair $(1, \mu'_1) < (\omega''_1, \mu'_{\omega''_1})$—by Step 3—and the higher NAM pair $(\omega''_1, \mu'_{\omega''_1})$ and $(\omega'_{\mu''_1}, \mu''_1)$. NAM pairs above PAM pairs violate Step 2 (left panel of Figure A1). ∎

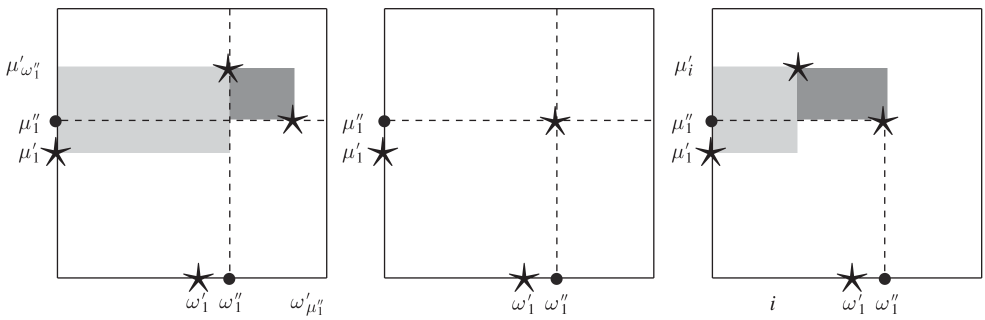

Figure A1. Steps 3–5 in the Induction Proof

Notes: In the counterfactual logic in Steps 3–5, stars and circles denote respective proposed matched pairs at θ' and θ'', respectively. Step 3 establishes that the index of the partner for the lowest man (woman) under θ'' must be exactly one higher than the index for the lowest man (woman) under θ'. The left panel depicts the NAM pair (dark gray) above the PAM pair (light gray) in Step 4. The middle panel depicts the conclusion of Step 4: man μ''_1 and woman ω''_1 must match under θ'. The right panel depicts the NAM pair above the PAM pair in Step 5-(a).

The middle panel of Figure A1 depicts the takeout of Steps 3–4. We iteratively use this matching pattern to show how $(\ddagger\ddagger)$ greatly restricts the matching at θ' and θ''.

Step 5: $\mu'_1 \geq \mu'_i = \mu''_i - 1$ *for* $i = 1,\ldots,\omega'_1$ *and* $\omega'_1 \geq \omega'_j = \omega''_j - 1$ *for* $j = 1,\ldots,\mu'_1$.

PROOF:

We proved this for $i = 1$ and $j = 1$ and now prove the claimed ordering $\mu'_1 \geq \mu'_i = \mu''_i - 1$ for $i = 2,\ldots,\omega'_1$. By symmetry, $\omega'_1 \geq \omega'_j = \omega''_j - 1$ for $j = 2,\ldots,\omega'_1$.

Part (a): $\mu'_i < \mu'_1$ *for* $i = 2,\ldots,\omega'_1$. If not, then $\mu'_i \geq \mu'_1$ for some $2 \leq i \leq \omega'_1$. And since $\mu'_i = \mu'_1$ entails overmatching, we have $\mu'_i > \mu'_1$ for $i = 2,\ldots,\omega'_1$. Thus, μ' involves a PAM pair $(1,\mu'_1) < (i,\mu'_i)$. We claim that (i,μ'_i) and (ω''_1,μ''_1) constitutes a higher NAM pair, violating Step 2. Indeed, $i \leq \omega'_1 < \omega''_1$ (by the premise above and Step 3, respectively). Also, $\mu'_i > \mu''_1$ since we have assumed $\mu'_i > \mu'_1$ and deduced $\mu'_1 = \mu''_1 - 1$ in Step 3 and, in Step 4, that μ''_1 is matched to ω''_1 at θ', and we just showed $\omega''_1 > i$. (See the right panel of Figure A1.) ∎

Part (b): $\mu'_i < \mu''_i$ *for* $i = 2,\ldots,\omega'_1$. If not, then $\mu'_i \geq \mu''_i$ for some $2 \leq i \leq \omega'_1$. Since $\mu'_i \geq \mu''_i$, if we remove couple (i,μ'_i) at θ' and couple (i,μ''_i) at θ'', then the resulting matching is PQD higher at θ'', by Fact 1 and $\mathcal{P}_n$. In the resulting matching, woman 1's partner is thus not higher at θ'' than θ'. But $\mu''_1 = \mu'_1 + 1$ by Step 3, and $\mu'_1 > \mu'_i \geq \mu''_i$, by part (a) and the premise of (b). Both removed men μ'_i and μ''_i are then strictly below μ'_1. So woman 1's partner is still 1 higher at θ'' than θ'. Contradiction. ∎

Part (c): $\mu_i' \geq \mu_i'' - 1$ *for* $i = 2, \ldots, \omega_1'$. If not, then $\mu_{i^*}' < \mu_{i^*}'' - 1$ for some $2 \leq i^* \leq \omega_1'$. Remove couple (ω_1'', μ_1'') at θ' (matched, by Step 4) and the couple $(\omega_1'', 1)$ at θ''. By Fact 1 and Assumption $\mathcal{P}_n$, the resulting matching is PQD higher at θ''.

But since $\omega_1'' > \omega_1'$ by Step 3, all women $i = 1, \ldots, \omega_1'$ remain. Each has a weakly lower partner at θ' than θ'' since we started with $\mu_i' < \mu_i''$ for $i = 1, \ldots, \omega_1'$ by Step 3 for $i = 1$, and part (b) for $i > 1$. Also, woman $i^* \leq \omega_1'$ has a strictly lower partner, as $\mu_{i^*}' < \mu_{i^*}'' - 1$. The resulting matching cannot be PQD higher at θ''. Contradiction. ∎

Step 6: *The matching* μ'' *is NAM among men and women at most* $\omega_1'' = \mu_1'' \geq 2$.

PROOF OF $\omega_1'' = \mu_1''$:

By Steps 3 and 5, we get $\mu_1'' = \mu_1' + 1 \geq \mu_i''$ for $i = 1, \ldots, w_1' = \omega_1'' - 1$ and $\mu_1'' \geq 2 > 1 = \mu_{\omega_1''}''$. So in matching μ'', women $i \leq \omega_1''$ match with men $j \leq \mu_1''$. Hence, $\mu_1'' \geq \omega_1''$. Ditto, by Steps 3 and 5, $\omega_1'' \geq \omega_j''$ for $j = 1, \ldots, \mu_1''$, and in matching ω'', men $j \leq \mu_1''$ match with women $i \leq \omega_1''$. Hence, $\mu_1'' \leq \omega_1''$. Thus, $\mu_1'' = \omega_1'' \geq 2$. ∎

PROOF OF $\mu_i'' = \mu_1'' - i + 1$ FOR $1, \ldots, \omega_1''$:

This is an identity at $i = 1$ and true at $i = \omega_1''$, by $\omega_1'' = \mu_1''$ (just proven) and $\mu_{\omega_1''}'' = 1$. So, henceforth, assume $i \in \{2, \ldots, \omega_1'' - 1\}$. We claim that for all such i, $\mu_1' \geq \mu_i''$. Indeed, by Steps 3 and 5, $\mu_1'' = \mu_1' + 1 \geq \mu_i''$; and since we do not overmatch, $\mu_1'' \neq \mu_i''$ for $i \neq 1$. Since $\mu_1' \geq \mu_i''$, Step 5 yields equality $\omega_j' = \omega_j'' - 1$ at $j = \mu_i''$, and so $\omega_{\mu_i''}' = \omega_{\mu_i''}'' - 1 = i - 1$. But then since $\omega_{\mu_{i-1}'}' = i - 1$ and each woman has a unique partner, $\omega_{\mu_i''}' = i - 1$ implies $\mu_i'' = \mu_{i-1}'$. As $\mu_{i-1}' = \mu_{i-1}'' - 1$ by Step 5 and $i \leq \omega_1'' - 1 = \omega_1'$ (by our premise and Step 3), we have $\mu_i'' = \mu_{i-1}'' - 1$. ∎

An n-type pure matching μ is NAM* if $\mu_n = n$ and $\mu_i = n - i$ for $i = 1, \ldots, n-1$, i.e., NAM among types $1, \ldots, n-1$, so that NAM* = NAM3 when $n = 3$.

Step 7: *The matching* μ' *is NAM* among men and women at most* $\omega_1'' = \mu_1'' \geq 2$.

PROOF:

Steps 3, 5, and 6 imply $\mu_i' = \mu_i'' - 1 = \mu_1'' - i$ for $i = 1, \ldots, \omega_1' = \omega_1'' - 1$. Couple (ω_1'', μ_1'') matches under μ', by Step 4. So μ' is NAM* for types $1, \ldots, \mu_1'' = \omega_1''$. ∎

By Steps 6–7, μ'' is NAM and μ' is NAM* on types $1, \ldots, \omega_1'' = \mu_1'' \equiv k \geq 2$. Since NAM* $\succ_{PQD}$ NAM, if $k < n + 1$, then Premise $\mathcal{P}_k$ fails. Step 8 finishes the proof by showing that NAM at θ'' and NAM* at θ' is also impossible for $k = n + 1$ types.

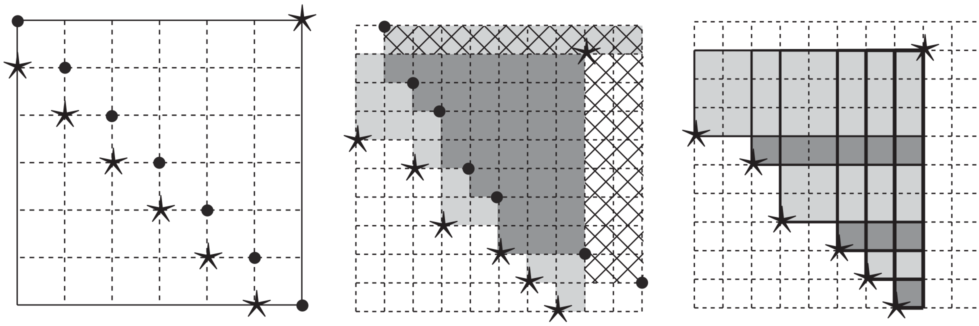

Figure A2. Step 8 of Induction Proof

Notes: Left panel: NAM for θ'' (circles) and NAM* for θ' (stars) with $n+1$ types. Adding $k-1$ couples weakly higher at θ' than θ'' produces the matches in the middle panel. Let K^G, K^L, K^T, K^R be the gray, light gray, top crosshatched, and right crosshatched regions. By (C2), the NAM* minus NAM difference is $\mathbb{S}^{n+k}(K^G \cup K^L \mid \theta') > 0$, as NAM* is optimal for θ'. But $\mathbb{S}^{n+k}(K^L \mid \theta') < 0$, as K^L is the union of rectangles, each below a NAM pair for θ''. So $\mathbb{S}^{n+k}(K^G \mid \theta') > 0$. By (C2), the NAM* minus NAM difference is $\mathbb{S}^{n+k}(K^G \cup K^R \cup K^T \mid \theta'') < 0$, negative by NAM optimal for θ''. Finally, $\mathbb{S}^{n+k}(K^T \mid \theta'), \mathbb{S}^{n+k}(K^R \mid \theta') > 0$, as each crosshatched region lies above a PAM pair for θ'. So $\mathbb{S}^{n+k}(K^G \mid \theta'') < 0$. But as $\mathbb{S}^{n+k}(K^G \mid \theta') > 0$, this contradicts summed rectangular synergy upcrossing in θ. Right panel: Illustration for Step 8(c).

NAM for men $\{i_1, \ldots, i_\ell\}$ and women $\{j_1, \ldots, i_\ell\}$ is $\{(i_1, j_\ell), (i_2, j_{\ell-1}), \ldots, (i_\ell, j_1)\}$. Rematching to NAM*, $\{(i_1, j_{\ell-1}), (i_2, j_{\ell-2}), \ldots, (i_\ell, j_\ell)\}$ changes payoffs by

$$\sum_{u=1}^{\ell-1} \left(f_{i_u, j_{\ell-u}} - f_{i_u, j_{\ell+1-u}}\right) + f_{i_\ell, j_\ell} - f_{i_\ell, 1} = \sum_{u=1}^{\ell-1} \left[\left(f_{i_\ell, j_{\ell+1-u}} - f_{i_\ell, j_{\ell-u}}\right) - \left(f_{i_u, j_{\ell+1-u}} - f_{i_u, j_{\ell-u}}\right)\right].$$

So *the payoff of NAM* less that of NAM on any subset of* ℓ *types equals* (suppressing the superscript on S)

$$\text{(C2)} \qquad \sum_{u=1}^{\ell-1} S\left(i_u, j_{\ell-u}, i_\ell, j_{\ell+1-u}\right).$$

Step 8: *NAM at* $\theta'' \Rightarrow \sim$*NAM* at* θ' *for summed rectangular synergy dyads in* $\mathcal{D}^*_{n+1}$.

Part (a): Contradiction Assumption. For $n+1$ types, posit NAM* and NAM uniquely optimal at θ' and θ'' (Figure A2, left panel). Induce summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$ by removing $k-1 \geq 0$ optimally matched couples with higher partners at θ' than θ'' (our earlier building block (f)) from a summed rectangular synergy dyad $\left(\mathbb{S}^{n+k}(K \mid \theta'), \mathbb{S}^{n+k}(K \mid \theta'')\right) \in \hat{\mathcal{D}}_{n+k}$. The θ' matching here is NAM* for men $\mathbf{i}' = (i'_1, \ldots, i'_{n+1})$ and women $\mathbf{j}' = (j'_1, \ldots, j'_{n+1})$, while the θ'' matching with these $n+k$ types is NAM for men $\mathbf{i}'' = (i''_1, \ldots, i''_{n+1})$ and women $\mathbf{j}'' = (j''_1, \ldots, j''_{n+1})$, with $(\mathbf{i}', \mathbf{j}') \leq (\mathbf{i}'', \mathbf{j}'')$ (Figure A2, middle panel).

Part (b): Couple sets U', U'' with $\mathbb{S}^{n+k}(U''|\theta'') < 0 < \mathbb{S}^{n+k}(U'|\theta')$. For rectangles $r'_u \equiv (i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u})$ and $r''_u \equiv (i''_u, j''_{n+1-u}, i''_{n+1}, j''_{n+2-u})$, define "upper sets":

- $U' \equiv \cup_{u=1}^{n} r'_u$, the union of the gray and light gray rectangles in the middle panel of Figure A2.
- $U'' \equiv \cup_{u=1}^{n} r''_u$, the union of the gray and the two crosshatched regions.

As NAM* is uniquely optimal for the subsets of men $\mathbf{i}'$ and women $\mathbf{j}'$ at θ', it payoff-dominates NAM. Given linearity of summed rectangular synergy at $\ell = n+1$ in (C2),

$$\mathbb{S}^{n+k}(U'|\theta') = \sum_{u=1}^{n+1} S^{n+k}(r'_u|\theta') = \sum_{u=1}^{n+1} S^{n+k}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u}|\theta') > 0.$$

Likewise, NAM uniquely optimal for subsets $\mathbf{i}''$ and $\mathbf{j}''$ at θ'' implies $\mathbb{S}^{n+k}(U''|\theta'') < 0$.

Part (c): $\mathbb{S}^{n+k}(K^G|\theta') > 0$ for $K^G \equiv U' \cap U''$. First, $U' = \cup_{u=1}^{n}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+1})$, i.e., a union of rectangles with fixed northeast corner (Figure A2, right panel). Likewise, we have $U'' \equiv \cup_{u=1}^{n} r''_u$. Since $(\mathbf{i}', \mathbf{j}') \leq (\mathbf{i}'', \mathbf{j}'')$ (part (a)), if $(i,j) \in U' \setminus U'' = U' \setminus K^G$ (light gray in Figure A2, middle panel), then $(i'_{u^*}, j'_{n+1-u^*}) \leq (i,j)$, and $i \leq i''_{u^*}$ or $j \leq j''_{n+1-u^*}$, with at least one strict, at some u^*. So couple (i,j) is below the meet of the θ'' matched NAM pair $(i''_{u^*}, j''_{n+2-u^*})$ and $(i''_{u^*+1}, j''_{n+1-u^*})$. As rectangular synergy is upcrossing in types, $s_{ij}(\theta'') < 0$. Then $s_{ij}(\theta') < 0$, as synergy is upcrossing in θ. Then $\mathbb{S}^{n+k}(U' \setminus K^G|\theta') < 0$, as this holds for all $(i,j) \in U' \setminus K^G$. As summed rectangular synergy is additive and $\mathbb{S}^{n+k}(U'|\theta') > 0$ (part (b)), $\mathbb{S}^{n+k}(K^G|\theta') = \mathbb{S}^{n+k}(U'|\theta') - \mathbb{S}^{n+k}(U' \setminus K^G|\theta') > 0$.

Part (d): $\mathbb{S}^{n+k}(K^G|\theta'') < 0$. Since $(\mathbf{i}', \mathbf{j}') \leq (\mathbf{i}'', \mathbf{j}'')$ (part (a)), define rectangles $K^T \equiv (i''_1, j'_{n+1}, i'_{n+1}, j''_{n+1})$ and $K^R \equiv (i'_{n+1}, j''_1, i''_{n+1}, j'_{n+1})$ (respectively, top and right crosshatched regions, Figure A2, middle panel). Then $U'' \setminus K^G = K^T \cup K^R$. As summed rectangular synergy is linear,

$$\text{(C3)} \qquad \mathbb{S}^{n+k}(K^G|\theta) = \mathbb{S}^{n+k}(U''|\theta) - \mathbb{S}^{n+k}(K^T|\theta) - \mathbb{S}^{n+k}(K^R|\theta).$$

Rectangle K^T is above the rectangle defined by the θ' PAM pair (i'_1, j'_n) and (i'_{n+1}, j'_{n+1}). So $\mathbb{S}^{n+k}(K^T|\theta'') > 0$, as summed rectangular synergy is upcrossing on rectangles and θ. Likewise, K^R is above the rectangle defined by the θ' PAM pair (i'_n, j'_1) and (i'_{n+1}, j'_{n+1}). So $\mathbb{S}^{n+k}(K^R|\theta'') > 0$. Then $\mathbb{S}^{n+k}(K^G|\theta'') < 0$, as $\mathbb{S}^{n+k}(U''|\theta'') < 0$ by part (b) and (C3).

Since $\mathbb{S}^{n+k}(K^G|\theta') > 0$ (part (c)), we cannot have $\left(\mathbb{S}^{n+k}(K|\theta'),\mathbb{S}^{n+k}(K|\theta'')\right) \in \hat{\mathcal{D}}_{n+k}$; thus, by part (a), we have contradicted dyads $\left(\mathbb{S}^{n+1}(K|\theta'),\mathbb{S}^{n+1}(K|\theta'')\right) \in \mathcal{D}^*_{n+1}$ and thus conclude that NAM at θ'' and NAM* at θ' is impossible.[20] ∎

C. *Proof of Proposition 3 for a Continuum of Types*

Step 1: *Uniquely optimal finite type matchings exist for a payoff perturbation with summed rectangular synergy upcrossing in* θ.

PROOF:

Let $\mathcal{X}^n = \{x_1^n,\ldots,x_n^n\}$ and $\mathcal{Y}^n = \{y_1^n,\ldots,y_n^n\}$ be equal quantile increments, with $G(x_1^n) = H(y_1^n) = 1/n$ and $G(x_i^n) = G(x_{i-1}^n) + 1/n$ and $H(y_j^n) = H(y_{j-1}^n) + 1/n$. Let G^n and H^n be cdfs on $[0,1]$, stepping by $1/n$ at $\mathcal{X}^n$ and $\mathcal{Y}^n$ (respectively). Put $f_{ij}^n(\theta) = \phi(x_i^n,y_j^n|\theta)$. The set $\mathcal{M}^n(\theta)$ of pure optimal matchings is nonempty, by Lemma 2.

Since unique optimal matchings are pure, we restrict to pure matchings. These are uniquely defined by the male partner vector $\mu = (\mu_1,\ldots,\mu_n)$. Call the pure matching $\hat{M}$ *lexicographically higher* than M *if and only if* its male partner vector $\hat{\mu}$ lexicographically dominates μ. Let $\bar{M}^n(\theta)$ (respectively, $\bar{\mu}^n(\theta)$) be the optimal pure matching highest in the lexicographic order and $\underline{M}^n(\theta)$ (respectively, $\underline{\mu}^n(\theta)$) the lowest. Easily, each is well defined.

Fix $\theta'' \succ \theta'$. Let $\iota(j) = \bar{\mu}_j^n(\theta') - 1$ and pick $\varepsilon > 0$. Perturb synergy down at θ':

$$s_{ij}^{n\varepsilon}(\theta') \equiv s_{ij}(\theta') - \varepsilon^j \mathbf{1}\left\{(i,j) = (\iota(j),j)\right\}. \tag{C4}$$

We prove that $\bar{M}^n(\theta')$ is uniquely optimal at θ' for any production function with ε-perturbed synergy (C4), for all small $\varepsilon > 0$. Similar logic will prove that $\underline{M}^n(\theta'')$ is uniquely optimal at θ'' with $s_{ij}^{n\varepsilon}(\theta'') \equiv s_{ij}(\theta'') + \varepsilon^j \mathbf{1}\left\{(i,j) = \left(\underline{\mu}_j^n(\theta''),j\right)\right\}$ for all small $\varepsilon > 0$.

Pick a matching M that is not optimal at $\varepsilon = 0$. Since $\bar{M}^n(\theta')$ is optimal at $\varepsilon = 0$, $\bar{M}^n(\theta')$ yields a higher payoff than M for all small $\varepsilon > 0$.

As $\bar{\mu}^n(\theta')$ is the lexicographically highest optimal matching at θ', another optimal μ obeys $\left(\bar{\mu}_1^n(\theta'),\ldots,\bar{\mu}_{\ell-1}^n(\theta')\right) = (\mu_1,\ldots,\mu_{\ell-1})$, and first diverges at $\bar{\mu}_\ell^n(\theta') > \mu_\ell$, for some woman $\ell < n$. Using $M_{ij} = \sum_{k=1}^j \mathbf{1}\{\mu_k \le i\}$, equation (5), and (C4), the payoff $\bar{M}^n(\theta')$ exceeds that of $M \in \mathcal{M}^n(\theta')$ by $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta')\left[\bar{M}_{ij}^n(\theta') - M_{ij}\right]$. This expands to

$$\sum_{j=1}^{n-1} \varepsilon^j\left[M_{\iota(j)j} - \bar{M}_{\iota(j)j}^n(\theta')\right] = \varepsilon^\ell + \sum_{j=\ell+1}^{n-1}\varepsilon^j \sum_{k=\ell+1}^{j}\left[\mathbf{1}\{\mu_k \le \iota(j)\} - \mathbf{1}\{\bar{\mu}_k^n \le \iota(j)\}\right].$$

Altogether, $\lim_{\varepsilon\to 0}\varepsilon^{-\ell}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta')\left[\bar{M}_{ij}^n(\theta') - M_{ij}\right] = 1 > 0$. ∎

[20] This last step assumes upcrossing synergy sums on connected *join semi-lattices* (sets that contain the join of any pair of elements). All of our results only require this weaker time series assumption.

Step 2: *If* $\theta' \succ \theta'$, *then* $\bar{M}^n(\theta'') \succeq_{PQD} \underline{M}^n(\theta')$ *for all n.*

PROOF:

Since $S^{n\varepsilon}(r|\theta)$ is continuous in ε, there exists $\hat{\varepsilon}_n > 0$ such that, for all $r = (i_1, j_1, i_2, j_2)$ and $0 \leq \varepsilon < \hat{\varepsilon}_n$, if $S^{n0}(r\,|\,\theta) \lesseqgtr 0$, then $S^{n\varepsilon}(r\,|\,\theta) \lesseqgtr 0$. By the contrapositives,

$$\text{(C5)}\quad S^{n\varepsilon}(r\,|\,\theta) \geq 0 \Rightarrow S^{n0}(r\,|\,\theta) \geq 0 \quad \text{and} \quad S^{n\varepsilon}(r\,|\,\theta) \leq 0 \Rightarrow S^{n0}(r\,|\,\theta) \leq 0.$$

We claim that $S^{n\varepsilon}(r\,|\,\theta)$ *is strictly upcrossing in r for all* $0 < \varepsilon < \hat{\varepsilon}_n$. For if not, then $S^{n\varepsilon}(r''|\,\theta) \leq 0 \leq S^{n\varepsilon}(r'\,|\,\theta)$ for some $r'' \succ_{NE} r'$. But then $S^{n0}(r''\,|\,\theta) \leq 0 \leq S^{n0}(r'\,|\,\theta)$ by (C5), contradicting $S^{n0}(r\,|\,\theta)$ strictly upcrossing in r, as follows from Step 1.

Continuum summed rectangular synergy is upcrossing in θ by assumption; and thus, finite summed rectangular synergy $\sum_{k=1} S^{n0}(r_k\,|\,\theta)$ for all finite approximations. Then, perturbed summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k\,|\,\theta)$ is upcrossing in θ since synergy $s_{ij}^{n\varepsilon}(\theta'')$ is nonincreasing in ε and $s_{ij}^{n\varepsilon}(\theta'')$ is nondecreasing in ε by construction (C4).

So for $\varepsilon \in (0, \hat{\varepsilon}_n)$, rectangular synergy $S^{n\varepsilon}(r\,|\,\theta)$ is strictly upcrossing in r and summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k\,|\,\theta)$ upcrossing in θ, for couple sets $K \subseteq \mathbb{Z}_n^2$. Given $\bar{M}^n(\theta''), \underline{M}^n(\theta')$ uniquely optimal, $\underline{M}^n(\theta'') \succeq_{PQD} \bar{M}^n(\theta'), \forall n$, by Proposition 3. ∎

Step 3: *There exists a subsequence of matchings* $\{M^{n_k}(\theta)\}$ *that converges to an optimal matching in the continuum model.*

PROOF:

Define step function $\phi^n(x, y\,|\,\theta) = f_{ij}^{n\varepsilon_n}(\theta)$ for $(x, y) \in [x_{i-1}^n, x_i^n) \times [y_{j-1}^n, y_j^n)$, where $\varepsilon_n = \hat{\varepsilon}_n / n$. Then $\{G^n\}$ and $\{H^n\}$ weakly converge to G and H as $n \to \infty$, while ϕ^n uniformly converges to ϕ. By Theorem 5.20 in Villani (2008), their optimal matching cdfs have a limit point $M^\infty(\theta)$ optimal in the continuum model.[21] ∎

Step 4: $M^\infty(\theta'') \succeq_{PQD} M^\infty(\theta')$ *for all* $\theta'' \succeq \theta'$.

PROOF:

Fix $\theta'' \succeq \theta'$, and let $\{n_k\}$ be a subsequence along which the sequence of finite type matchings $\{M^{n_k}(\theta')\}$ converges to $M^\infty(\theta')$, as defined in Step 3. Now, since cdfs $\{G^{n_k}\}$ and $\{H^{n_k}\}$ weakly converge to G and H, and $\phi^{n_k}(x, y\,|\,\theta'')$ converges uniformly to $\phi(x, y\,|\,\theta'')$, there exists a subsequence $\{n_{k_\ell}\}$ of $\{n_k\}$, along which the sequence of finite type matchings $\{M^{n_{k_\ell}}(\theta'')\}$ converges to $M^\infty(\theta'')$ by Theorem 5.20 in Villani (2008). Further, by Step 2, $M^{n_{k_\ell}}(\theta'') \succeq_{PQD} M^{n_{k_\ell}}(\theta')$. But then, the limits must be

[21] Namely, fix a sequence $\{\phi_k\}$ of continuous and uniformly bounded production functions converging uniformly to ϕ. Let $\{G_k\}$ and $\{H_k\}$ be cdf sequences and M_k an optimal matching for ϕ, given G_k and H_k. If G_k and H_k weakly converge to G and H, then some subsequence of $\{M_k\}$ weakly converges to a matching M^* optimal for ϕ, G, and H.

ordered $M^{\infty}(\theta'') \succeq_{PQD} M^{\infty}(\theta')$ by Theorem 9.A.2.a in Shaked and Shanthikumar (2007). ∎

D. *Marginal Rectangular Synergy: Proof of Proposition 4*

A nonnegative function $\sigma : Z \mapsto \mathbb{R}_+$ on lattice Z is *log-supermodular* (*LSPM*) if

(C6) $$\sigma(z \wedge z')\sigma(z \vee z') \geq \sigma(z)\sigma(z'), \quad \forall z, z' \in Z.$$

CLAIM 1: *The indicator function* $\mathbf{1}\{x \in [u(x_1), u(x_2)]\}$ *is log-supermodular in* (x, x_1, x_2) *for all nondecreasing functions u.*

PROOF:

Define $(u_i, u_i') \equiv (u(x_i), u(x_i'))$ for $i \in \{1,2\}$. If both $x \in [u_1, u_2]$ and $x' \in [u_1', u_2']$, then $x \vee x' \in [u_1 \vee u_1', u_2 \vee u_2']$ and $x \wedge x' \in [u_1 \wedge u_1', u_2 \wedge u_2']$; and thus, $\mathbf{1}\{x \vee x' \in [u_1 \vee u_1', u_2 \vee u_2']\}\mathbf{1}\{x \wedge x' \in [u_1 \wedge u_1', u_2 \wedge u_2']\} = 1$. ∎

Now, assume marginal rectangular synergy is upcrossing in types. The steps for downcrossing marginal rectangular synergy are symmetric.

Step 1: *If marginal rectangular synergy is strictly upcrossing, then rectangular synergy is strictly upcrossing.*

PROOF:

We prove the continuum case, which implies the finite type result. By Claim 1, the function $\mathbf{1}\{x \in [x_1, x_2]\}$ is log-supermodular function in (x, x_1, x_2). By Karlin and Rubin's classic 1956 result, if $\Delta_x(x \mid y_1, y_2, \theta)$ is upcrossing in x, then the last integral in (10) is upcrossing in x_1 and x_2, and so in (x_1, x_2). Symmetrically, rectangular synergy is upcrossing in (y_1, y_2) when the y-marginal rectangular synergy is upcrossing in y. Altogether, rectangular synergy $\mathcal{S}$ is upcrossing in types if both MPIs are upcrossing.

Now assume $\Delta_x(x \mid y_1, y_2)$ is strictly upcrossing; and so, if $S(x_1', y_1, x_2', y_2) = 0$, then $\Delta_x(x_1' \mid y_1, y_2) < 0 < \Delta_x(x_2' \mid y_1, y_2)$. So $\mathcal{S}_{x_1}(x_1', y_1, x_2', y_2) = -\Delta_x(x_1' \mid y_1, y_2) > 0$, and $\mathcal{S}_{x_2}(x_1', y_1, x_2', y_2) = \Delta_x(x_2' \mid y_1, y_2) > 0$. Then $\mathcal{S}(x_1'', y_1, x_2'', y_2) > 0$ for all $(x_1'', x_2'') > (x_1', x_2')$. By symmetric reasoning, $\mathcal{S}$ strictly upcrosses in (y_1, y_2). ∎

Step 2: *The optimal matching is unique in the continuum type model.*

PROOF:

By Theorem 5.1 in Ahmad, Kim, and McCann (2011), there is a unique optimal matching when (i) G is absolutely continuous, (ii) ϕ is C^2, and (iii) the critical points of (what they call a "twist difference") $\phi(x, y_2) - \phi(x, y_1)$ include at most one local max and one local min, for all y_1, y_2. Our continuum types model imposes (i) and (ii). We claim that (iii) follows from marginal rectangular synergy $\Delta_x(x \mid y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x, for $y_2 > y_1$. In particular, if $y_2 > y_1$, then $\Delta_x(x \mid y_1, y_2)$ is upcrossing in x, and any critical point of the twist difference is a global minimum. Similarly, then any critical point is a global maximum if $y_2 < y_1$. ■

Step 3: *Sorting increases in θ.*

PROOF:

Propositions 3 and 4 share the time series assumption. By Step 1, the cross-sectional premise of Proposition 4 implies the cross-sectional premise of Proposition 3. Finally, the optimal matching is generically unique for any finite type model and is unique for continuum type models by Step 2. By Proposition 3, sorting rises in θ. ■

E. *A Generalization of Proposition 5*

With a continuum of types, *synergy is proportionately upcrossing* if

$$\phi_{12}^{-}(z \wedge z', \theta)\phi_{12}^{+}(z \vee z', \theta') \geq \phi_{12}^{-}(z, \theta')\phi_{12}^{+}(z', \theta) \tag{C7}$$

for $z = (x, y)$, $z' = (x', y')$, and $\theta' \succeq \theta$, where meet $\wedge$ and join $\vee$ assume the vector order. For a finite number of types, synergy is proportionately upcrossing if $s_{ij}(\theta)$ obeys an inequality analogous to (C7) for arguments $z = (i, j)$ and $z' = (i', j')$, and for $\theta' \succeq \theta$.

Synergy is proportionately upcrossing if it is increasing in θ and monotone in types. Indeed, $(z \vee z', \theta') \succeq (z', \theta) \Rightarrow \phi_{12}^{+}(z \vee z', \theta') \geq \phi_{12}^{+}(z', \theta)$, and $(z, \theta') \succeq (z \wedge z', \theta) \Rightarrow \phi_{12}^{-}(z \wedge z', \theta) \geq \phi_{12}^{-}(z, \theta')$. And, easily, the product of a proportionately upcrossing and LSPM function is proportionately upcrossing. All told, we generalize Proposition 5:

PROPOSITION 6: *Assume synergy is upcrossing in θ, synergy is one-crossing in types, and proportionately upcrossing synergy. Sorting increases in θ in generic finite type models, or with continuum types if synergy strictly one-crosses in types.*

FINITE TYPES PROOF:

We verify the premise of Proposition 3. By Theorem 1, total synergy $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbf{1}\{(i,j) \in Z\}$ on any set of couples $Z \subseteq \mathbb{Z}_n^2$ is upcrossing in $t = \theta$. So summed rectangular synergy $\sum_k S(r_k | \theta)$ is upcrossing in θ for any nonoverlapping set of rectangles $\{r_k\}$. Next, rectangular synergy $S(r|\theta) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbf{1}\{(i,j) \in r\}$ is upcrossing in r by Theorem 1 with $t = r \in \mathbb{R}^4$. By Claim 1, the indicator function $\mathbf{1}\{(i,j) \in r\} = \mathbf{1}\{i \in [i_1, i_2]\}\mathbf{1}\{j \in [j_1, j_2]\}$ is LSPM in (i, j, r) since LSPM is preserved by multiplication.[22] Then $s_{ij}(\theta)\mathbf{1}\{(i,j) \in r\}$ obeys inequality (C7) in $z = (i, j)$ and r since $s_{ij}(\theta)$ obeys (C7) for fixed θ. Rectangular synergy upcrosses in r, by Theorem 1. ■

[22] Theorem 1 assumes $t \in \mathcal{T}$, a poset. Here, we exploit the fact that the space of rectangular *sets* of couples is a sublattice of $\mathbb{Z}^2$, even though the PQD order on *distributions* over couples is not a lattice.

CONTINUUM OF TYPES PROOF:

We apply Proposition 4. By Theorem 1, total synergy $\int_Z \phi_{12}(x,y\,|\,\theta)dx\,dy$ is upcrossing in $t = \theta$ for any measurable set $Z \subseteq [0,1]^2$. Thus, summed rectangular synergy $\sum_k S(R_k\,|\,\theta)$ is upcrossing in θ for any nonoverlapping set of rectangles $\{R_k\}$. Next, the x-marginal rectangular synergy $\int \phi_{12}(x,y)\mathbf{1}\{y \in [y_1,y_2]\}dy$ is strictly upcrossing in x. Let $x'' > x'$. Posit for a contradiction:

$$\text{(C8)} \quad \int \phi_{12}(x'',y)\mathbf{1}\{y \in [y_1,y_2]\}dy \le 0 \le \int \phi_{12}(x',y)\mathbf{1}\{y \in [y_1,y_2]\}dy.$$

As synergy $\phi_{12}(x,y)$ is strictly upcrossing in x and y, by (C8), there exist zeros $y',y'' \in (y_1,y_2)$ such that $\phi_{12}(x',y) \lesseqqgtr 0$ for $y \lesseqqgtr y'$ and $\phi_{12}(x'',y) \lesseqqgtr 0$ for $y \lesseqqgtr y''$. Easily, these zeros are ordered $y'' < y'$. But then inequalities in (C8) are simultaneously impossible, for

$$\begin{aligned}
0 &\le \int \phi_{12}(x',y)\mathbf{1}\{y \in [y_1,y_2]\}dy < \int \phi_{12}(x',y)\mathbf{1}\{y \in [y_1,y'']\}\mathbf{1}\{y \in [y',y_2]\}dy \\
&\Rightarrow 0 < \int \phi_{12}(x'',y)\mathbf{1}\{y \in [y_1,y'']\}\mathbf{1}\{y \in [y',y_2]\}dy \\
&< \int \phi_{12}(x'',y)\mathbf{1}\{y \in [y_1,y_2]\}dy
\end{aligned}$$

by Theorem 1 since $\int \phi_{12}(x,y)\lambda(y)dy$ is upcrossing in $t = x$ for any nonnegative $\lambda(y)$—because $\phi_{12}(x,y)$ is proportionately upcrossing in types and upcrossing in y. ∎

F. *Type Distribution Shifts: Proof of Corollary 1*

Throughout, we without loss of generality assume types shift up in the parameter θ.

Step 1: *Summed Rectangular Quantile Synergy is Upcrossing in θ.*

For any finite disjoint set of rectangles $\{R_k\}$ in $[0,1]^2$, let $Z \equiv \cup_k R_k$ and define the pdf

$$\lambda(x,y\,|\,\theta) \equiv \frac{\mathbf{1}\{(G(x\,|\,\theta),H(y\,|\,\theta)) \in Z\}}{\int\int \mathbf{1}\{(G(s\,|\,\theta),H(t\,|\,\theta)) \in Z\}ds\,dt}.$$

We claim that the associated cdf $\Lambda(x,y\,|\,\theta) \equiv \int^y\int^x \lambda(s,t\,|\,\theta)ds\,dt$ is nonincreasing in θ. Indeed, the indicator function $\mathbf{1}\{(s,t) \le (x,y)\}$ is log-supermodular in (s,t,x,y) by Claim 1. Recalling that rectangles R_k are defined by quantiles $[p_1,p_2] \times [q_1,q_2]$, we rewrite

$$\begin{aligned}
\mathbf{1}\{(G(s\,|\,\theta),H(t\,|\,\theta)) \in R_k\} = \mathbf{1}\Big\{(s,t) \in &\left[G^{-1}(p_1\,|\,\theta),G^{-1}(p_2\,|\,\theta)\right] \\
&\times \left[H^{-1}(q_1\,|\,\theta),H^{-1}(q_2\,|\,\theta)\right]\Big\},
\end{aligned}$$

which is log-supermodular in (s,t,θ) by $G^{-1}(p\,|\,\theta),H^{-1}(q\,|\,\theta)$ nondecreasing in θ and Claim 1. Thus, since log-supermodularity is preserved by multiplication, integration

(Karlin and Rinott 1980), and summation (over R_k), $\int\int \mathbf{1}\{(G(s|\theta), H(t|\theta)) \in Z\}\mathbf{1}\{(s,t) \leq (x,y)\} ds\,dt$ is log-supermodular in (x, y, θ). Consequently, $(x, y) \leq (x', y')$ implies that the ratio

$$\frac{\int\int \mathbf{1}\{(G(s|\theta), H(t|\theta)) \in Z\}\mathbf{1}\{(s,t) \leq (x,y)\} ds\,dt}{\int\int \mathbf{1}\{(G(s|\theta), H(t|\theta)) \in Z\}\mathbf{1}\{(s,t) \leq (x',y')\} ds\,dt} \text{ is nonincreasing in } \theta.$$

Finally, since $\Lambda(x, y|\theta)$ is this ratio evaluated at (x', y') equal to the highest types on each side of the market, Λ is nonincreasing in θ.

Now, define total quantile synergy (11) on the set Z in the continuum model

$$\begin{aligned}\Upsilon(\theta) &\equiv \int\int \varphi_{12}(p, q|\theta)\mathbf{1}\{(p,q) \in Z\} dp\,dq \\ &= \int\int \phi_{12}(x,y)\mathbf{1}\{(G(x|\theta), H(y|\theta)) \in Z\} dx\,dy\end{aligned}$$

by the change of variables $x = G^{-1}(p|\theta)$ and $y = H^{-1}(q|\theta)$; and thus, $dx = dp/g(G^{-1}(p|\theta))$ and $dy = dq/h(H^{-1}(q|\theta))$. Then using the fact that the cdf $\Lambda(x, y|\theta)$ is first-order increasing in θ and $\phi_{12}(x, y)$ is nondecreasing, we find

$$\begin{aligned}0 \leq \Upsilon(\theta) \Rightarrow 0 &\leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta) dx\,dy \\ &\leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta') dx\,dy \Rightarrow 0 \leq \Upsilon(\theta').\end{aligned}$$

Identical steps prove the result for models with finite types.

Step 2: Quantile marginal rectangular synergy (strictly) upcrosses in quantiles.

We prove case (b) (continuum types). Case (a) follows from symmetric logic.

Nondecreasing synergy is proportionately upcrossing; and thus, $\Delta_x(x|y_1, y_2)$ strictly upcrosses in x as shown in Appendix C.E. Given $G(x|\theta)$ absolutely continuous, $g > 0$; and so,

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(G^{-1}(p|\theta)|H^{-1}(q_1|\theta), H^{-1}(q_2|\theta))/g(G^{-1}(p|\theta))$$

is strictly upcrossing in p. Similarly, $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in q. All told, we've seen that quantile sorting increases in θ, by Step 1 and Proposition 4. ∎

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