Frictional Matching Models[*]

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Abstract

This article offers a self-contained graduate lecture on developments frictional matching models 1990–2010, exploring how frictions skew the matches that occur. This literature turned exploiting new tools from monotone methods under uncertainty, and seeing how this journey plays out is instructive in itself for economic theory.

Contents

[*]I have learned from my many coauthors so much that I now forget which insights are theirs and which mine.

payoffs	Y_1	Y_2	Y_3
X_1	$2, 23$	$4, 24$	$\mathbf{6, 25}$
X_2	$4, 16$	$\mathbf{8, 18}$	$12, 20$
X_3	$\mathbf{6, 9}$	$12, 12$	$18, 15$

sums	Y_1	Y_2	Y_3
X_1	**25**	28	31
X_2	20	**26**	32
X_3	15	24	**33**

Figure 1: **Match Payoffs and Payoff Sums.** We indicate the stable (blue) matching for the NTU match payoffs at left. Once monetary or utility transfers are allowed (the TU case at right), the corresponding payoff sums are the relevant benchmark, and the stable matching switches (now red).

1 Frictionless Matching Benchmarks

1.1 A Motivational Example

Let's consider a matching market between two sides of the market, called X's and Y's. The Gale-Shapley algorithm (1962) will discover the stable matching. Consider the payoffs (in utils) in Figure 1. If the X's do the proposing, then everyone first asks Y_3, since he is best for all. Then Y_3 accepts X_1, sealing that match. At the next round, X_2 and X_3 ask Y_2, who then accepts X_2. Finally, X_3 and Y_1 match. The final matching is (X_1, Y_3), (X_2, Y_2), (X_3, Y_1). Or, if instead Y's propose, each asks X_1, who accepts Y_3. Each Y then asks X_2, who chooses Y_2. Finally, X_3 and Y_1 match. The same matching arises in this example — and so there is a unique stable matching.

Next observe how these matches are inefficient. For instance, X_3 is willing to sever his match with Y_1, and pay Y_3 up to $18-6 = 12$ to join forces with him; furthermore, this deal sweetener exceeds the loss $25 - 15 = 10$ that Y_3 suffers from quitting his match with X_1. The above unique NTU matching is thus unstable once transfers are allowed. Intuitively, the total payoffs (at right in Figure 1) determine matching decisions once individuals can offer side-payments — and, as they say, money changes everything!

While side-payments have the ring of illegality, at issue here is the world with wages and other plain vanilla transfers. These are driven by *outside options* — namely, the least amount that someone can *guarantee* himself in the matching model by exercising his best other match. In this setting, the outside options are wages. Imagine that two agent firms (i, j) can costlessly form that pay out wages using the match payoff sums $f(i, j)$. Let the Walrasian auctioneer call out wages for everyone until all markets clear for the individuals. Call the wages of the X and Y sides w_i^X and w_j^Y. Then the

PAM firms make nonnegative profits exactly when $w_i^X + w_i^Y \leq f(i,i)$ for $i = 1, 2, 3$. Meanwhile, the non-PAM firms will not form only if wages over-exhaust output, namely, $w_i^X + w_j^Y \geq f(i,j)$ for all payoffs $f(i,j)$ $(i \neq j)$ at the right in Figure 1. One can show that these inequalities have a solution.[1]

1.2 Conditions for Assortative Matching

The more general matching model has two distinct sides of the market — at one end of the spectrum, men and women in a social setting, and at the other, workers and firms in an employment context. A simpler matching setting is the "unisex" model of symmetric partnerships.

Individuals can be of several observable types — the matching appeal of the social mate, the productivity of the worker or the excellence of the firm. Any individual of type x matches like another other such individual, and thus, we can simply refer to the types. Types are simply scalars $x \in [0, 1]$, and have an atomless distribution with cdf $L(x)$ and finite density $\ell(x) \equiv L'(x)$. We assume that a type x earns $f(x, y)$ when matched with a type y. Throughout, the function $f : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is a nonnegative and for simplicity, continuous and twice differentiable on $\mathbb{R}^2$.

This is in many ways a ridiculously simplified setting. Firstly, beauty is not in the eye of the beholder: There is no disagreement about a worker's productivity across jobs, nor a firm's appeals to different workers. Second, there is no role for choice, such as effort, apart from the matching decision. Relaxing these restrictions is an important avenue for future research.

In the symmetric unisex case, the production function obeys $f(x, y) \equiv f(y, x)$. For as in the earlier example, this arises when matched individuals can share outputs. *Positive assortative matching* (PAM) occurs when types sort into matches according to their quantile. In this uni-sex environment, each type $x \in [0, 1]$ matches with another type x. By contrast, with *negatively assortative matching* (NAM), each type x matches with the "opposite percentile" type $y(x)$ — namely, the one for whom $L(y(x)) \equiv 1 - L(x)$.

A passing result in Becker 1973 considered the NTU setting — namely,

[1]For we know on the one hand, that there is solution to the social planner's problem, and on the other hand, that it solves the Kuhn-Tucker optimization conditions. If we let each type's wage be its shadow value to the planner (i.e., the Lagrange multiplier), then the complementary slackness conditions yield the desired inequalities.

the stable matches determined by the Gale-Shapley algorithm.[2] Consistent with the earlier example, Becker found out that the "market outcome" in the NTU case is PAM exactly when preferences over partners are increasing: $f_2(x,y) > 0$. Essentially, the market clears "top to bottom" in this case.

Becker (1973) focused on the transferable utility (TU) world, allowing utility side payments. He then deduced that PAM is both efficient and a competitive equilibrium when types are productive complements, possessing a positive cross partial derivative $f_{12}(x,y) > 0$. Economists have also seized on the lattice-theoretic label *supermodular* for this property (Topkis, 1998).

To see Becker's efficiency claim,[3] let's consider any pair of types $x < y$. Integrating $f_{12} > 0$ on any rectangle $[x,y] \times [x,y]$ yields a positive discrete cross partial difference $f(y,y) - f(x,y) > f(y,x) - f(x,x)$. In other words, $f(x,x) + f(y,y) > 2f(y,x)$. This implies that mixed matches (x,y) are inefficient: For a mass $\epsilon > 0$ of such mixed matches near (x,y) can create equal $\epsilon/2$ masses of matches near (x,x) and (y,y) with higher total output.

We now prove Becker's claim that PAM is a competitive equilibrium. Let's introduce the *match surplus function* $s(x,y)$ for easier comparison later:

$$s(x,y) = f(x,y) - w(x) - w(y)$$

This measures the amount by which the inside match option exceeds the sum of the two best outside options. Since every type x will find the best match y, we know that he solves the first order condition $s_2(x,y) = 0$. Then this "ideal partner", say $y(x)$, is rising in x precisely because $s_{12}(x,y) \equiv f_{12}(x,y) > 0$. In the unisex matching world, the only increasing function that clears the market is $y(x) = x$. So try the candidate wage profile $w(x) \equiv f(x,x)/2$, namely equal sharing of the output, which is obviously feasible. Second, given these wages, no other matches are profitable:

$$w(x) + w(y) \equiv f(x,x)/2 + f(y,y)/2 > f(x,y) \tag{1}$$

This follows because of symmetry and supermodularity of $f(x,y)$.

Conversely, consider *submodular* production, with $f_{12} < 0$. Then inequality (1) reverses, and the equilibrium (and efficient) outcome now in-

[2]Eeckhout (2000) formally proves this, among other things.

[3]As so often happens, mathematicians got there long before economists, and with far greater generality, but without the essential economic context. In this case, Lorentz (1953) deduced the formal content of Becker's Pareto claim.

volves NAM. Define the production function $g(x, y) \equiv f(x, 1-y)$, thus reversing the order on types. Then $g_{12} > 0$, and therefore PAM is optimal for the production function $g(x, y)$. Logically, NAM is optimal for $f(x, y)$.

The distinction between the TU and NTU allocation predictions seen in the motivational example emerges for any production functions that are submodular and increasing, or supermodular and decreasing. In the first case, NTU predicts PAM and TU predicts NAM; in the second case, the opposite holds. For instance, types in the example were $1, 2, 3$, and the X-payoffs $2XY$, and the Y-payoffs $XY - 8X + 30$. Easily, the own payoffs were falling in the partner types, but the shared payoff sums were supermodular.

The importance of Becker's focus on complementarity was underscored in Kremer's (1993) paper. He found that production functions where success depends multiplicatively on all types — such as when everyone must successfully perform a task — were automatically complementary; thus Kremer found that Becker's sorting theorem applies in these settings. Kremer and Maskin (1996) instead motivate non-complementary cases with examples where the "strongest link" matters — capturing matches with a defined "second banana" role. This setting yields some interesting discontinuous efficient matching patterns. Adding more economic structure, Ackerberg and Botticini (2002) explore matching in a contractual setting between principals (landowners) whose projects vary in their riskiness and agents (tenants) of varying risk tolerances. Serfes (2005) explores a more general setting, where either PAM or NAM emerges. Here, positive sorting means that poor tenants (and so very risk averse) cultivated risky vines, while wealthier tenants took charge of safer cereal production.

2 Matching with Search Frictions

In Becker's frictionless matching world, the long side of the market can get capriciously shafted. For instance, with 10^9 women and $10^9 + 1$ men, if all matches yield payoff 1 and being unmatched yields nothing, then men earn a zero wage. But if one man dies, as so often happens, then a 50-50 split may emerge. The search world that we now explore does not suffer from this somewhat counterfactual fragility.

2.1 Anonymous Search for Partners

Anonymous search is a natural formulation of search frictions that builds on the story underlying the consumer price search literature (Lippman and McCall, 1976). Proceeding in continuous time, we imagine that unmatched individuals find it hard to locate partners. The simplest story might be the *linear search technology*. It assumes that potential partners arrive at some exogenous and fixed "rendezvous" rate $\rho > 0$ — and so with chance $\rho\, dt$ in any infinitesimal length dt interval. These partners are randomly and representatively drawn from the pool of unmatched individuals on the other side of the market. But it is more tractable to analyze the *quadratic search technology*, in which ρ is scaled by the mass of unmatched partners. Or equivalently, one can imagine that randomly-chosen potential partners for any unmatched individual should arrive at rate ρ, but if that partner happens to be matched, then he misses the meeting.

Since there is a continuum of atomless (negligible) individuals, we can intuitively ignore mixed strategies, and simply assume that the strategies are *acceptance sets* $A(x) \subseteq [0, 1]$. Having in mind that everyone is both consumer and consumption good, we must keep track of the inverse *opportunity set* $\Omega(x) \equiv \{y \in [0, 1] | x \in A(y)\}$ of each type x. So the sets $A(x)$ and $\Omega(x)$ correspond to the preferences and opportunities of type x. The *matching set* consists of the set of mutually agreeable matches, namely, $M(x) = A(x) \cap \Omega(x)$. All matches in this intersection solve the double coincidence of wants, since both parties to the match are now willing.

In a further simplification, let us explore a world without on-the-job search. Namely, we venture that everyone is either matched and thus unavailable, or unmatched and searching. So the opportunity cost of matching is that new partners cease to arrive; this intuitively gives individuals an incentive to *decline* some matches, and avoid trivialities.

Almost all successful research on equilibrium search and matching has assumed a steady-state model. For even the simplest of nonstationary environments can be notoriously intractable, and should only be attacked if the underlying theoretical exercise really turns on the nonstationarity.[4] I follow this pattern, which conveniently makes unnecessary any time subscripts.

There are two primary ways to secure a steady-state. First, we could

[4]Shimer and Smith (2001) underscores how hard a nonstationary search model can be to analyze.

venture that matches dissolve at some fixed rate $\delta > 0$ — or equivalently, with chance approaching δdt in any small dt interval, where $\delta > 0$. Second, we could assume eternal matches (so that $\delta = 0$), and then posit a steady inflow of unmatched individuals that replaces the newly matched.

We assume that $f(x, y)$ now describes the flow payoffs, discounted at some interest rate $r > 0$. We can then define two interrelated values: First, the expected present (Bellman) value $V(x)$ of payoffs to x when initially unmatched, assuming optimal behavior; second, the expected present value $V(x|y)$ of payoffs to x when initially matched with y. Since the unmatched value is a pure option on getting matched, its return is the expected arrival rate of the expected surplus $s(x|y) = V(x|y) - V(x)$ from matching with an acceptable type $y \in M(x)$. If we let $U(y)$ denote the stationary cumulative distribution function of unmatched individuals, then we discover $rV(x) = \rho \int_{y \in M(x)} [V(x|y) - V(x)]U'(y)dy$.

We find our analysis easier if we use the *average* present *unmatched value* $v(x) = rV(x)$ and the *average matched value* $v(x|y) = rV(x|y)$, since they will share the same flow payoff units. Our earlier accounting expression for the value then becomes:

$$v(x) = (\rho/r) \int_{y \in M(x)} [v(x|y) - v(x)]U'(y)dy \tag{2}$$

Next, the average present value $v(x|y)$ of type x from matching with y includes a flow payoff $f(x, y)$, as well as a δ arrival rate of a capital loss of $v(x) - v(x|y)$. This yields the implicit equation

$$v(x|y) = f(x, y) + (\delta/r)[v(x) - v(x|y)] \tag{3}$$

2.2 Assortative Matching

As with Becker (1973), we ask what assumptions on productive interaction yield PAM for all levels of search frictions, and for all type distributions. More to the point, since possible mates are drawn from an atomless continuum, matching sets can no longer be point-valued. In this case, what exactly does PAM mean? We now explore a natural definition of PAM in this setting, that subsumes the PAM definition in the Walrasian setting. It should have the same econometric properties — that the expected partner of type x must be weakly increasing in x. We find a mathematically elegant

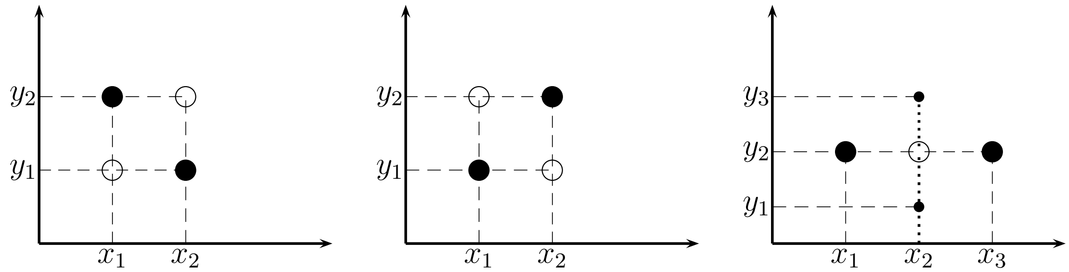

Figure 2: **Assortative Matching.** The left two panels respectively depict the definitions of PAM and NAM: If the pairs indicated by filled dots match, then the pairs indicated by hollow dots match as well. The right panel depicts the proof of convexity. If low and high types, x_1 and x_3, match with $y_2 \in (y_1, y_3)$, then so must a middle type $x_2 \in (x_1, x_3)$, given PAM or NAM.

lattice definition that achieves this. This definition is necessary and sufficient for the above econometric property to hold for all type distributions.

Let us reconsider the basic insight that PAM means that high types should match with other high types, and low types with other low types. One way of formulating this is that if high and low types are matched, namely the pairs (x_1, y_2) and (x_2, y_1), with $x_1 < x_2$ and $y_1 < y_2$, then so are the matches of like types (x_1, y_1) and (x_2, y_2). (Equivalently, the matching set is increasing function of types in the "strong set order".) Figure 2 illustrates this and the parallel definition of NAM.[5]

A useful property of this definition is that given PAM or NAM, every type has a convex matching set. This proof is graphically obvious, as seen in Figure 2. This implies that the matching set can be written $M(x) = [\theta(x), \psi(x)]$, with lower and upper bounds $\theta(x)$ and $\psi(x)$. Next, since the matching sets of the types on the vertical axis are themselves convex, the lower bound function $\theta(x)$ is quasiconvex, and the upper function $\psi(x)$ is quasiconcave. This offers an easier attack on the assortative matching characterization. For *if matching sets are convex, then PAM arises exactly when the lower and upper bounds are weakly increasing.* For such monotone bounds obviously ensures the lattice definition of PAM in Figure 2; conversely, that PAM picture fails at the extremes when the upper or lower bound fall somewhere. This suggest an immediate definition that *if matching sets are convex, and the lower and upper bounds are strictly*

[5]At a formal level, we ask that the set of mutually agreeable matches be a *lattice* for PAM: For all pairs of partners z, z' in $\mathbb{R}^2$, if z and z' are each in the matching set, so is the meet $z \wedge z'$ and join $z \vee z'$ (respectively, the vector-max and vector-min).

increasing whenever possible, then matching obeys strict PAM.

3 NTU Matching

Let us start in the world of NTU matching, as this will prove a springboard into the harder TU matching exercise. We will assume for now that an equilibrium exists, and describe who matches with whom. In fact, we will essentially construct an equilibrium in the process.

For simplicity, let's assume that preferences are monotone, with $f_2 > 0$. In this case, the acceptance set is $A(x) = [\theta(x), 1]$, for some cut-off partner $\theta(x)$. The opportunity set is therefore $\Omega(x) = \{y \in [0,1] | x \geq \theta(y)\}$. Then by our earlier discussion, PAM arises exactly when $\theta(x)$ is nondecreasing. In words, higher types are choosier. Optimality is then easily captured by the equality of inside and outside options, or $f(x, \theta(x)) = v(x)$. In other words, individuals should match with all types that provide at least the unmatched value.

3.1 Block Segregation

There is one low-hanging fruit in this domain — so low-hanging, that it was repeatedly re-discovered in a "great minds thinking alike" rush of research in the 1990s. Indeed, suppose that we can express output multiplicatively, as $f(x, y) = h_1(x)h_2(y)$. For instance, everyone might just care about his partner's type, whereupon $f(x, y) = y$. This implies that all types x share the same cardinal preferences over match partners y. For we can create an affine transformation of match payoffs $f(x', y) = [h_1(x')/h_1(x)]f(x, y)$ — that includes the unmatched option, whose flow payoff is zero.[6] And in a world of uncertainty, cardinal preferences govern risky choices, such as the decision to accept a match, or press on and search. As such, any two individuals with the same opportunity sets must make the same choices.

So motivated, consider the highest type $x = 1$, who is desired by all. Faced with search frictions, his optimal *reservation partner* or *threshold partner* is $\theta(1) < 1$. Then everyone in the interval $[\theta(1), 1]$ shares the opportunity set of type 1, as well as the same cardinal preferences. *Ipso*

[6] With explicit flow search costs, this fails, since it is not true that $-c = [h_1(x')/h_1(x)](-c)$ for all x, x'. Thus, block segregation would not arise. See Chade for more discussion.

facto, they will choose the same cut-off partner $\theta(1)$. Thus, the interval of individuals $[\theta(1), 1]$ must constitute a closed matching set. But now this logic can be iterated, considering the preferences of individuals just slightly below $\theta(1)$. They too share a threshold partner $\theta(2) < \theta(1)$, and so on. What emerges is a unique equilibrium of "block segregation", a partition of $[0, 1]$ with class boundaries $\theta(1) > \theta(2) > \cdots$. It is easy to see that there will be only finitely many such boundaries exactly when $f(0, 0) > 0$. For as the threshold partner vanishes, the chance of lesser types finding a willing partner vanishes too. If the bottom type provides strictly positive payoff, then it is best to accept him and end the search.

Allocation results yield easy insights into wages and values. And in this case, block segregation forces a discontinuous value function.

3.2 Strictly Increasing Matching Sets

Whenever continuous fundamentals (the preferences and the search technology) lead to a discontinuous economic outcome, one's curiosity is naturally piqued. Block-segregation is in this way a striking matching outcome. This provides an equilibrium economic rationale for the numerous Victorian classist fables. But the stark perfectly non-communicating nature of these classes almost demands a larger picture.

Let us ponder what is special about block segregation payoff functions of the form $f(x, y) = h_1(x)h_2(y)$. While motivated from decision theory, might some mathematical structure be relevant here? In particular, the function $\log f(x, y)$ is additively separable in x, y, or equivalently has a zero cross-partial derivative. Any such function is also known as log-modular. Could it be that stepping just outside this class offers the missing bigger picture? What can be said of strictly log-supermodular functions, for which $\log f(x, y)$ has a positive cross partial?

To answer this question, let's consider two equivalent expressions for the value function. First, ponder how it relates to the *inside option*. Inspired by the log-supermodularity insight, we work with logarithms throughout. Assume that the threshold $\theta(x)$ is differentiable. Differentiating the log-optimality condition $\log v(x) \equiv \log f(x, \theta(x))$ in the type x yields:

$$\frac{v'(x)}{v(x)} = \frac{f_1(x, \theta(x))}{f(x, \theta(x))} + \theta'(x)\frac{f_2(x, \theta(x))}{f(x, \theta(x))} \tag{4}$$

Next, we proceed likewise from the purely accounting definition of the values as an *outside option* — namely, the present value of future matches. It helps to ignore deaths, putting $\delta = 0$, so that the matched value in (3) is $v(x|y) = f(x,y)$. We could then re-write the Markov recursive equation (2). Instead, focus on the highest types for whom the world is their oyster, since everyone wishes to match with them. Let's employ a cut-off partner θ. Then the resulting *policy value* $v_\theta(x)$ solves an analogous equation:

$$v_\theta(x) = (\rho/r) \int_{Y \geq \theta} [f(x,Y) - v_\theta(x)] U(dY) = \frac{\rho \int_{Y \geq \theta} f(x,Y) U(dY)}{r + \rho U(Y \geq \theta)} \tag{5}$$

Since the best cut-off is the optimal threshold $\theta = \theta(x)$, the partial derivative of the policy value $v_\theta(x)$ in θ vanishes at $\theta = \theta(x)$. So if we totally differentiate $\log v(x) \equiv \log v_{\theta(x)}(x)$ in x, then we arrive at an Envelope Theorem implication:

$$\frac{v'(x)}{v(x)} = \frac{\int_{Y \geq \theta(x)} f_1(x,Y) U(dY)}{\int_{Y \geq \theta(x)} f(x,Y) U(dY)} \tag{6}$$

Jointly, the optimization and accounting lessons (4) and (6) imply

$$\frac{f_1(x,\theta(x))}{f(x,\theta(x))} + \theta'(x) \frac{f_2(x,\theta(x))}{f(x,\theta(x))} = \frac{\int_{Y \geq \theta(x)} f_1(x,Y) U(dY)}{\int_{Y \geq \theta(x)} f(x,Y) U(dY)} \tag{7}$$

We now finish the argument with an elementary insight about ratios. Note that the inequality $3/4 < 5/6$ implies $3/4 < (3+4)/(5+6) < 5/6$. This logic underlies the easy proof that if $a(t), b(t) > 0$ are smooth functions, and $[a(t)/b(t)]' > 0$, then

$$\frac{a(t_0)}{b(t_0)} < \frac{\int_{t_0}^{t_1} a(t)dt}{\int_{t_0}^{t_1} b(t)dt} < \frac{a(t_1)}{b(t_1)} \qquad \forall\, t_0 < t_1 \tag{8}$$

If f_1/f is strictly increasing in y, then this inequality implies that the right side of (7) exceeds the first term on its left side. From this, we deduce that $\theta'(x) > 0$. This says that higher agents are more selective. Observe that this argument is tight: For if f is strictly log-submodular, then past some point, even higher types are willing to accept lower types: $\theta'(x) < 0$.

The argument that the lower threshold rises in the type can be finished

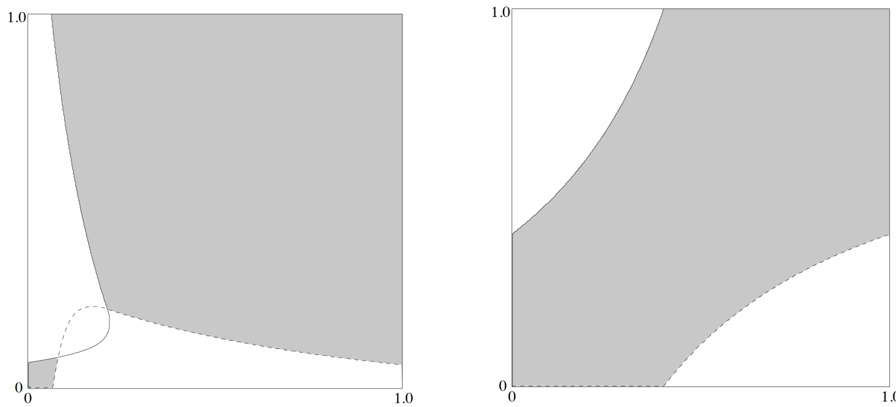

Figure 3: **NTU Matching with Anonymous Search.** At left is the graph of the matching set with payoff function $f(x, y) = x + xy + y$. Preferences are not only increasing in partners, as in Becker's corresponding result requires, but even supermodular: $f_{12} > 0$. At right is the graph of the matching set with a log-supermodular $f(x, y) = e^{xy}$. Here, we find assortative matching.

by inverting the matching set, and deducing that the upper bound $\psi(x)$ falls as the type x falls because $\theta(x)$ is increasing. One can show that this shifting upper bound reinforces the logic above.

Theorem 1 (PAM and NTU) *If x earns $f(x, y) > 0$ in a match with y, where $f_2(x, y) > 0$, then the equilibrium NTU matching is block segregation if f is log-modular, and strict PAM if f is strictly log-supermodular.*

The left panel of Figure 3 illustrates the theorem, while the right panel shows the necessity of the log-supermodularity assumption. So quite unlike the frictionless NTU result in Becker (1973), monotonicity alone does not deliver PAM. It is not enough that one prefers that higher types hold out for them longer. In fact, even Becker's supermodular condition for PAM in the Walrasian setting is not enough. Rather, the payoff of matching up must rise proportionately faster at higher types. For as an outside option, the value is the price of the agent's time. Then log-supermodularity alone[7] guarantees that the value of an agent's time rises faster than the value of matching with a fixed reservation partner; hence, the only way to equalize the inside and outside options is for the reservation partner to rise.

[7] Log-supermodularity is stronger than supermodularity if f is increasing in both types, since $f_{12}f > f_1 f_2$ and $f_1, f_2 > 0$ imply $f_{12} > 0$.

4 TU Matching

Transferable utility demands that we understand the accounting of shared surplus. We first derive a useful identity. Since matches dissolve at rate δ:

$$v(x|y) + v(y|x) \equiv f(x,y) + (\delta/r)[v(x) + v(y) - v(x|y) - v(y|x)]$$

This implies that the average total match surplus for both parties equals

$$[v(x|Y) - v(x)] + [v(Y|x) - v(Y)] = \frac{r}{\delta + r}[f(x,y) - v(x) - v(y)] \tag{9}$$

Since side payments are allowed in a TU model, parties agree to match precisely when there is non-negative surplus. Define the average *surplus function* $s(x,y)$ and the *matching set* $M(x)$ as follows:

$$s(x,y) \equiv f(x,y) - v(x) - v(y) \geq 0 \quad \Leftrightarrow \quad y \in M(x) \tag{10}$$

The unmatched value serves a similar outside options role in this search model with transferable utility as the wage in Becker's model: Individuals agree to match exactly when their joint match value weakly exceeds the sum of their values. But in a search setting, this surplus of inside over outside options will generally be strict. How match surplus is split is therefore critical. Equal division is a neutral benchmark:

$$v(y|x) - v(x) = v(x|y) - v(y) \tag{11}$$

This is often dubbed the *Nash bargaining solution*; but since no bargaining is modeled, it is more natural to appeal to simplicity. Indeed, any other split would need motivation, justifying why one class of agents is stronger than another (addressed in §**??**). Modifying the value expression (2) in light of (11), we discover:

$$v(x) \quad = \quad (\rho/r)\int_{Y \in M(x)} [v(x|Y) - v(x)] \tag{12}$$

Using (9) and the Nash split (11), we may rewrite the value function (12) as:

$$v(x) \quad = \quad \frac{1}{2}\frac{\rho}{\delta + r}\int_{Y \in M(x)} (f(x,Y) - v(x) - v(Y)) \tag{13}$$

Observe that (12) resembles the optimality equation for NTU, but with a fraction 1/2 scaling the surplus, and matches now discounted by $r + \delta$, given match dissolutions. If β is the lead factor in (12), we can rewrite it as:

$$v(x) = \beta \int_Y \max\langle 0, f(x,Y) - v(x) - v(Y)\rangle \tag{14}$$

This fundamental equation lies at the heart of the TU search-matching paradigm. The value function resembles a potential,[8] and is uniquely defined given the production function $f(x,y)$. Understanding such potentials is a recurrent and interesting open problem in dynamic economic theory.

The value function solving the equation (14) is continuous — and in this respect, the TU model critically differs from the NTU model. As a result, *the match surplus vanishes around the boundary of the matching set.*[9] This intuitively means that marginal changes in the matching set have no partial effect on match value (12), since those matches yielded vanishingly little surplus. Abbreviating $2\beta \equiv \rho/(r+\rho)$ yields a key derivative:

$$v'(x) = \beta \int_Y (\max\langle 0, f_1(x,Y) - v'(x)\rangle) = \frac{\beta \int_{Y \in M(x)} f_1(x,Y)}{1 + \beta U(Y \in M(x))} \tag{15}$$

Given (15), any value function solving (14) is increasing and differentiable.

Equation (15) resembles the NTU value equation (5), except that the marginal value v' replaces the value v, and the marginal product f_1 appears instead of f. This observation originally inspired the attack on PAM in the TU model that follows. The key log-supermodularity assumption intuitively should apply to the own-marginal product f_1 in this setting.

In light of our earlier discovery that PAM follows if matching sets are convex and have monotone bounds, our plan of attack is:

A. We show that if all matching sets are convex, then supermodularity

[8] Sergiu Hart and Andreu Mas-Colell, "Potential, Value, and Consistency", Econometrica, Vol. 57, No. 3 (May, 1989), pp. 589-614.

[9] By contrast, the own surplus $s(x|y) \equiv f(x,y) - v(x)$ in an NTU match rises as the partner y approaches the top of the matching interval $M(x)$. Just as well, this too follows from the Envelope Theorem logic, the matching set is jointly optimal.

implies PAM *provided*[10] the marginal product of 0 vanishes: $f_2(0, y) \equiv 0$.

B. We argue that matching sets are convex when own marginal products are log-supermodular in both types (and one other missing ingredient obtains).

4.1 Convex Matching and Supermodularity $\Rightarrow$ PAM

We proceed by contradiction, since that gives us an extra ingredient in the proof to work with. Suppose that matching sets are all convex, production is supermodular, but that PAM fails. There are two ways this can occur.

First, some matching set $M(x_1)$ may have a higher upper bound than another set $M(x_2)$, with $x_2 > x_1$, say $y_1 = \psi(x_1) > \psi(x_2) = y_2$. In this case, the value function would have to fall, since we could deduce that lower types have higher surplus. For production is supermodular exactly when match surplus is supermodular, as $s_{12} = f_{12} > 0$. So the higher type x_2 sees his match surplus rise faster in his parter's type than does the lower type x_1. Now, type x_2 has a lower upper partner than type x_1. Since he derives zero surplus with the highest match partner, integrating $s_2(x, y)$ down from y_1, his surplus must be lower with every parter.

Second, PAM mail fail if the lower bound on the matching set $M(x)$ dips. For since the value function is increasing, we have $s_2(0, y) = f_2(0, y) - v'(y) = 0 - v'(y) < 0$. Thus, the "ideal partner" of type 0 (who gives her highest surplus) is 0. And since match payoffs are nonnegative, his matching set must include 0. But as remarked, convex matching sets have a quasi-convex lower bound $\theta(x)$. Since $\theta(x)$ is initially weakly increasing, it is always weakly increasing. This completes the PAM proof.

4.2 Convex Matching Sets

The natural route to establishing convex matching sets is to prove that the surplus function is quasi-concave in one's partner's type. To this end, let's toss aside some algebraic complexities, and simply assume that all types match; this way changes in the matching set can be ignored.[11] This reduces

[10] We show why this is needed in Figure....

[11] And let's not be troubled by the fact that in this case, matching sets are trivially convex, since the proof in Shimer and Smith (2000) shows that this difficulty is just an annoyance.

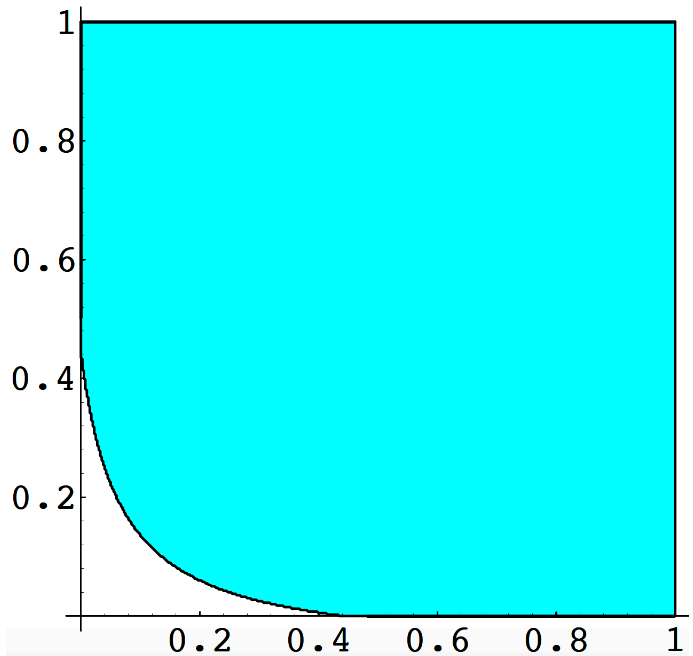

Figure 4: **Complementarity and Nonsorting.** This depicts matching sets for $f(x,y) = x + y + xy$, $\rho = 50r$, $\delta = r/2$, and $L(x) = x$ on $[0,1]$. Although production is supermodular ($f_{12} > 0$) and matching sets are convex, NAM and not PAM arises — because $f(0,0) = 0$ and $f(0,y) > 0$ for $y > 0$ force $0 \notin M(0)$.

the implicit equation (15) for marginal value to $v'(x) = \gamma E_Y f_1(x,Y)$, for a constant $\gamma < 1$ rising in the meeting rate ρ and falling in the interest rate r.[12]

A. Convexity for High Types. The match surplus of type x has slope $s_2(x,y) = f_2(x,y) - v'(y)$ in his partner's type y. Given symmetry $f(x,y) \equiv f(y,x)$, the marginal value is also $v'(y) \equiv \gamma E_Y f_2(X,y)$ when everyone matches. Hence:

$$s_2(x,y) = f_2(x,y) - \gamma E_X f_2(X,y) > E_X[f_2(x,y) - f_2(X,y)] \tag{16}$$

since $\gamma < 1$. An easy implication of supermodularity is that $f_2(1,y) - f_2(x',y) > 0$ whenever $x' < 1$, and so $s_2(1,y) > 0$. Given continuity of the marginal surplus, we have $s_2(x,y) > 0$ for high enough types $x < 1$. Namely, *highest types see their match surplus rising in their partner's type y,*

[12]Specifically, $\gamma = \rho U(0 \leq Y \leq 1)/[r + \rho U(0 \leq Y \leq 1)]$, which clearly lies between 0 and 1.

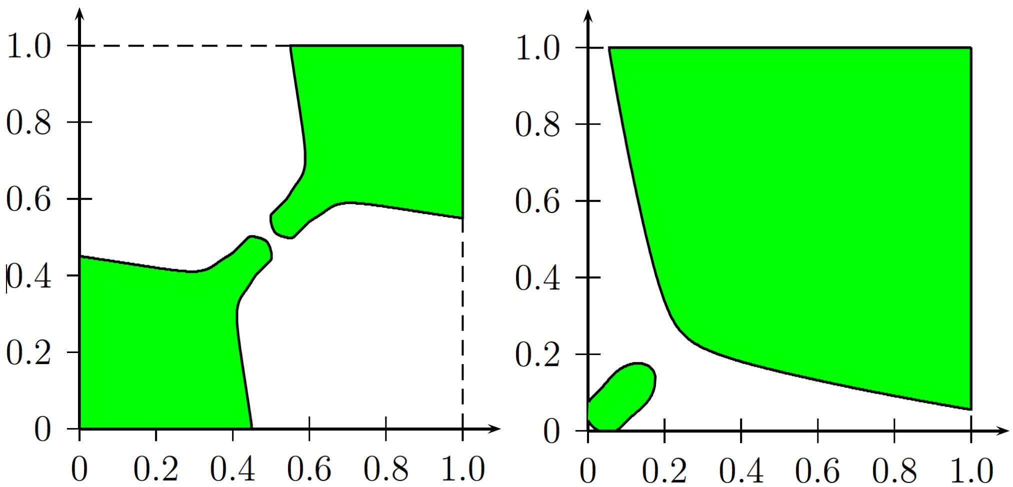

Figure 5: **Non-Convex Matching** The left and right panels respectively depict matching sets for the SPM production functions $f(x,y) = (x+y-1)^2$ and $f(x,y) = (x+y)^2$ with $\delta = r$ and a uniform type distribution. Matching is easier at left than at right, with a meeting rate $\rho = 100r$ versus $\rho = 35r$.

when production is supermodular. In this case, the matching set of any high type is a convex upper set in $[0, 1]$.

Figure 5 shows how matching sets of lower types may fail to be convex. The right panel considers $f(x,y) = (x+y-1)^2$. Types near $x = 1/2$ might not even match with peer types in one case — for $x = 1/2$ produces nothing when matched with her own type, and obviously receives no transfer. The right panel finds a similar convexity failure with the monotonic function $f(x,y) = (x+y)^2$. In this case, the matching sets of types $x \in [0.06, 0.18]$ include low and high types, but not middle types on the diagonal.

B. Convexity for Low Types. The match surplus of lower types need not always rise in their partner's type: Very high types may require too much compensation. Convexity can only be deduced from a quasiconcave surplus function. Since the surplus function is smooth, an easy sufficient condition for quasiconcavity is that its derivative obey a *single-crossing property*: If $s_2(x, \bar{y}) = 0$ for some $\bar{y}$, then $s_2(x, y) < 0$ respectively for all $y > \bar{y}$. Since $s_2(x, y) = f_2(x, y) - \gamma E_X[f_2(X, y)]$ by (16), this is equivalent to a *level crossing property* for a known function:[13]

$$\frac{E_X[f_2(X,y)]}{f_2(x,y)} \gtreqless 1/\gamma \qquad \text{for } y \gtreqless \bar{y} \tag{17}$$

The ideal partner y of type x is the one with whom match surplus is maximal, and thus for which the marginal value equals the own marginal product: $v'(y) = f_2(x,y)$. Our log-supermodularity assumption guarantees a unique ideal partner, since higher partners yield lower match surplus.

We must prove inequality (17) for all types x lower than those included in part A. We can prove it for the lowest type $x = 0$, showing that the left side of (17) rises in y. Emulating the logic used in the NTU model after the critical equation (7), this level crossing property holds when $f_2(x,y)/f_2(0,y)$ is rising in y. Since $X > 0$ with probability one, it suffices that the marginal product f_2 be log-supermodular. And by continuity, this holds for types near $x = 0$. So far, we have seen that *if the marginal product of production is log-supermodular, then the least types have convex matching sets.*

Immediately, we see the problem with the sorting failures in Figure 5 — for when $f(x,y) = (x+y)^2$, the ratio $f_2(x,y)/f_2(0,y) = 1 + x/y$ falls in the partner's type y, since $\log f(x,y) = 2\log(x+y)$ is strictly submodular. Not surprisingly, we found a failure of matching set convexity for type $x = 0$.

C. A Single-Crossing Property for Gambles. To finish the argument, we must take a detour. We formulate and prove a useful lemma, that is the basic building block for the preservation of monotonicity under uncertainty. Notably, this turns out to be a famous result in information economics.

Lemma 1 (SCP for Gambles) *Define $h(x,y) > 0$ on $[0,1]$, and let the partial derivative $h_1(x,y) > 0$ be log-supermodular. Fix an arbitrary density for x on $[0,1]$. For any $y \in [0,1]$, let $\bar{x}$ be the "certainty equivalent" of X solving $h(\bar{x},y) = E_X[h(X,y)]$, then $h(\bar{x},z) \geq E_X[h(X,z)]$ for all $z \geq y$. Further, the opposite inequality follows if $h_1(x,y)$ is log-submodular.*

For a classic application, consider Diamond and Stiglitz's (1974) analysis of increasing risk and risk aversion. Their main result, Theorem 3, proved

[13] When the level to be crossed through is zero, then this is known as a single-crossing property (SCP), and is commonly exploited in comparative statics analysis in economics. Any SCP is an ordinal property — and holds under weaker conditions. We will eventually need a SCP to finish our proof, but for now, we require this stronger cardinal property.

that if a (thrice smooth) utility function $U(x, y)$ is increasing in wealth x, and the Arrow-Pratt coefficient of risk aversion $-U_{xx}/U_x = -(\log(U_x))_x$ increases in the parameter y, then the certainty equivalent $\bar{x}$ of any wealth gamble X rises in y.

To see how this follows from Lemma 1, observe that the coefficient of risk aversion can be written as $-(\log(U_x))_x$. Hence, it weakly increases in y iff $-(\log(U_x))_{xy} \geq 0$. This says that the marginal utility of income U_x is log-submodular in (x, y). In words, marginal utility of income decrements grow proportionately larger at greater levels of risk aversion. This is the intuitive reason why the certainty equivalent of any gamble rises in y, since the utility gains from more favorable gamble outcomes are worth less.

Because of its importance, we will now explain why Lemma 1 is true. Define

$$H(z) \equiv E_X \left(\phi(X)[h(X, z) - h(\bar{x}, z)] \right)$$

where

$$\phi(t) \equiv \frac{h(t, y) - h(\bar{x}, y)}{h(t, z) - h(\bar{x}, z)} = \frac{\int_{\bar{x}}^{t} h_1(x, y)dx}{\int_{\bar{x}}^{t} h_1(x, z)dx}$$

Since $h(\bar{x}, y) = E_X[h(X, y)]$ and $\phi(x) \equiv 1$ when $y = z$, we have $H(y) = 0$. It suffices to show that $H(z) \leq 0$. To do this, we return to the ratio inequality in (8). Intuitively, the ratio in (8) was increasing in t when $[a(t)/b(t)]' > 0$ and decreasing when $[a(t)/b(t)]' < 0$ (indeed, we can just differentiate). In this context, since $z \geq y$, we must have $\phi'(t) \leq 0$ when h_1 is log-supermodular and $\phi'(t) \geq 0$ when h_1 is log-submodular. Now, the covariance of an increasing function $\phi(x)$ and a decreasing function $x \mapsto h(x, z) - h(\bar{x}, z)$ is negative. In other words, we have:

$$H(z) \leq E_X[\phi(X)]E_X[h(X, z) - h(\bar{x}, z)] = 0$$

D. Convexity for All Types. Our twin application of monotone methods has fallen short of deducing a convex matching set for "middling types". Is it true that every type is either high enough for supermodular production to yield a rising surplus function, or low enough for log-supermodular marginal products to suffice? We now finish the proof of convexity, empowered by the single-crossing property for gambles.

Let's recall the earlier sufficient condition for a quasi-concave function.

We need to show that for all types $x \in [0,1]$:

$$f_2(x,y) - v'(y) = 0 \Rightarrow f_2(x,z) - v'(z) \leq 0 \qquad \text{whenever } z > y \tag{18}$$

We reinterpret our earlier analysis as checking assertions (a) and (b) below.

(a) The premise of (18) fails for high types $x < 1$, since surplus is rising.

(b) The implication (18) holds for low types $x > 0$.

(c) Every type $x \in (0,1)$ is "low" or "high".

Now, since $f_{12} > 0$, there is a unique cut-off $\bar{x}$ so that $f_2(\bar{x},y) = E_X f_2(X,y)$. Define the partial derivative function $h \equiv f_2$. Differentiating, we find that $h_1 = f_{12} > 0$ by strict production supermodularity. We now must introduce one final assumption on production — that the cross partial f_{12} is log-supermodular. In this case, Lemma 1 asserts $f_2(\bar{x},z) \geq E_X f_2(X,z)$ for all $z \geq y$. So if $x = \bar{x}$, then:

$$\frac{f_2(x,z)}{f_2(x,y)} \leq \frac{v'(z)}{v'(y)} = \frac{E_X f_2(X,z)}{E_X f_2(X,y)} \qquad \text{whenever} \quad z \geq y \tag{19}$$

It turns out that the threshold $\bar{x}$ is the critical type separating the high types in case (a) for whom the surplus function rises in the partner's type, and the low types in case (b) with single-peaked preferences. For if $x \geq \bar{x}$, then by (16), the surplus $s(x,y)$ has strictly positive slope in y, and thus (18) is a valid syllogism. On the other hand, the left side of (19) rises in x since the marginal product f_2 is log-supermodular. Thus, the inequality (19) holds if $x < \bar{x}$. And we have shown that all types have convex matching sets. This gives a flavor of the essence of the proof of the following assortative matching characterization:

Theorem 2 (PAM with TU) *If matched types x and y produce flow output $f(x,y) > 0$, where f is symmetric, then the equilibrium TU matching obeys strict PAM if f is supermodular, f_1 and f_{12} are log-supermodular, and $f_2(0,\cdot) \equiv 0$.*

An interesting open problem is to prove that log-supermodularity of f_{12} is needed, or to dispense with this proof ingredient altogether. It clearly plays a subtle role in the proof — essentially acting as a SCP that itself sends us to one of two cases. In principle, if it fails, we should find for some level of search frictions a hole in the middle of the search set, for middling types. Shimer and Smith were not able to produce such an example.

An interesting benchmark to consider is the the simplest model with two types of agents, low and high, and thus match payoffs $c > b > a$, as described below:

	L	H
H	b	c
L	a	b

The frictionless Walrasian world explored by Becker (1973) yields PAM whenever the supermodular inequality holds: $a + c > 2b$. Burdett and Coles (1999) found that in the TU search model, PAM occurs when this holds, and as long as a match between two low types produces at least 2/3 as much as a high-low match: $a > 2b/3$. Deducing this offers some nontrivial practice with the search analysis.

This restriction on the two type setting can only be understood in light of the boundary condition in Theorem 2. For the log-supermodularity of the marginal products only binds on a model with three or more types — for which there is a 2×2 array of marginal products (output differences). Writing the log-supermodularity condition in its product form, it requires that this matrix have a non-negative determinant. By the same token, the log-supermodular cross partial derivative is only restrictive in a model with four or more types.

5 Existence of Search Equilibrium

A Walrasian equilibrium is a *pair* of *prices* and *allocations*, such that the allocations are optimal given prices, and prices clear markets, given the allocations. The existence proof method developed in Debreu (1952) and Arrow and Debreu (1954) parallelled Nash's (1950) proof: They first deduced conditions that delivered an upper hemicontinuous and convex-valued map from prices to prices (where Nash mapped strategies to strategies); they next applied Kakutani's Fixed Point Theorem (1941) to secure a fixed point.

In the frictional world, an equilibrium is both conceptually and technically a more complicated object. A *search equilibrium* is now a *triple*

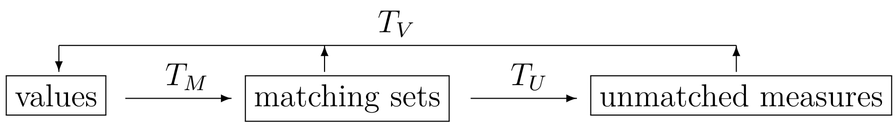

Figure 6: **Big Picture of Search Existence Proof Logic.** Search equilibrium requires that the three maps T_M (optimality), T_U (implied unmatched density), and T_V (value accounting) be both well-defined and continuous.

(v, M, u) — namely, the value function v, the matching set function M, and the new equilibrium object is the unmatched measure u (simply, here a density). So the value function acts like a wage function (the "price"), since it represents the outside options, while the matching set $M(\cdot)$ is the allocation. The unmatched density u captures the friction.

Existence no longer follows from the logic of Nash (1950). For now we must keep track not only of the mass of realized trades, but also of the mass of unmatched traders. We have so far suppressed the unmatched density $u(x)$ from all integrals, since its exact form did not affect any proofs. For this existence proof discussion, we now introduce it. The unmatched density $u(x) \leq \ell(x)$ obeys the implicit equation:

$$\delta[\ell(x) - u(x)] = \rho u(x) \int_{M(x)} u(y)dy \tag{20}$$

So motivated, search equilibrium turns on three maps T_M, T_V, T_U. First, given any value function v, write the optimal matching set as $M = T_M(v)$, as defined by (10) in the TU setting, for example. Next, the matching set M yields a density of unmatched agents $u = T_U(M)$, implied by (20). Finally, the triple of unmatched density u, matching set function M, and value function v yields a new value function $v = T_V(v, M, u)$. Each map must be well-defined and continuous in some fashion for us to apply a suitable fixed point theorem, and secure a fixed point of the composite map $v \mapsto T_V(v, T_M(v), T_U(T_M(v)))$ — as schematically depicted in Figure 6.

Showing that T_M is continuous — namely, nearby value functions lead to nearby matching sets is not too hard in the TU model — since the matching set is an upper contour set of the surplus function, via (10).[14] Equally well,

[14] It is somewhat trickier in the NTU case, since value functions are discontinuous. Smith (2006) uses a bounded variation norm to deal with this richer space of value functions.

the map T_V yielding continuation value functions is intuitively continuous, since it is defined by an integral. That $u = T_U(M)$ exists and is continuous, is absent from the Walrasian analysis, and is the toughest.[15]

For this reason, Bloch and Ryder (2000) made a clever and obviously false assumption that matched pairs are replaced instantaneously by unmatched "clones". Many papers have since made this assumption, and it is obviously an innocuous sin for any result that holds for all unmatched densities.

Since it is novel to the frictional matching world, we now explore why T_U is continuous. Consider the finite n-type case, with a symmetric matrix of matching chances $M = [m_{ij}]$. With $n = 2$ types, we can see that the steady-state equation (20) reduces to the vector equation $\delta\ell = \delta u + \rho A(m, u)$, where

$$A(m,u) = \begin{bmatrix} m_{11}u_1^2 + m_{12}u_1u_2 \\ m_{21}u_1u_2 + m_{22}u_2^2 \end{bmatrix}$$

We want to invert this, and express u as a function of M. Differentiating, we discover the directional derivative:

$$D_uA(m,u) = \begin{bmatrix} 2m_{11}u_1 + m_{12}u_2 & m_{12}u_1 \\ m_{21}u_2 & m_{21}u_1 + 2m_{22}u_2 \end{bmatrix}$$

Towards inverting this, observe that $D_uA(m, u)$ is a positive definite matrix. For if we consider any $x' = (x_1, x_2) \in \mathbb{R}^2$, since $m_{21} = m_{12}$ and $m, u \geq 0$, we have:

$$x'D_uA(m,u)x = (2u_1x_1^2)m_{11} + (u_2x_1^2 + (u_1+u_2)x_1x_2 + u_1x_2^2)m_{12} + (2u_2x_2^2)m_{22} \geq 0$$

exploiting symmetry $m_{ij} \equiv m_{ji}$, because:

$$u_2x_1^2 + (u_1+u_2)x_1x_2 + u_1x_2^2 = (\sqrt{u_2}x_1 + \sqrt{u_1}x_2)^2 + 4x_1x_2(\sqrt{u_2} - \sqrt{u_1})^2 \geq 0$$

So the derivative $D_uA(m, u)$ is positive definite, and thus so too is the sum $\delta I + \rho D_uA(m, u)$. Finally, any positive definite matrix is invertible. Then by the Implicit Function Theorem, the steady-state equation (20), written

[15] Shimer and Smith call this the "fundamental matching lemma". In a clever piece of sleuthing (personal communication), Georg Noldeke has since found that this also arises in chemical reaction networks (Craciun and Feunberg, 2005). Noldeke claims that Martin Feinberg has lecture notes from 1979 which cover the quadratic matching technology.

$B(m,u)=0$, implicitly well-defines a smooth function $u=T_U(M)$ since the derivative $B_u(m,u)\equiv \delta I+\rho D_u A(m,u)$ is smooth and invertible.

The above existence proof logic suggests the algorithm for computing the equilibrium — use tatonnement via value adjustment, analogous to Walrasian wage adjustment

6 Explicitly Costly Search

There are two traditional search friction stories that one might entertain. On the one hand, one might imagine that search takes a while, and thus the cost of search is foregone match payoffs. This is what we have explored — the *time cost of search* story. But the original price search model imagined that the act of search was critical cost, as happens when searching for a low gas price. This is *explicitly costly search.* It is best modeled in discrete time, and arguably not the most plausible description of search for match partners. This is fundamentally an easier model to analyze, since the cost of search is the same for all partners.

Table 1: **Summary of the Assortative Matching Literature.** The left columns owe to Becker (1973), where TU means that wages are competitively set. The top middle entry is found in Morgan (1995), while Alp (2006) is the middle bottom entry. The right bottom entry is Shimer and Smith (2000). Smith (2006) derives the top right entry.

	No Search	Fixed Cost Search	Opportunity Time Cost Search
NTU	$f_2>0$	$f_2>0, f_{12}>0$	$f_2>0, (\log f)_{12}>0$
TU	$f_{12}>0$	$f_{12}>0$	$f_{12}>0, (\log f_1)_{12}>0, (\log f_{12})_{12}>0$

This table ignores the boundary conditions.