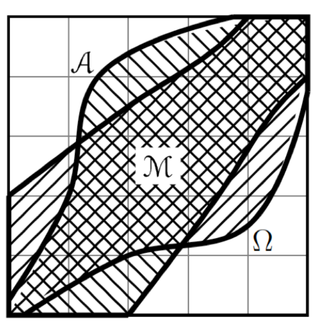
Assortative Matching and Search

- We explore two cases:

1. NTU (nontransferable utility) where payoffs are exogenously fixed
2. TU (transferable utility) where payoffs are reflect an endogenous surplus split

Matching

- flow payoffs $f(x, y)$
- *Acceptable types* $A(x) \subseteq [0,1]$
- inverse *opportunity set* $\mathbf{\Omega}(x) = \{y \mid [0,1] | x \in A(y)\}$
- *mutually agreeable matches* $M(x) = A(x) \cap W(x)$
- Everyone is both consumer and consumption good alike in a matching model



Search Frictions

- potential partners arrive at some fixed *rendezvous rate* ρ > 0
  ➔ with chance ρ *dt* in any infinitesimal length *dt* interval
- *Interest rate* r>0
- To secure a steady state, maybe matches dissolve at fixed rate δ > 0
  ➔ chance δ *dt* in any small *dt* interval
- The model is the same if we multiple (ρ,r,δ) by any k>0

Bellman Values

- expected present value *V*(*x*) of payoffs to *x* when unmatched
- expected present value *V*(*x*|*y*) of payoffs to *x* when matched with *y*
- *acceptable types y* in *A(x)* obey *V*(*x*|*y*) ≥ *V*(*x*)
- expected surplus *s*(*x*|*y*) = *V*(*x*|*y*) - *V*(*x*) to x of matching withy in A(x)
- *U*(*y*) is the stationary cdf of unmatched individuals

$$rV(x) = \rho \int_{y \in M(x)} [V(x \mid y) - V(x)] U'(y) dy$$

Bellman Equations

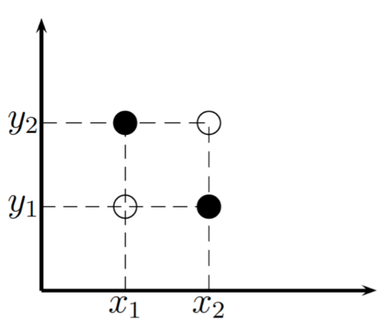
- average present values
  - $v(x) = rV(x)$
  - $v(x|y) = rV(x|y)$
- $v(x|y)$ weights
  - An initial flow payoff $f(x, y)$
  - An arrival rate δ of a capital loss of $v(x)$ - $v(x|y)$

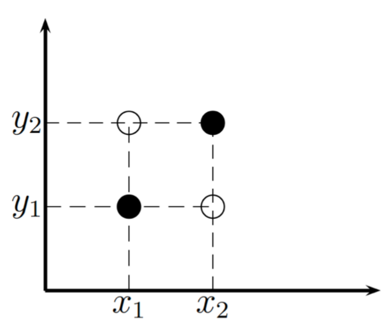
$$v(x \mid y) = f(x, y) + (\delta / r)[v(x) - v(x \mid y)].$$

$$v(x) = (\rho / r)\int_{y \in M(x)} [v(x \mid y) - v(x)]U'(y)dy.$$

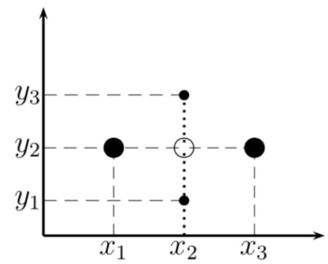
Assortative Matching: PAM vs NAM

- PAM / NAM = *negative / positive assortative matching*
- PAM: If mixed high & low types are matched, (x_1, y_2) and (x_2, y_1), with $x_1 < x_2$ and $y_1 < y_2$, then so are likes (x_1, y_1) and (x_2, y_2)
- NAM is the opposite





PAM or NAM ➔ Convex Matching Sets



- This insight now offers us a simpler way to establish sorting
- For convex matching sets imply $M(x) = [\theta(x), \psi(x)]$, with a quasi-convex lower bound $\theta(x)$, and quasi-concave upper bound $\psi(x)$.
- So PAM iff the lower and upper bounds are weakly increasing
- This gives a direct recipe for deducing PAM

NTU matching

- Assume monotone preferences, with $f_2 > 0$, so that in a frictionless setting, the Gale-Shapley stable outcome is PAM.
- With search frictions, intuitively, the acceptance set is $A(x) = [\theta(x),1]$ for some cutoff partner $\theta(x)$ reminiscent of a reservation wage
- The opportunity set is therefore $W(x) = \{y \mid [0,1] | x \geq \theta(y)\}$
- PAM iff $\theta(x)$ is nondecreasing (higher types are "choosier")
- Optimal matching requires that inside option pays the expected outside option: $f(x,\theta(x)) = v(x)$

Block Segregation

- Assume the unmatched status has flow payoff zero
- If either $f(x, y)=h_1(x)h_2(y)$, such as when $f(x, y) = y$, then $f(x', y) = [h_1(x')/h_1(x)]f(x, y)$ and thus utility of x and x' are
- Everyone wishes to match with the highest type *x* = 1.
- Faced with search frictions, her optimal reservation partner is $\theta(1)< 1$
- Then everyone in the interval $[\theta(1), 1]$
  - shares type 1's opportunity set
  - Has the same cardinal preferences.
  - Ipso facto, they will choose the same cutoff partner type $\theta(1)$

Preferences Leading to Block Segregation

- iterating, what unfolds is a unique partition of [0,1] with class boundaries $\theta(1)> \theta(2)>\ldots$
- There are finitely many boundaries exactly when $f(0, 0) > 0$
- Notice that $f(x, y) = h_1(x)h_2(y)$ iff f is log-modular
- What if we consider strictly LSPM functions?
- Assume a differentiable threshold $\theta(x)$.
- Differentiating the optimality equation $\log v(x) = \log f(x,\theta(x))$ in *x* yields

$$\frac{v'(x)}{v(x)} = \frac{f_1(x,\theta(x))}{f(x,\theta(x))} + \theta'(x)\frac{f_2(x,\theta(x))}{f(x,\theta(x))}.$$

Optimal Choosiness

- δ = 0 implies $v(x|y) = f(x,y)$
- The *policy value* $v_\theta(x)$ solves the recursion equation

$$v_\theta(x) = (\rho / r)\int_{y\geq\theta}[f(x,y) - v_\theta(x)]U'(y)dy = \frac{\rho\int_{y\geq\theta} f(x,y)U'(y)dy}{r + \rho[1-U(\theta)]}.$$

- The partial derivative of the policy value $v_\theta(x)$ in θ vanishes at θ = θ(x)
- Differentiating log $v(x)$ = log $v_{\theta(x)}(x)$ in x, the Envelope Theorem gives:

$$\frac{v'(x)}{v(x)} = \frac{\int_{y\geq\theta(x)} f_1(x,y)U'(y)dy}{\int_{y\geq\theta(x)} f(x,y)U'(y)dy}.$$

How Choosiness Varies in Type

$$\frac{f_1(x,\theta(x))}{f(x,\theta(x))} + \theta'(x)\frac{f_2(x,\theta(x))}{f(x,\theta(x))} = \frac{\int_{y\geq\theta(x)} f_1(x,y)U'(y)dy}{\int_{Y\geq\theta(x)} f(x,y)U'(y)dy}.$$

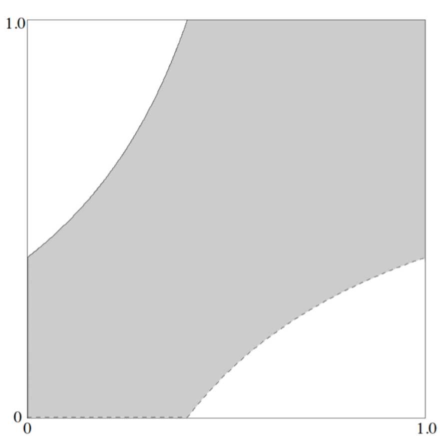
- The inequality 3/4 < 5/6 implies 3/4 < (3 + 5)/(4 + 6) < 5/6
- Similarly, if $a(t)$, $b(t)$ > 0 are smooth functions, and $[a(t)/b(t)]'$ > 0, then

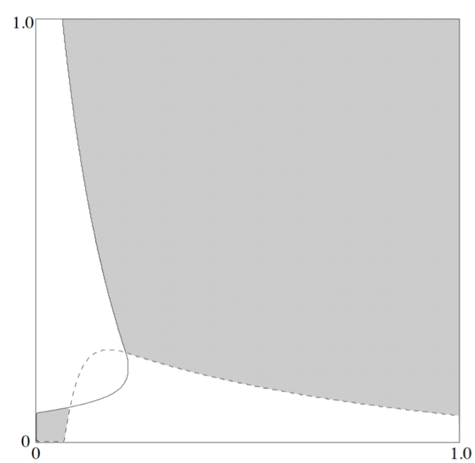
$$\frac{a(t_0)}{b(t_0)} < \frac{\int_{t_0}^{t_1} a(t)dt}{\int_{t_0}^{t_1} b(t)dt} < \frac{a(t_1)}{b(t_1)} \qquad \forall\, t_0 < t_1.$$

Sorting in the Marriage Model with Transferable Utility and Search Frictions

Theorem 1 (PAM and NTU): Assume x earns $f(x, y) > 0$ in a match with y, where $f_2(x, y) > 0$. Then the equilibrium matching is block segregation if f is log-modular and is strict PAM if f is strictly LSPM.

PAM with $f(x,y)=e^{xy}$ and no PAM with $f(x,y)=x+xy+y$





Transferable Utility Matching

- Match surplus clearly determines a mutual matching decision with TU:

$$s(x,y) \equiv f(x,y) - v(x) - v(y) \geq 0 \quad \Leftrightarrow \quad y \in M(x).$$

- Equal surplus division ("Nash bargaining solution")

$$v(y \mid x) - v(x) = v(x \mid y) - v(y).$$

- Unmatched surplus can be rewritten as an integral of match surplus (recall time scale invariance of parameters):

$$v(x) = \frac{1}{2}\frac{\rho}{\delta + r}\int_{y \in M(x)} \big(f(x,y) - v(x) - v(y) \big) U'(y)dy.$$

Value and Marginal Value with TU

- The TU search model turns on solving functional equations like those for "potentials":

$$v(x) = \beta \int \max\langle 0, f(x,y) - v(x) - v(y) \rangle U'(y)dy.$$

- Since surplus vanishes at the edge of the matching set:

$$v'(x) = \beta \int \big(\max\langle 0, f_1(x,y) - v'(x) \rangle \big) U'(y)dy$$

$$= \frac{\beta \int_{y \in M(x)} f_1(x,y) U'(y)dy}{1 + \beta \int_{y \in M(x)} U'(y)dy}.$$

Solution Recipe

Claim 1: If matching sets are convex, then SPM implies PAM if $f_2(0, y)$ º 0.

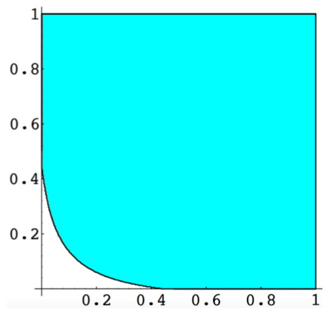
Claim 2: Matching sets are convex when own-marginal products f_1 are LSPM, and cross partials f_{12} are LSPM

Convex Matching and Supermodularity ➔ PAM

- If PAM fails then $\psi(x_1) > \psi(x_2)$ for some $x_2 > x_1$
- Since surplus is supermodular, as $s_{12} = f_{12} > 0$, type x_1 sees her match surplus rise more slowly in her partner's type than does x_2.
- Integrating $s_2(x, y)$ down from any given $y=\psi(x_1)$, the surplus of type x_1 is lower with every partner than the surplus of type x_2.
- But x_1 has a higher upper partner (yielding zero surplus) than type x_2.
- Hence, $v(x_1) > v(x_2)$
- Contradiction to f increasing

Using the Zero Marginal Product Condition

- Since $s_2(0, y) = f_2(0, y) - v'(y) = 0 - v'(y) < 0$, the highest surplus partner of x = 0 is zero.
- So the lower bound θ(*x)* is initially weakly increasing
- Convex matching sets have a quasi-convex lower bound function θ(*x*)
- Hence, the lower bound is everywhere weakly increasing



SPM ➔ Convexity for high types

- The match surplus of type *x* has slope $s_2(x, y) = f_2(x, y) - v'(y)$
- Assume all types match, so that matching set changes can be ignored
- $v'(x) = \gamma \; E_Y f_1(x, Y)$, for some γ<1
- γ rises in the rendezvous rate ρ and falls in the interest rate *r*

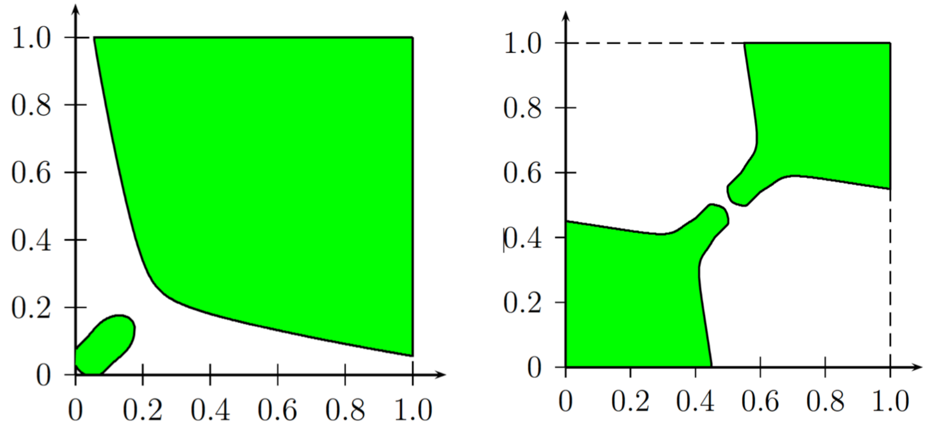
$$s_2(x,y) = f_2(x,y) - \gamma E_X f_2(X,y) > E_X[f_2(x,y) - f_2(X,y)]$$

- By SPM, $f_2(1, y) - f_2(x', y) > 0$ whenever $x' < 1$, so $s_2(1, y) > 0$
- So the highest types match surplus rises in their partner's type *y*

➔ Their matching set is a convex upper set in [0, 1]

SPM Need Not Guarantee PAM

- $f(x,y)=(x+y)^2$. and $f(x,y)=(x+y-1)^2$.



LSPM of Marginal Products ➔ Convexity for Low Types

- Now assume low types x
- An easy sufficient condition for quasi-concavity of match surplus is that its derivative downcrosses
- This would also ensure that the "ideal partner" for type x (maximal surplus, i.e. with $v'(y) = f_2(x, y)$) increases
- But $s_2(x, y) = f_2(x, y) - v'(y)$ downcrosses if $\gamma E_Y f_2(X,y)/f_2(x, y)$ increases
- This holds when $f_2(x, y)/f_2(0, y)$ increases in y, i.e. LSPM
- Notice how this explains the failure for $f(x,y)=(x+y)^2$.
- For $f_2(x, y)/f_2(0, y) = 1+x/y$ *falls* in y, since log f(x,y)=2 log(x+y) is LSBM

The Final LSPM Condition

- Observe that

$$f_2(x,y) - v'(y) = 0 \Rightarrow f_2(x,z) - v'(z) \le 0 \qquad \text{whenever } z > y.$$

- The first logic argues the premise fails for high types
- The second logic argues the implication holds for low types
- Is every type high enough for first logic, or low enough for second logic?

A Single Crossing Property for Gambles

- **Lemma**: If h(x,y)>0 and h_1>0 is LSPM. Assume E[h(X,y)]= h(x*,y). Then E[h(X,z)] ≥ h(x*,z) for all z ≥ y.
- This implies Diamond-Stiglitz (1973): *A global increase in the Arrow Pratt risk aversion coefficient lowers the certainty equivalence of any gamble.*
  - If greater y lowers risk aversion u(x,y) for money, then u is LSPM.
  - By the lemma, someone who is indifferent about a gamble at low y, is strictly willing to gamble at a higher y.
- In our search setting, set h=f_2 . Then h_1>0 is and h_1= f_{12} is LSPM.
- The lemma asserts that if the denominators coincide, then numerators are ordered:

$$\frac{f_2(x,z)}{f_2(x,y)} \le \frac{v'(z)}{v'(y)} = \frac{E_X f_2(X,z)}{E_X f_2(X,y)} \qquad \text{whenever} \quad z \ge y.$$

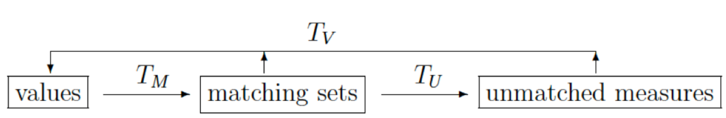
PAM in TU Search and Matching Models

Theorem 2 (PAM with TU): If *x* and *y* jointly produce symmetric output $f(x, y) > 0$, then the equilibrium TU matching obeys strict PAM if *f* is supermodular, f_1 and f_{12} are log-supermodular, and $f_2(0, y) = 0$ for all *y*.

Search Equilibrium Existence

- A search equilibrium is a triple (*v*, *M*, *u*)—namely,
  - the value function *v*
  - the matching set function *M*
  - the unmatched density *u*
- It obeys three functional equations: the last one is this steady-state condition, that the unemployment inflow balances the newly matching:

$$\delta[\ell(x) - u(x)] = \rho u(x) \int_{M(x)} u(y)dy.$$



Two Type Illustration of Logic

- With n = 2 types, the steady-state equation reduces to

$$\delta\ell = \delta u + \rho A(m,u) \quad \text{where} \quad A(m,\ u) = \begin{bmatrix} m_{11}u_1^2 + m_{12}u_1u_2 \\ m_{21}u_1u_2 + m_{22}u_2^2 \end{bmatrix}.$$

- By the Implicit Function Theorem, there is a unique and continuous solution y(x) to F(x,y)=0 if F_y is invertible. Thus, there is a continuous map m→u(m) provided I + ρ $D_uA(m, u)$ is invertible
- This holds because $D_uA(m, u)$ is a positive definite matrix, since

$$D_u A(m,u) = \begin{bmatrix} 2m_{11}u_1 + m_{12} & m_{12}u_1 \\ m_{21}u_2 & m_{21}u_1 + 2m_{22}u_2 \end{bmatrix}.$$

- and $x' D_u A(m,u)x = (2u_1x_1^2)m_{11} + (u_2x_1^2 + (u_1+u_2)x_1x_2 + u_1x_2^2)m_{12} + (2u_2x_2^2)m_{22}$ ≥0, as:

$$u_2x_1^2 + (u_1+u_2)x_1x_2 + u_1x_2^2 = (\sqrt{u_2}x_1 + \sqrt{u_1}x_2)^2 + 4x_1x_2(\sqrt{u_2} - \sqrt{u_1})^2 \geq 0.$$

Frictional vs Frictionless Sorting

	No Search	Fixed Cost Search	Opportunity Time Cost Search
NTU	$f_2 > 0$	$f_2 > 0, f_{12} > 0$	$f_2 > 0, (\log f)_{12} > 0$
TU	$f_{12} > 0$	$f_{12} > 0$	$f_{12} > 0, (\log f_1)_{12} > 0, (\log f_{12})_{12} > 0$

Steady State Dynamics in Chemistry Resemble those in a Search Model:

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"Life is an equilibrium state of synthesis and degration of proteins" - Yoshinori Ohsumi
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