The Comparative Statics of Sorting[*]

Axel Anderson[†] Lones Smith[‡]
Georgetown Wisconsin

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Abstract

We create a general and tractable theory of increasing sorting in pairwise matching models with transferable utility. Our partial order, *positive quadrant dependence*, subsumes Becker (1973) as the extreme cases with most and least sorting. It implies sorting by correlation of matched partners, or distance between partners. Our theory turns on *synergy* — the cross partial difference or derivative of match production. This reflects basic economic forces: diminishing returns, technological convexity, insurance, and match learning dynamics.

We prove that sorting increases if match synergy globally increases, and is also cross-sectionally monotone or single-crossing. Our theorems shed light on major economics sorting papers, affording immediate proofs and new insights. They open the door to fast predictions for new pairwise sorting models in economics.

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[†]Email: aza@georgetown.edu; web page: www.sites.google.com/view/axel-anderson

[‡]Email: lones.smith@wisc.edu; web page: www.lonessmith.com

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1 Introduction

Assortative matching is the allocational theme in the vast literature on decentralized matching. This finding has seen application in marriage, employment, partnerships, optimal assignment, and pairwise trade. Becker (1973) showed that it emerges when match types are complementary. The power of this conclusion is also its weakness — higher "men" match with higher "women," *without exception*. Since it is an ideal, how should we understand sorting deviations? For instance, with search frictions, Shimer and Smith (2000) deduced sorting under stronger complementarity assumptions. But they found that matching set are centered about Becker's frictionless match partner.

While search and information frictions undeniably distort sorting, match productivity is surely the major driving force for the *changes in who matches with whom*. How does sorting vary across economic environments? Currently, our only general matching theory is Becker's — but it allows just two conclusions: either positive or negative assortative matching. His premises are also restrictive: a globally positive or globally negative cross partial difference or derivative. Chade, Eeckhout, and Smith (2017) explore many natural and some well-cited economic matching settings where both assumptions fail. The lack of a predictive general theory that applies to such cases has greatly limited the analytic reach of the matching literature in economics. Theory papers have all explicitly solved for the matching, and this too has focused excessive attention on the extreme solvable cases of perfect positive or negative sorting.

This paper fills this void: We develop a tractable general theory of sorting changes in the frictionless pairwise matching model with either finitely many or a continuum of types and *transferable utility* (TU). *We provide comparative statics predictions for the planner's problem, or equivalently, equilibria, without ever solving for either.* We shed light on influential economics matching papers since Becker (1973) without everywhere complementary types, and relax this major match payoff restriction for future work.

We argue that the *positive quadrant dependence* (PQD) partial order captures an economic meaning of "more assortative" (**Lemma 1**). This stochastic order ranks matching measures by the mass in the southwest quadrant. Rising in the PQD order, (*i*) the average distance between matched types falls, (*ii*) the correlation of matched types increases, *and* (*iii*) the regression coefficient of women on their partners' types increases. In other words, our sorting comparative statics findings are of direct empirical relevance in economics. By contrast, we show in §B that *no coherent sorting theory can emerge based on covariance, correlation, or average distance between partners.*

To illustrate the PQD order, consider the six possible complete matchings among three men and three women (Figure 1). Each man matches with a weakly closer partner in PAM than in NAM1 or NAM3, in turn each closer than in PAM2 or PAM4, and

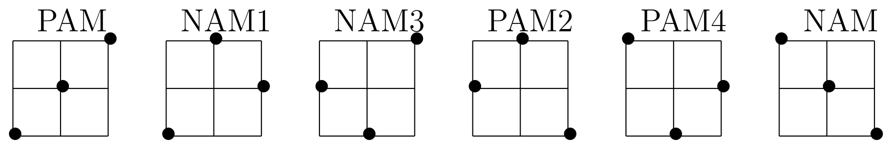

Figure 1: **Pure Matchings with 3 Types.** The possibilities are: negative and positive assortative matching (NAM and PAM), negative sorting in quadrants 1 and 3 (NAM1 and NAM3), and positive sorting in quadrants 2 and 4 (PAM2 and PAM4).

finally than in NAM. Meanwhile, the matchings NAM1 and NAM3, as well as PAM2 and PAM4, are incomparable. We thus have a partial order:

$$\text{PAM} \succ_{PQD} [\text{NAM1, NAM3}] \succ_{PQD} [\text{PAM2, PAM4}] \succ_{PQD} \text{NAM} \tag{1}$$

We introduce an assumption on production functions that is a local version of Becker's complementarity. *Synergy* is the cross partial difference of production with finitely many types, and the cross partial derivative with continuous types. To highlight its central role, we derive a formula rewriting total match output (4) as a weighted average of match synergy. This means that any matching characterization must turn on synergy. Becker (1973) deduces positive sorting for globally positive synergy, and negative sorting for globally negative synergy. We subsume a vast array of intermediate cases, where synergy changes sign, and so greatly expand the reach of matching theory.

Rephrasing Becker (1973), globally positive synergy implies assortative matching. So then is sorting greater with more synergistic production? A simple three-type example refutes this conjecture — the optimal matching oscillates between NAM1 and NAM3 as synergy rises in Figure 3. So on the one hand, an increasing sorting theory must build on production synergy, but on the other, sorting need not increase even if synergy everywhere does. This highlights the difficulty of our comparative statics goal.

Our sorting results posit that synergy changes sign at most once, from negative to positive, *and* is cross-sectionally well-behaved. **Proposition** 0 says that sorting increases if synergy increases, and is monotone in partner types. But synergy is not monotone in typical matching models. All results follow from **Proposition** 1, as it has the weakest assumptions. It asserts that sorting increases if (a) the total synergy on all unions of rectangular partner sets changes sign only from negative to positive, and (b) the total synergy on any type rectangle changes sign just once as it shifts north-east. Next, **Proposition** 2 replaces (b) with the total synergy on type line segments. Finally, **Proposition 3** is our easiest to check general sorting result, ideal for the continuum type matching models. It posits that synergy changes sign only from negative to positive, both over time and cross-sectionally. To ensure these pointwise conditions aggregate, we introduce a new *proportional upcrossing* condition. This local inequality ensures that positive synergy increases proportionately more than absolute negative synergy.[1] That monotone functions are proportional upcrossing proves **Proposition 0**.

Distributional shifts can also greatly impact sorting: A rise in high types of women may have profound macro consequences on sorting. **Corollary 2** repurposes our increasing sorting theory for type distribution shifts. We argue when type distribution shifts are formally equivalent to productive synergistic shifts, so that our theory applies.

To see how much we expand the predictive reach of matching theory, assume 100 men and 100 women. Becker (1973) applies for two synergy sign combinations. We encompass $2 \cdot 99^2$ sign combinations — and ones that specifically arise in applications.[2]

We prove **Proposition 1** for finitely many types, and deduce all other sorting results by corollary. The proof in §D.2 by induction on the number of types is a major part of the paper, and never solves for an optimum. Rather it chases down failures of the comparative static to the possible shift from NAM to the n-type version of NAM3.

Economic Applications of Our Theory. Becker's work has sparked a vast literature on the transferable utility matching paradigm. But his sorting conclusion requires complementary partner types, which fails in many recent explored models.

We argue in §6 that our theory sweeps in many old and new economic models:

1. The typical economic force of *diminishing returns* lowers synergy and so sorting.
2. Match synergy is greater for *"weakest link" technologies* — namely, where the lesser type impacts payoffs more. We argue that this force formally makes the technology more convex in types — just as $\min(x, y)$ is more convex than $x + y$.
3. The opposite case of a *"strongest link"* matching technology captures insurance — to wit, the greater type matters more. These technologies are least synergistic.
   - (a) Insurance has this flavor, since the higher type transfers money to the lesser — as with *adverse selection*. To showcase our theory, we consider Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending, and more strongly prove that sorting is monotone in the model parameters.[3]
   - (b) Insurance also optimally arises with *moral hazard.* In a principal-agent matching model, Serfes (2005) found that negative sorting — more risk averse agents with safer projects — arises for a low effort disutility, but positive sorting for a high disutility. Our theory allows a quick stronger characterization that sorting globally increases in the disutility of effort.

[1] We prove in a multi-dimensional extension of Karlin and Rubin's 1956 upcrossing preservation. This is related to the problem of aggregating the single-crossing property (Quah and Strulovici, 2012).

[2] For our upcrossing assumption, a sign change can occur after any of 99 men and 99 women.

[3] Legros and Newman (2002) showed that supermodular production does not induce supermodular match payoffs with imperfect credit constraints. Our nowhere decreasing theory applies to their model.

4. Our theory also speaks to dynamic matching with evolving types. In a model of *mentor-protege workplace learning*, the greater type lifts the lesser. This strongest link technology lowers match synergy.[4] Next, assume new agents inherit traits from old ones. *Recessive gene dynamics* have the flavor of weakest link matching, and are more synergistic, while *dominant gene dynamics* are less synergistic.
5. Finally, our sorting theory applies to *double auctions*, building on the fact that Becker yields negative sorting: low cost sellers trade with high value buyers.

Longer proofs and new monotone comparative statics results are in Appendices.

2 Becker's Marriage Model and Planner's Result

Our model is standardly adapted from Becker and the pairwise matching literature with two groups (men and women, firms and workers, buyers and sellers) or one (partnership model). To subsume both finite and continuum type models, we posit a unit mass of "women" and "men" with respective *types* $x, y \in [0,1]$ and cdfs G and H. We assume absolutely continuous type distributions G and H, and for the finite type model, G and H are discrete measures with equal weights on female types $0 \le x_1 < x_2 < \cdots < x_n \le 1$ and male types $0 \le y_1 < y_2 < \cdots < y_n \le 1$. In the finite types case, assume equal sized women and men relabeled as $i, j \in \{1, 2, \ldots, n\}$, respectively.

We assume a C^2 production function $\phi > 0$, so that types x and y jointly produce $\phi(x, y)$. In the finite type model, the output for match (i, j) is $f_{ij} = \phi(x_i, y_j) \in \mathbb{R}$. Production is *supermodular* or *submodular* (SPM or SBM) for all $x' < x''$ and $y' < y''$ if:

$$\phi(x', y') + \phi(x'', y'') \ge (\le)\ \phi(x', y'') + \phi(x'', y') \tag{2}$$

Strict supermodularity (respectively, strict SBM) asserts globaly strict inequality in (2). And production is *modular* (or additive) when (2) always holds with equality.

Like Becker's, our theory does not explore an extensive margin (whether to match). A *matching* is a bivariate cdf $M \in \mathcal{M}(G, H)$ on $[0,1]^2$ with marginals G and H. In the finite type case, G and H put equal unit weight on $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. A finite matching is a nonnegative matrix $[m_{ij}]$, with cdf $M_{i_0 j_0} = \sum_{1 \le i \le i_0, 1 \le j \le j_0} m_{ij}$, and unit marginals $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0 j}$ for all men i_0 and women j_0. In a *pure matching*, $[m_{ij}]$ is a matrix of 0's and 1's, with everyone matched to a unique partner.

There are two perfect sorting flavors. In *positive assortative matching* (PAM), any woman type of x at quantile $G(x)$ pairs with a man of type y at the same quantile $H(y)$,

[4] Bayesian updating need not inherit supermodularity in Anderson and Smith (2010). Supermodularity is often not preserved in our work with evolving human capital (Anderson and Smith, 2012).

and thus the match cdf is $M(x, y) = \min(G(x), H(y))$. In *negative assortative matching* (NAM), complementary quantiles match, and so $M(x, y) = \max(G(x) + H(y) - 1, 0)$. Matched types are *uncorrelated* given uniform matching, and so $M(x, y) = G(x)H(y)$.

The *partnership* (or unisex) model is a special case where types x and y share a common distribution, $G = H$, the production function ϕ is symmetric ($\phi(x, y) = \phi(y, x)$), and so too is the optimal matching distribution $M(x, y) \equiv M(y, x)$. In this case, PAM simply reduces to the matching $y = x$, or match with the same type.

A social planner maximizes total match output, namely $\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(\theta)m_{ij}$ with finite types, or more generally $\int_{[0,1]^2} \phi(x, y|\theta)M(dx, dy)$, where we index output $\phi(x, y|\theta)$ by a (*often suppressed*) *state* $\theta \in \Theta$, a partially ordered set. As optimal matching $\mathcal{M}^*(\theta)$ solves:

$$\mathcal{M}^*(\theta) = \arg \max_{M \in \mathcal{M}(G,H)} \int_{[0,1]^2} \phi(x, y|\theta)M(dx, dy) \tag{3}$$

Needless to say, the optimal matching is of little value without a welfare theorem. But after proving existence of $\mathcal{M}^*$, Gretsky, Ostroy, and Zame (1992) then decentralize it as a competitive equilibrium. So *all our theory applies to equilibrium sorting patterns.*

Problem (3) has been solved in just three general cases: All feasible matchings are optimal with additive production, while Becker solved for SBM and SPM production:[5]

Becker's Sorting Result. *Given SPM (SBM) production ϕ, PAM (NAM) is an optimal matching. Given strict SPM (SBM), these pairings are uniquely optimal.*

For an intuition, assume finitely many types and SPM (2). A maximum of (3) obviously exists. To see uniqueness, not that if ever women $x' < x''$ and men $y' < y''$ are negatively sorted into matches (x', y'') and (x'', y'), then total output is raised by rematching them as (x', y') and (x'', y''). A proof for any number of types is in §3.

Without SBM or SPM, solving the general social planner's problem (3) is a hard open question. We bipass this, and ask how the optimal matching $\mathcal{M}^*(\theta)$ changes in θ. We derive its comparative statics in θ when output $\phi(x, y|\theta)$ is neither SPM or SBM. Hereafter, a *time series* property suggestively refers to changes in the state θ,[6,7] and a *cross-sectional property* to production changes over the type space. We then apply our finding in several matching models across economics, without SPM or SBM output.

Throughout the paper, we present finite type and continuum type results together, as synergy is a common theme. We draw both intuition and our overall inductive proof logic from the finite type case, and derive the continuum type results by taking limits.

[5]Koopmans and Beckmann (1957) decentralize the solution as a competitive equilibrium assuming TU. Legros and Newman (2007) show that some NTU models can be mapped into the TU paradigm.

[6]The term time-series is used to distinguish variation *across* matching markets from changes across types within a market. The state could also represent geographic differentiation in matching markets.

[7]Equivalently, our theory compares sorting for two production functions ϕ_1 and ϕ_2 (i.e. $\theta_1 < \theta_2$).

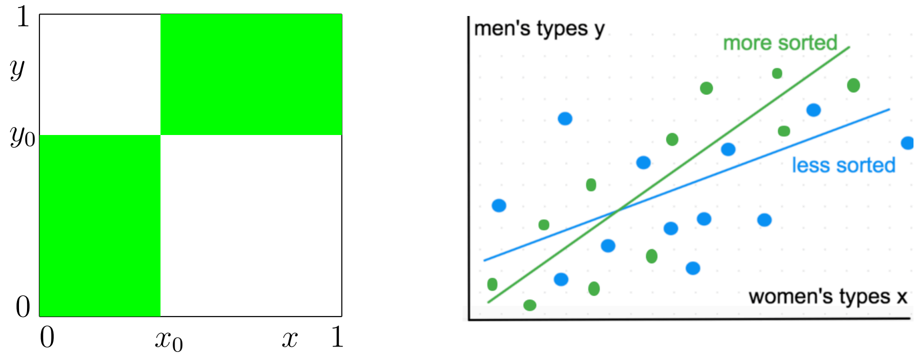

Figure 2: **PQD Order.** Left: PQD increases for cdfs on $[0,1]^2$ raise the probability mass on all lower left rectangles (corners $(0,0)$ and (x_0, y_0)), and so on all upper right rectangle (corners (x_0, y_0) and $(1,1)$). Right: We schematically depict Lemma 1(c).

3 Synergy and Sorting Measurement

A. Synergy. We now introduce a local measure of Becker's restrictive assumption supermodularity. In finite type models, *synergy* is the cross partial difference of output:

$$s_{ij}(\theta) = f_{i+1j+1}(\theta) + f_{ij}(\theta) - f_{i+1j}(\theta) - f_{ij+1}(\theta)$$

The central importance of synergy is revealed by expressing match output as a weighted sum of match synergies. Lemma 2 in §A doubly sums match output by parts:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} [f_{nj+1} - f_{nj}]\, j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij} M_{ij} \qquad (4)$$

So any two production functions with identical synergies share the optimal matching. For instance, if production is linear, then synergy vanishes, and all match distributions yield the same output. Becker focused on the SPM case with globally nonnegative synergy. We henceforth link changes in the synergy to changes in the optimal matching.

B. Positive Quadrant Dependence (PQD). *PQD* is a partial order on bivariate probability distributions $M_1, M_2 \in \mathcal{M}(G,H)$. Say that matching measure M_2 is *PQD higher than* M_1, or $M_2 \succeq_{PQD} M_1$, if $M_2(x,y) \geq M_1(x,y)$ for all types x, y. So M_2 puts more weight than M_1 on all lower (southwest) orthants. As M_1 and M_2 share marginals, M_2 puts more weight than M_1 on all upper (northeast) orthants too (Figure 2).

As (1) notes, PQD only partially orders the six possible pure matchings on three types. It helps to note that all match cdf's are sandwiched above NAM and below PAM:

$$\max(G(x) + H(y) - 1, 0) \leq M(x,y) \leq \min(G(x), H(y)) \qquad (5)$$

The second inequality says that the mass of matched men and women in $[0,x] \times [0,y]$

is at most the supply of men or women. The first inequality is more subtle — or $1 - M(x,y) \leq \min(1 - G(x) + 1 - H(y), 1)$, says the mass of men and women in $[0,x] \times [0,y]$ not matched is at most the supply of men plus the supply of women.

Becker's Result follows from the bounds (5) and either summation (4) with finitely many types, or Lemma 4 in §F.2 with a continuum of types. For SPM output implies all $s_{ij} \geq 0$, and so by (4) output is highest when the cdf $M(x,y)$ is maximal. So PAM dominates all other matchings. Similarly, SBM implies globally nonpositive synergy, $s_{ij} \leq 0$, and thus output is highest when the match cdf $M(x,y)$ is minimal, namely, for NAM. More generally, the PQD and SPM orders coincide in $\mathbb{R}^2$, i.e. *increases in the PQD order increase (reduce) the total output for any SPM (SBM) function* ϕ:[8]

$$M_2 \succeq_{PQD} M_1 \quad \Leftrightarrow \quad \int \phi(x,y) M_2(dx,dy) \geq \int \phi(x,y) M_1(dx,dy) \qquad \forall \phi \text{ SPM} \tag{6}$$

The PQD sorting measure implies some more typical economically relevant measures for measured traits $u(x)$ and $v(y)$ of women x and men y, increasing in x and y:

Lemma 1. *Fix increasing functions u and v. Given a PQD order upward shift:*
(*a*) *the average geometric distance $E[|u(X) - v(Y)|^\gamma]$ for matched types falls, if $\gamma \geq 1$;*
(*b*) *the covariance $E_M[u(X)v(Y)] - E[u(X)]E[v(Y)]$ across matched pairs rises;*
(*c*) *the coefficient in a linear regression of $v(y)$ on $u(x)$ across matched pairs rises.*

Proof of (*a*): By inequality (6) it suffices that $|u(x) - v(y)|^\gamma$ is SBM for all $\gamma \geq 1$. Since $-\psi(u - v)$ is SPM for all convex ψ, by Lemma 2.6.2-(*b*) in Topkis (1998), we have $-|u - v|^\gamma$ SPM for all $\gamma \geq 1$. So, $|u(x) - v(y)|^\gamma$ is SBM for all increasing u and v.

Proof of (*b*): Since the marginal distributions on X and Y is constant for all $M \in \mathcal{M}(G,H)$, and $u(x)v(y)$ is supermodular for all increasing u and v, the covariance $E_M[XY] - E[X]E[Y]$ between matched types increases in the PQD order by (6).

Proof of (*c*): The coefficient $c_1 = cov(u(X)v(Y))/var(v(X))$ in the univariate match partner regression $v(y) = c_0 + c_1 u(x)$ increases in the PQD order, by part (*b*). □

PQD is an *ordinal sorting ranking*, like PAM — not dependent on type sizes.[9] So if educational sorting rises in the PQD order, then this holds regardless of whether it is measured in highest degree, schooling years, or even log years. But for non-PQD comparable matching changes, whether these cardinal sorting measures rise or fall depends on the type sizes. The sorting conclusion can easily reverse if the choice of

[8] Lehmann (1973) introduces the PQD order, and Cambanis, Simons, and Stout (1976) prove that the PQD and SPM orders are equivalent in $\mathbb{R}^2$.

[9] Posit a uniform type distribution on $[0,1]$. Assume that any $x \leq 1/2$ matches with $x + 1/2$. Since it is increasing *on the domain of larger match partners*, Legros and Newman (2002) call this matching "monotone". Yet this matching maximizes the average distance between partners.

Match Payoffs

	x_1	x_2	x_3
y_3	9	**14**	18
y_2	5	2	**14**
y_1	**1**	5	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	16	**24**
y_2	**5**	3	16
y_1	1	**5**	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	**20**	30
y_2	5	6	**20**
y_1	**1**	5	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	22	**36**
y_2	**5**	7	22
y_1	1	**5**	9

Cross Partial Differences of Match Payoffs

	x_1x_2	x_2x_3
y_2y_3	8	-8
y_1y_2	-7	8

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	9	-5
y_1y_2	-6	9

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	10	-4
y_1y_2	-3	10

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	11	-1
y_1y_2	-2	11

Figure 3: **Sorting Need Not Rise in Synergy.** Top: the unique efficient matching alternates between NAM1 and NAM3. Bottom: match synergies (cross payoff differences) strictly increase as we move right, but sorting does not PQD rise. Sorting by two common cardinal measures can move contrarily. If $x \in \{1, 2, 3\}$ and $y \in \{0.5, 1.8, 3\}$, *NAM1 to NAM3 shifts reduce both covariance* and *average distance between partners.*

cardinal measure changes (see Figure 3 and Appendix B). Lemma 1 assures us that *our theory yields a common implication for changes in all such cardinal sorting statistics.*

4 Increasing Sorting

Becker implies that globally negative synergy respectively leads to NAM, and globally positive synergy leads to PAM. One might then surmise that sorting increases if synergy everywhere increases. This natural conjecture fails: In Figure 3, synergy strictly increases at each step, and yet the uniquely optimal matching oscillates between the non PQD-comparable NAM1 and NAM3. But it is true that the optimal matching *cannot fall in the PQD order if synergy globally rises.*[10] For if matching M' payoff dominates another PQD lower matching M, so that $M' \geq M$, then it still dominates if synergy is higher: for by (4), the payoff gap $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(M'_{ij} - M_{ij})$ is nonnegative.

The reason for the increasing sorting failure in Figure 3 is that the space of matching cdf's is not a lattice (Müller and Scarsini, 2006), invalidating monotone comparative statics. We surmount this challenge by adding *cross-sectional* assumptions on synergy.

4.1 Strictly Monotone Synergy in Types

First consider the simplest case: *synergy is (strictly) monotone in types* if synergy is either non-decreasing (increasing) or non-increasing (decreasing) in (x, y). Say that *sorting increases* in θ if $M_2 \succeq_{PQD} M_1$ for all $M_1 \in \mathcal{M}^*(\theta_1)$, $M_2 \in \mathcal{M}^*(\theta_2)$ and $\theta_2 \succeq \theta_1$.

[10] An online Appendix F.1 derives a more general theory of comparative statics on posets, and proves this known math result that the optimal matching (as in Figure 3) *never falls* in the PQD order.

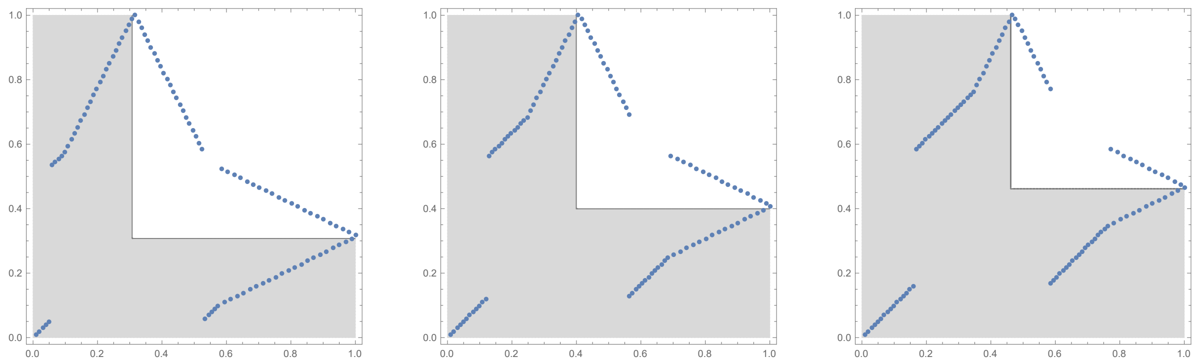

Figure 4: **Non-pure Matching Example.** We numerically depict the matching support for the synergy function $\alpha - \beta \min\{x_i, x_j\}$. *All matching plots depict optimally matched (blue) pairs for a uniform distribution on a finite* 100×100 *matching array.* In each graph, synergy is positive (negative) on the shaded (unshaded) regions. Left to right plots assume $(\alpha, \beta) = (0.4, 1.3)$, $(0.4, 1)$, and $(0.6, 1.3)$.

Proposition 0. *Assume synergy is non-decreasing in* θ. *Sorting is increasing in* θ *for:* (a) *generic finite type models if synergy is monotone in types and* (b) *continuum types model if synergy is strictly monotone in types.*

To illustrate this first sorting result, consider the synergy function $\phi_{12}(x, y) = \alpha - \beta \min\{x, y\}$ — since it alone fixes the optimal matching, by (4). Synergy is monotone in types (non-decreasing when $\beta \leq 0$ and non-increasing when $\beta \geq 0$), and increases in α and falls in β. By Proposition 0, sorting increases in α and falls in β (Figure 4).

Increasing sorting emerges despite potential complexity of the optimal matching. For instance, the optimal matching is not simply described by a cutoff in the type space with PAM above, and NAM below, this cutoff (or vice versa). The matching here switches back and forth between locally positive and locally negative sorting. This finite type plots also suggests that the optimal matching need not be pure (one-to-one) in the continuum limit. But none of our continuum type sorting results require purity.

4.2 One-Crossing Rectangular Synergy in Types

While the conditions in Proposition 0 are quick to check, they do not hold in many economic applications. We instead derive and prove stronger result with a weaker and more commonly met single-crossing premise: A function Υ is *upcrossing in* t[11] on a partially ordered set T if $\Upsilon(t) \geq (>)0$ implies $\Upsilon(t') \geq (>)0$ for all $t' \succeq t$, *downcrossing in* t if $-\Upsilon$ is upcrossing, and *one-crossing in* t if it is upcrossing *or* downcrossing. Strict

[11]The single crossing property usually implies a two dimensional functional domain. To avoid this confusion, and clarify the direction, we instead use the suggestive terms *upcrossing* and *downcrossing*.

versions of these conditions require that weak inequalities imply strict inequalities. For example, Υ is *strictly upcrossing* if $\Upsilon(t) \geq 0$ implies $\Upsilon(t') > 0$, for all $t' > t$.

Index type space *rectangles* $[i_1, i_2] \times [j_1, j_2]$ by opposite corners, $r \equiv (i_1, j_1, i_2, j_2)$. *Rectangular synergy* then sums the synergy $s_{ij}(\theta)$ over all its smallest rectangles:

$$S(r|\theta) \equiv \sum_{i=i_1}^{i_2-1} \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) = f_{i_1 j_1}(\theta) + f_{i_2 j_2}(\theta) - f_{i_1 j_2}(\theta) - f_{i_2 j_1}(\theta)$$

This coincides with the economic notion of a "sorting premium", or the surplus from positively sorting couples $(i_1, j_1) < (i_2, j_2)$ vs. negatively sorting them as (i_1, j_2) and (i_2, j_1). For a type continuum, $S(R|\theta) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y|\theta) dx dy$ for $R = (x_1, y_1, x_2, y_2)$.

Summed rectangular synergy adds synergy on any finite set of disjoint rectangles: $\{r_k\}$ with finite types, or $\{R_k\}$ with a continuum of types. So equipped, our key time-series assumption asserts that summed rectangular synergy is upcrossing in θ. Easily, summed rectangular synergy is upcrossing in θ if synergy is non-decreasing in θ.

Our first cross-sectional assumption uses the northeast partial order on rectangles: $r \succeq_{NE} r'$, if diagonally opposite corners of r are weakly higher than r'. Rectangular synergy is *one-crossing* in types if $S(r|\theta)$ is upcrossing (downcrossing) in r, for all θ.

Proposition 1. *Assume (A1)* summed rectangular synergy is upcrossing in θ *and (B1)* rectangular synergy is one-crossing in types. *If there is a unique optimal matching at $\theta_2 \succ \theta_1$, then sorting is PQD higher at θ_2 than θ_1.*

For finite type models, the optimal matching is generically unique by Koopmans and Beckmann (1957). We prove uniqueness in continuum type models in §D.4.

Proposition 1 is our key result. That rectangular synergy is one-crossing in types precludes the example in Figure 3, where sorting was nonmonotone. Appendix §D.2 gives the tricky induction proof on the number of types. Here, *we show how time series and cross-sectional logic jointly rule out anything but a PQD rise with three types.*

Three-Type Proof: Step 1. The optimal matching cannot PQD fall. Consider PAM4 and NAM. Now, $(2, 1)$ and $(3, 2)$ are matched pairs under PAM4, while NAM sorts them as $(2, 2)$ and $(3, 1)$. So the PAM4 payoff exceeds the NAM payoff by rectangular synergy for women $2, 3$ and men $1, 2$, i.e. by $s_{21}(\theta)$. If NAM and PAM4 are uniquely optimal respectively at states $\theta'' \succ \theta'$, then $s_{21}(\theta'') < 0 < s_{21}(\theta')$, violating upcrossing summed rectangular synergy. The payoff difference between any two PQD ranked 3-type matchings (1) is the summed rectangular synergy for some set of couples.

Three-Type Proof: Step 2. Next, we rule out non-PQD comparable shifts, namely, NAM1 to NAM3, and PAM2 to PAM4, or vice versa. Figure 5 traces the logic ruling out these two transitions, using both time series and cross-sectional assumptions.

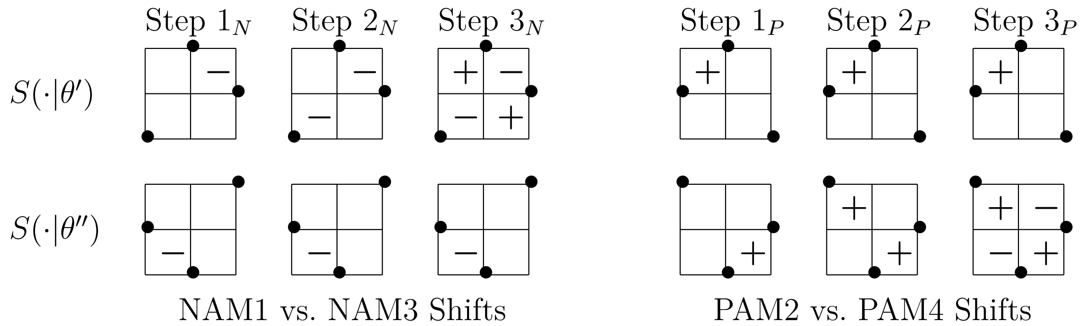

Figure 5: **Proof Synopsis for $n = 3$ Types: Precluding PQD-Unranked Shifts.** Posit nonzero synergies. We rule out NAM1 at θ' and NAM3 at θ'', or PAM2 at θ' and PAM4 at θ''. The synergy signs in Steps 1_N and 1_P reflect local optimality. Step 2_N deduces $s_{11}(\theta') < 0$ via upcrossing synergy from θ'' to θ'. Given PAM on rectangles $r = (1,1,2,3), (1,1,3,2)$ at θ', local optimality implies $S(r|\theta') > 0$. As rectangular synergy sums synergies, synergy signs in Step 3_N follow. So $S(r|\theta')$ is not one-crossing in r, a contradiction. Next, Step 2_P deduces $s_{12}(\theta'') > 0$ via upcrossing synergy from θ' to θ''. Given NAM on rectangles $r = (1,1,2,3), (1,1,3,2)$ at θ', local optimality implies $S(r|\theta') < 0$. Since rectangular synergy is the sum of synergies, we can fully sign s_{ij}. This sign pattern in Step 3_P violates $S(r|\theta'')$ one-crossing in r, a contradiction.

4.3 One-Crossing Marginal Rectangular Synergy in Types

We now provide a stronger, but easier to check, cross-sectional assumption to deliver increasing sorting. Specifically, the *x-marginal rectangular synergy* $\Delta_i(i|j_1, j_2)$ is the sum of synergy over men in the interval $[j_1, j_2 - 1]$ and the *y-marginal rectangular synergy* $\Delta_j(j|i_1, i_2)$ is sum of synergy over women in the interval $[i_1, i_2 - 1]$, i.e.:

$$\Delta_i(i|j_1, j_2, \theta) \equiv \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) \qquad \text{and} \qquad \Delta_j(j|i_1, i_2, \theta) \equiv \sum_{i=i_1}^{i_2-1} s_{ij}(\theta)$$

For a type continuum, the marginal rectangular synergy is an integral $\Delta_x(x|y_1, y_2, \theta) \equiv \int_{y_1}^{y_2} \phi_{12}(x, y)dy$ or $\Delta_y(y|x_1, x_2, \theta) \equiv \int_{x_1}^{x_2} \phi_{12}(x, y)dx$. These sums and integrals are *one-crossing* if they are respectively both upcrossing or both downcrossing in x and y.

Proposition 2. *Assume (A1)* summed rectangular synergy is upcrossing in θ *and (B2)* marginal rectangular synergy is one-crossing. *Sorting rises in θ in generic finite type, or continuum, types models with a strictly one-crossing marginal rectangular synergy.*

In §D.4, we integrate one-crossing marginal rectangular synergy (B2) to deduce one-crossing rectangular synergy (B1). Hence, *Proposition 1 implies Proposition 2.*

Next, we apply optimal transport theory to establish that the continuum optimal matching is unique when marginal rectangular synergy is strictly one-crossing.

4.4 Purely Local Assumptions on Synergy

We now develop a fully local approach to synergy aggregation that simultaneously secures the needed cross-sectional and time series conditions in Proposition 2.

We posit an extra assumption to ensure that any one-crossing synergy property aggregates to rectangles. While we wish to apply it to a signed synergy function, we take inspiration from log-supermodularity, since it is preserved by integration (Karlin and Rinott, 1980). Denote by $f^+ \equiv \max(f,0)$ and $f^- \equiv -\min(f,0)$ the positive and negative parts of a function f. With a continuum of types, *synergy is proportionately upcrossing* if:

$$\phi_{12}^-(z \wedge z',\theta)\phi_{12}^+(z \vee z',\theta') \geq \phi_{12}^-(z,\theta')\phi_{12}^+(z',\theta) \tag{7}$$

for $z=(x,y)$, $z'=(x',y')$, and $\theta' \succeq \theta$, where meet $\wedge$ and join $\vee$ assume the vector order.

For a finite number of types, synergy is proportionately upcrossing if $s_{ij}(\theta)$ obeys an inequality analogous to (7) for arguments $z=(i,j)$ and $z'=(i',j')$, and for $\theta' \succeq \theta$.

Monotonicity is not needed for proportionately upcrossing synergy; we only require that positive synergy absolutely increase in θ more than negative synergy does.[12]

Proposition 3. *Assume (A2)* synergy is upcrossing in θ*, (B3)* synergy is one-crossing in types, and (C1) proportionately upcrossing synergy*. Sorting increases in θ in generic finite type models, or with continuum types if synergy strictly one-crosses in types.*

We prove that *Proposition 2 implies Proposition 3.* We show that if synergy is upcrossing in θ and proportionately upcrossing, then summed rectangular synergy is upcrossing in θ, while if synergy is one-crossing in types and proportionately upcrossing, then marginal rectangular synergy is one-crossing in types.

As synergy is proportionately upcrossing if it is increasing in θ and monotone in types[13], *Proposition 3 implies Proposition 0,* finishing the logical result chain.

Appendix §C.2 derives a simple smooth condition for (7): synergy is proportionately upcrossing if it is *smoothly log-supermodular (LSPM)*, namely $\sigma=\phi_{12}$ obeys

$$\sigma_{ij}\sigma \geq \sigma_i\sigma_j \tag{8}$$

Corollary 1. *Assume a continuum of types, with synergy upcrossing in θ (A2), strictly one-crossing in types (B3), and smoothly LSPM (C2). Then sorting is increasing in θ.*

Figure 6 presents an example in which synergy is both upcrossing in θ and in types, but in which sorting *falls* in θ. To verify that synergy is not proportionately upcrossing

[12] Assume negative synergy at couple z, and positive at a higher couple $z' = z \vee z' \succeq z \vee z' = z$. Then (7) says that $\phi_{12}^+(z',\theta')/\phi_{12}^+(z',\theta) \geq \phi_{12}^-(z,\theta')/\phi_{12}^-(z,\theta)$.

[13] $(z \vee z',\theta') \succeq (z',\theta) \Rightarrow \phi_{12}^+(z \vee z',\theta') \geq \phi_{12}^+(z',\theta)$, and $(z,\theta') \succeq (z \wedge z',\theta) \Rightarrow \phi_{12}^-(z \wedge z',\theta) \geq \phi_{12}^-(z,\theta')$.

Match Payoffs: $f_{ij}(\theta') \to f_{ij}(\theta'')$

	x_1	x_2	x_3
y_3	6	7	**11**
y_2	**4**	6	6
y_1	0	**4**	6

$\to$

	x_1	x_2	x_3
y_3	6	**13**	18
y_2	**4**	9	10
y_1	1	4	**10**

Synergy: $s_{ij}(\theta') \to s_{ij}(\theta'')$

	x_1x_2	x_2x_3
y_2y_3	-1	4
y_1y_2	-2	-2

$\to$

	x_1x_2	x_2x_3
y_2y_3	2	4
y_1y_2	-1	-5

Figure 6: **Proportionately Upcrossing Failure.** The unique efficient matching falls from NAM3 to PAM2 as θ' shifts up to θ''. Note that synergy is upcrossing in θ and in types. But synergy is not proportionately upcrossing and sorting falls in θ.

in this figure, let $z = (2, 1)$, $z' = (2, 2)$, $t = \theta'$, and $t' = \theta''$. Then:

$$\phi_{12}^{-}(z \wedge z', t)\phi_{12}^{+}(z \vee z', t') = 2 \cdot 4 = 8 < 20 = 5 \cdot 4 = \phi_{12}^{-}(z, t')\phi_{12}^{+}(z', t)$$

This example violates both time-series and cross-sectional premises in Proposition 1. In particular, rectangular synergy is not upcrossing in types at θ'', since $2+(-1) > 0 > 4+(-5)$. And rectangular synergy is not upcrossing in θ, since $4+(-2) > 0 > 4+(-5)$. This latter failure is precisely why the optimal matching falls from NAM3 to PAM2.

5 Increasing Sorting and Type Distribution Shifts

Distributional shifts can be formally embedded in production functions, and thus allow us to use our comparative statics theory to deduce sorting predictions for changes in the type distributions $G(\cdot|\theta)$ and $H(\cdot|\theta)$. We say that X *types shift up (down)* in θ if $G(\cdot|\theta)$ stochastically increases (decreases) in θ, i.e. $G(\cdot|\theta') \leq G(\cdot|\theta)$ if $\theta' \succeq \theta$. Similarly, Y *types shift up (down)* in θ if $H(\cdot|\theta)$ stochastically increases (decreases) in θ.

The PQD order introduced in §3 only ranks matching distributions with the *same* marginals G and H. To overcome this, we consider sorting in quantile space. Label every type by its *quantile* in the distribution, so $p = G(X(p, \theta)|\theta)$ and $q = H(Y(q, \theta)|\theta)$. Next for any matching distribution consider the associated bivariate *copula* which defines the sorting by quantiles, namely $C(p, q) = M(X(p, \theta), Y(q|\theta))$. The copula is the matching distribution defined on quantiles (p, q) rather than types (x, y). We say that *quantile sorting* is higher at M'' than M' when the associated copulas are ranked $C'' \succeq_{PQD} C'$; namely, when C'' has more mass than C' in all lower and upper orthants in (p, q) space. The quantile sorting order generalizes the PQD order. For if M'' and M' share the same marginals, then $C'' \succeq_{PQD} C'$ if and only if $M'' \succeq_{PQD} M'$. And since all copulas have uniform marginals by definition, we can compare two copulas in the PQD order even if the associated matching distributions do not share marginals.

By Lemma 1, greater quantile sorting order reduces the average geometric distance

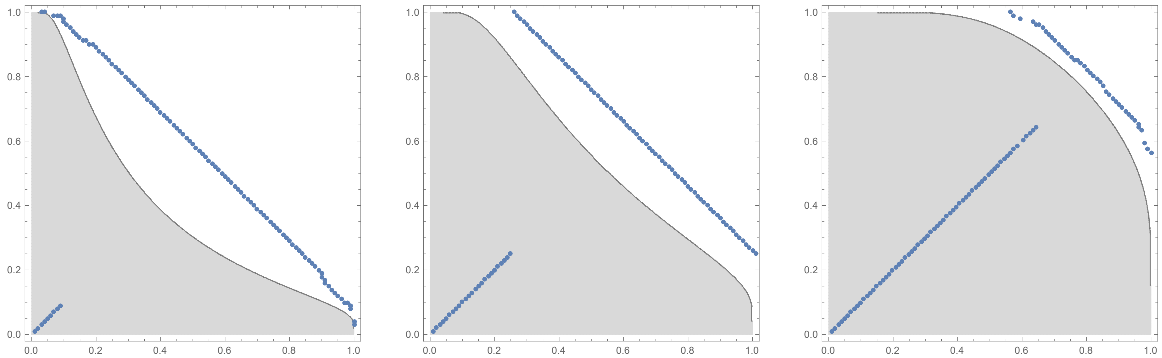

Figure 7: **Increasing Sorting with Type Shifts for Quadratic Production.** These graphs depict optimally matched quantile pairs (blue dots) given an exponential distribution on types $G(x|\theta) = 1 - e^{-x/\theta}$ and $H(y|\theta) = 1 - e^{-y/\theta}$, and quadratic production $xy - (xy)^2$. By Corollary 2, *quantile sorting increases as θ falls*, since synergy is falling in types. The plots depict $\theta = 1, 2/3, 1/3$ at left, middle, and right.

between matched quantiles, and raises the covariance across matched quantile pairs, and the coefficient in linear regression of male on female match partner quantiles.

Corollary 2. *Quantile sorting increases if types shift up (down):*
(*a*) *generically with finite types, if synergy is non-decreasing (non-increasing) in types;*
(*b*) *given G and H absolutely continuous, if synergy is increasing (decreasing) in types.*

For some insight into the proof in §D.6, consider the *quantile production function* $\varphi(p, q|\theta) \equiv \phi(X(p, \theta), Y(q, \theta))$ with *quantile synergy*:

$$\varphi_{12}(p, q|\theta) = \phi_{12}(X(p, \theta), Y(q, \theta))X_p(p, \theta)Y_q(q, \theta) \tag{9}$$

For concreteness, assume synergy ϕ is increasing in types, and that θ stochastically shifts up types. Then $\phi_{12}(X(p, \theta), Y(q, \theta))$ is increasing in quantiles p, q and θ. But we cannot conclude that *quantile synergy* is increasing in q and θ since (9) includes X_p and Y_q, which need not be monotone in q or θ. Given positive derivatives, quantile synergy is upcrossing in types and θ. We verify in §D.6 that the premise of Corollary 2 implies that of Proposition 2. Figure 7 depicts this result for quadratic production.

6 Economic Applications

In this section, we claim that standard economic matching models, and some new ones, naturally project into our synergistic matching framework. As such, this theory unifies the literature, extending the theory in those papers, since synergy captures the primary economic forces. Our outline focuses on the synergistic implications of economic forces.

6.1 Diminishing Returns

A key economic principle is that *diminishing returns reduces match synergies, whereas increasing returns amplifies them.* For just as convex transformations preserve SPM, concave transformations undercut it. For instance, while widget production may be complementary, firm profits quite often exhibit diminishing returns to more widgets.

We illustrate this with bi-quadratic production[14] $\phi(x, y) = \alpha xy + \beta(xy)^2$. Then the match synergy function $\phi_{12} = \alpha + 4\beta xy$ is strictly increasing in α and β. Becker implies PAM if $\alpha, \beta > 0$, and NAM if $\alpha, \beta < 0$. But if $\alpha > 0 > \beta$, then synergy is strictly monotone in types, and Proposition 0 predicts that sorting increases in (α, β).

With bi-cubic production $\phi(x, y) = \alpha xy + \beta(xy)^2 + \gamma(xy)^3$, synergy falls in types when $\beta, \gamma < 0$, and rises when $\beta, \gamma > 0$. But if $\beta\gamma < 0$, synergy may not be monotone in types; we can then only conclude nowhere decreasing sorting, by Corollary 3 in §F.2.

6.2 From Weakest to Strongest Link Technologies

Match partnerships differ by how differentially responsive output is to the types. One extreme case is the "weakest link" technology, namely, the SPM function $\min(x, y)$. Equally shared tasks, like jointly lifting a couch, have this flavor. Oppositely, the "strongest link" technology is the SBM function $\max(x, y)$. More generally, the lesser type matters more in a *weak link technology*. For a smooth technology ϕ, the rate of technical substitution therefore obeys $r(x, y) \equiv \phi_1(x, y)/\phi_2(x, y) \gtrless 1$ for $x \lesseqgtr y$. The opposite inequalities hold for a *strong link technology*, since the higher type matters more. Insurance exemplified this: the higher agent helps the lesser. We argue that *match synergies are higher with weak link, and lower with strong link, technologies.*

The CES technology $q(x, y) = (x^{-\rho} + y^{-\rho})^{-1/\rho}$ is a helpful tractable class. It is weak link and SPM when $\rho \geq -1$, and otherwise is strong link and SBM. The limits $\rho \to \pm\infty$ are the weakest and strongest link technologies, $\min(x, y)$ and $\max(x, y)$.

Assume diminishing returns to output q in production $\phi(x, y) = \alpha q(x, y) - \beta q(x, y)^2$, with $\alpha, \beta > 0$ and $\alpha > 2\beta q(1, 1)$ (or ϕ is increasing). Then synergy is everywhere negative when $\rho = -1$. In fact, production is SBM (yielding NAM) for all $\rho < \underline{\rho}$, and SPM (giving PAM) for $\rho > \bar{\rho}$, where $\bar{\rho} > \underline{\rho} > -1$. Our theory completes this picture. We use Proposition 2 in §E to show that sorting is increasing in ρ for all $\rho \in [0, \overline{\rho}]$, and so *synergy is falling in the elasticity of substitution* $1/(1 + \rho)$.

[14]More generally, let $\phi(x, y|\theta) = \psi(xy|\theta)$ for ψ increasing and concave with "relative risk aversion" $-z\psi''(z|\theta)/\psi'(z|\theta)$ decreasing in z and θ. Then we show in §E that synergy is upcrossing in types and θ and proportionately upcrossing: sorting increases in θ (as ψ becomes less concave) by Proposition 3.

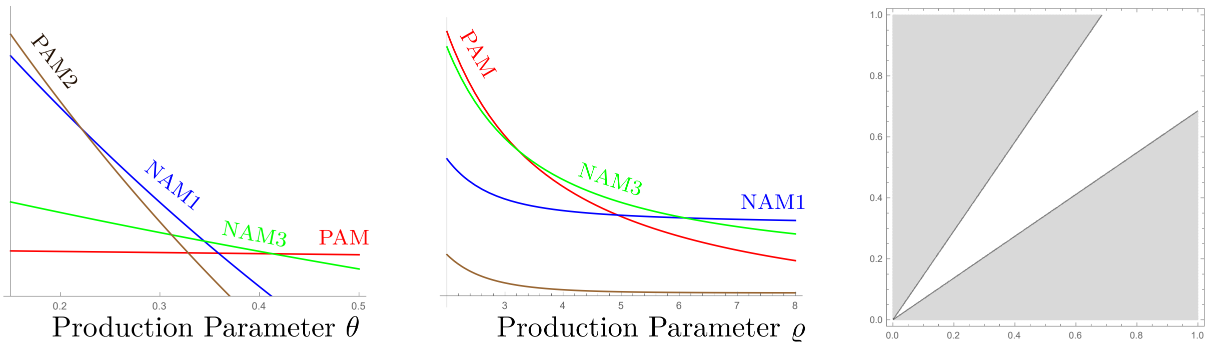

Figure 8: **Kremer-Maskin Payoffs.** We plot matching payoffs by three types $\{1, 2, 6.5\}$ with production function (10). Left: matching shifts from PAM2=PAM4, to NAM1, to NAM3, to PAM as θ rises ($\varrho = 100$). Middle: matching shifts from PAM, to NAM3, to NAM1, as ϱ rises ($\theta = 0.32$). Right: We plot the cross partial of $\phi(x, y|\theta, \varrho)$, which is not one-crossing in types. *So our sorting monotonicity theory is silent here.*

6.3 Task Assignment

Optimal ex post role assignment reduces match synergies. Kremer and Maskin (1996) assume that agents are assigned to the manager or deputy roles, where $x^\theta y^{1-\theta}$ is output if x is the manager and y the deputy, and $\theta \in [0, 1/2]$. As a unisex model, match output is then the maximum of two SPM functions $\max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\}$ — but is neither SPM nor SBM, since minimization preserves SPM, and maximization preserves SBM. We introduce a class of indexed smooth production functions converging to it as $\varrho \to \infty$:

$$\phi(x, y|\theta, \varrho) = x^\theta y^\theta \left(x^\varrho + y^\varrho\right)^{\frac{1-2\theta}{\varrho}} \tag{10}$$

Figure 8 explores a three type example in which sorting is not monotone in either θ or ϱ: The optimal matching switches between NAM1 and NAM3. For insight, note that $x^\theta y^{1-\theta}$ and $x^{1-\theta} y^\theta$ are both SPM. Rectangular synergy is then positive for rectangles iff they do not straddle the diagonal.[15] So synergy is not one-crossing in types, as Proposition 1 needs. Alternatively, the cross partial of the smooth approximating function $\phi(x, y|\theta, \varrho)$ in (10) is $+, -, +$ as types x or y increase (right panel of Figure 8).

Yet while sorting is not monotone, it cannot PQD fall as $(\theta, -\varrho)$ increases. See §F.3.

6.4 Adverse Selection

Adverse selection introduces a strongest link SBM matching insurance effect, and so reduces match synergies. Consider Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending with adverse selection. Borrowers vary by project success

[15] $\phi(x, x|\theta) + \phi(y, y|\theta) - 2\phi(x, y|\theta) = x + y - 2x^\theta y^{1-\theta} < 0$ for $y > x$ and $0 < \theta < 1/2$ and $1 < y/x < 1 + \varepsilon$.

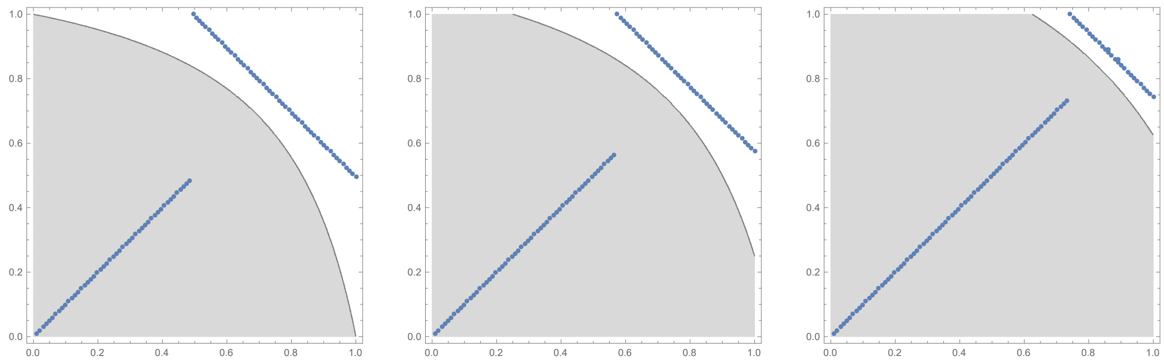

Figure 9: **Rising Sorting in a Group Lending Model.** These plots depict optimally matched man-woman pairs (blue dots) assuming a uniform distribution on 100 types for $\delta = 0.8$ when PAM is not optimal, or $\delta < c/(c+\pi-d)$ fails — so $(\pi, c, d) = (10, 0, 2)$ at left, $(\pi, c, d) = (10, 2, 2)$ in the middle, and $(\pi, c, d) = (4, 2, 2)$ at right.

chance x; a success pays $\pi > 0$ and a failure nothing. Pairs sign lending contracts, and project outcomes are independent. Borrowers but not banks observe each other's type.

After the project outcome, a borrower either repays the loan, or defaults. Each pays the debt $d > 1$ if both repay. If only one defaults, the other repays collateral plus debt $c + d > d$. In the paper, a borrower repays when his project succeeds if $\pi \geq c + d$. If both default, each loses access to credit markets. Borrowers discount future payoffs by $\delta \leq 1$,[16] defaulting if both projects fail. A project pair (x, y) has discounted value:

$$\phi(x, y) = x((\pi - d) - (1 - y)c) + y((\pi - d) - (1 - x)c) + \delta(1 - (1 - x)(1 - y))\phi(x, y) \quad (11)$$

The static payoff has the weakest link (SPM) flavor, since the collateral c is paid when precisely one of the two projects fails. But since the pair only loses access to credit markets when *both* projects fail, the continuation chance has a strongest link (SBM) flavor.[17] Indeed, synergy ϕ_{12} is globally positive if $\delta \leq \delta^* \equiv c/[c + (\pi - d)] < 1$. But with more patience, synergy is positive for low types and negative for high types.

In §E, we apply Proposition 2 to show that sorting rises in the collateral c, and falls in the net payoff $\pi - d$. For these respectively amplify and lessen the static weakest link force (Figure 9). While PAM is not optimal for $\delta \in (\delta^*, 1)$,[18] sorting is not monotone in the discount factor δ, as synergy is globally non-negative when $\delta \leq \delta^*$ *and* at $\delta = 1$.

[16] The discounted value is well defined for $\delta = 1$, since both projects eventually fail.

[17] Legros and Newman (2002) study matching with imperfect credit markets. Given financing, output is xy minus upfront fixed cost q. But creditors only finance pairs with $xy \geq \kappa > q$, with a weakest link flavor. So output $\phi = (xy - q)\mathbb{1}_{xy \geq \kappa}$ is neither SPM nor SBM. Synergy is not one-crossing in types, and sorting is not monotone in κ or q. By §F, sorting nowhere decreasing in $\theta = q/\kappa$.

[18] When $\delta \in (c/(c + \pi - d), 1)$, the symmetric synergy function $\phi_{12}(x, x)$ is strictly negative for x close to 1. Thus, cross matching types x and $x + \varepsilon$ beats sorting them, for high x and low ε.

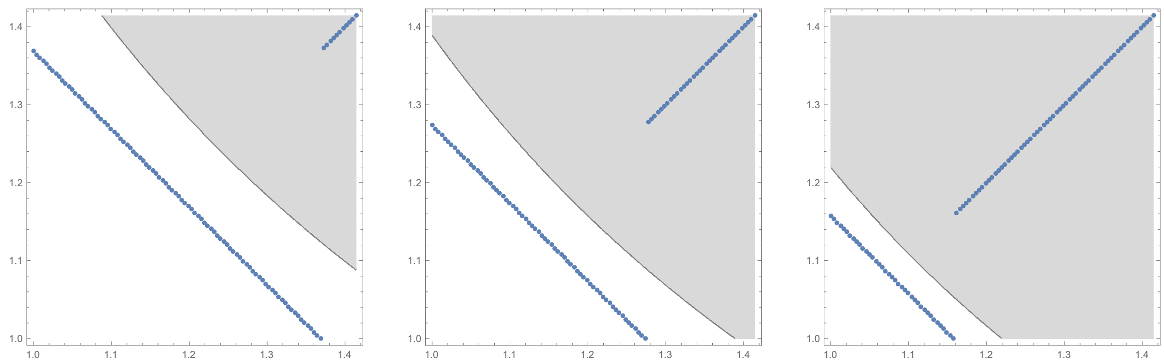

Figure 10: **Increasing Sorting in the Principal-Agent Model.** NAM is optimal for low dis-utility of effort θ, PAM for high θ, and the optimal matching is mixed for intermediate θ. These graphs depict optimally matched pairs (blue dots) for a discrete uniform distribution on 100 types of principals and agents. The left plot is drawn for $\theta = 0.65$, the middle for $\theta = 0.72$, and the right for $\theta = 0.82$. In all plots, the matching obeys local optimality — if the matching slopes up, then synergy is positive (shaded). In all plots, the reverse implication fails due to subtle global optimality considerations.

6.5 Moral Hazard with Endogenous Contracts

Serfes (2005) explores a pairwise matching principal-agent model. The output for any project is the sum of the agent's unobservable effort and a mean zero Gaussian error. Project variances $y \in [\underline{y}, \overline{y}]$ vary across principals, while agents differ by risk aversion parameter $x \in [\underline{x}, \overline{x}]$, and share a scalar dis-utility of effort $\theta > 0$. Contracts are signed after matching takes place; they specify the agent's wage as a function of realized output. Serfes derives (his equation (2)) the equilibrium expected output of an (x, y) match:

$$\phi(x, y|\theta) = \frac{1}{2\theta\,(1+\theta xy)} \quad \Rightarrow \quad \phi_{12}(x, y|\theta) = \frac{\theta xy - 1}{2\,(1+\theta xy)^3} \tag{12}$$

Serfes observes that synergy is globally negative for $\theta\bar{x}\bar{y} < 1$ and globally positive for $\theta\underline{x}\underline{y} > 1$. Thus, by Becker's Sorting Result, NAM obtains for $\theta < (\bar{x}\bar{y})^{-1}$ and PAM obtains for $\theta > (\underline{x}\underline{y})^{-1}$. This result reflects two counterveiling forces for sorting. First, if all contracts were the same, then efficient insurance across principal-agent pairs favors NAM: less risk averse agents work on higher variance projects. But the slope of the equilibrium wage contract is $(1 + \theta xy)^{-1}$; and thus, the incentives to provide effort are SPM for high types. The sign of synergy (12) implies that the insurance effect dominates for low types, and the incentive effect dominates for high types.[19] In other

[19] Ackerberg and Botticini (2002) investigate matching between landowners (principals) and tenants (agents) in 15th century Tuscany. Matched crop-tenant pairs exhibit positive covariance in crop types

words, we will argue that *optimal ex post contract effort reduces match synergies.*

We claim that sorting is increasing in θ when extremal types obey $\bar{x}\bar{y} \leq 2\underline{x}\underline{y}$ (‡). Assume $\theta' > \theta$. In both cases, sorting is weakly higher at θ' then θ. Assume $\theta'\underline{x}\underline{y} \leq 1 < \theta\bar{x}\bar{y}$. Then by (‡) we have $\theta'\bar{x}\bar{y} \leq 2$, and since the function $(t-1)/(1+t)^3$ is increasing for $t \in (0, 2]$, synergy (12) is increasing in θ, x, and y. Altogether, NAM obtains for $\theta \leq (\bar{x}\bar{y})^{-1}$, PAM for $\theta \geq (\underline{x}\underline{y})^{-1}$, and sorting is increases in θ between these two extremes, by Proposition 0, as in Figure 10. Since synergy increases in types when PAM is suboptimal, quantile sorting increases when types shift up, by Corollary 2.

6.6 Mentor-Protégé Learning Dynamics

Dynamic matching models with evolving types are best understood through the lens of match synergies. Assume pairwise matching in periods one and two.[20] Let $\phi^0(x, y)$ be the symmetric, increasing and SPM output function when types x and y match.

First assume that types transition in period two to $x' = \tau(x, y)$ and $y' = \tau(y, x)$. For instance, learning from co-workers is a key SBM force. In an extreme case, the lower type rises to the higher type: $\tau(x, y) = \max\{x, y\}$.[21] By contrast, negative peer effects may dominate among kids, with the extreme SPM case $\tau(x, y) = \min(x, y)$.

Next, in a procreation context, couple (x, y) produces offspring of type $\tau(x, y)$. Here, $\tau = \max$ captures *dominant type transmission* (one smart parent suffices for smarts), and $\tau = \min$ likewise *recessive inheritance*. All told, *learning or dominant type genetic transitions reduce match synergies; recessive genetics increase synergies*.

As PAM is optimal in period two, the social planner has dynamic match payoff:

$$\phi(x, y) = (1-\delta)\phi^0(x, y) + \tfrac{\delta}{2}\left[\phi^0(\tau(x, y), \tau(x, y)) + \phi^0(\tau(y, x), \tau(y, x))\right]$$

given discount factor δ. Simplify production to $\phi^0(x, y) = \sqrt{xy}$. Then dynamic synergy reduces to:

$$\phi_{12}(x, y) = (1-\delta)/\sqrt{xy} + \tfrac{1}{2}\delta\left[\tau_{12}(x, y) + \tau_{12}(y, x)\right] \qquad (13)$$

For the conjectured dynamic SPM neighborhood sorting application ($\tau_{12} \geq 0$), synergy is everywhere positive: The theory realistically predicts PAM. For dynamic workplace learning ($\tau_{12} < 0$), synergy is decreasing in the discount factor δ. Synergy is also decreasing in types (x, y).[22] Altogether, sorting is decreasing in δ, by Proposition 0.

(project variance y) and tenant wealth (risk aversion x). But since match sorting is imperfect (not PAM), our theory provides a framework for analyzing changes in crop-tenant matching across markets.

[20]Anderson and Smith (2012) allow an infinite horizon with stochastic type transitions.

[21]If naturally $\tau(x, x) = x$, we have $\tau(x, y) - x > y - \tau(y, x)$ iff $\tau(x, y) + \tau(y, x) > \tau(x, x) + \tau(y, y)$.

[22]Recall that $\rho < -1$; and thus, $\tau_{12}(x, y) = \alpha(1-\alpha)(1+\rho)(xy)^{-(1+\rho)}(\alpha x^{-\rho} + (1-\alpha)y^{-\rho})^{-\frac{1+2\rho}{\rho}}$ falls in (x, y). Thus, synergy (13) is a weighted average of two decreasing functions of (x, y).

6.7 Pairwise Trading with Multidimensional Characteristics

Our last pairwise matching application reaches outside the traditional realm of Becker sorting results to the classic unit trade model among buyers and sellers (Shapley and Shubik, 1971). Index sellers by a single characteristic y, and buyers by two traits (x, z). If buyers know (x, z) before trading, then this calls for a multidimensional version of our model — which is an open question.[23]

Assume instead that buyers only learn their second trait z after trade. Sorting depends on the (x, y) synergies, the (z, y) synergies, and the distribution of z given x. For instance, index sellers by home size y, for an increasing selling cost $c(y)$, and buyers by income x and future family size z. Buyer values are $x + \alpha xy + \beta zy$, where $\alpha, \beta \geq 0$. Assume family size z is random, with conditional expectation $Z(x)$.[24] Altogether, the match value is:

$$\phi(x, y|\theta) = \max(x + \alpha xy + \beta Z(x)y - c(y), 0) \tag{14}$$

Becker's Matching Theorem predicts that low cost sellers trade to high value buyers in the homogeneous double auction — the SBM special case $\alpha = \beta = 0$. Let us now focus on just those who do trade. Match synergy is $\alpha + \beta Z'(x)$ for all positive surplus trades.

If family size does not matter ($\beta = 0$) or positively covaries with income ($Z'(x) \geq 0$), then ϕ is SPM and thus PAM is optimal among those that trade. Also, if $\alpha = 0$ and family size negatively covaries with income $Z'(x) < 0$ (as is well documented across countries), then NAM is optimal among those that trade. Finally, assume $\alpha, \beta > 0$ and $Z'(x) < 0$. If Z' is also monotone, then synergy is monotone in x, increasing in α, and decreasing in β, and Proposition 0 predicts that *sorting rises in α and falls in β.*

7 Conclusion

Becker's finding that complementarity (or supermodularity) yields positive sorting launched the immense literature on pairwise matching. But an impassable wall of mathematical complexity has prevented any general theory for non-assortative matching — despite many economic models needing such a theory. This paper bipasses the solution of the optimal matching, and nevertheless derives the missing general theory for comparative statics. We argue that the PQD stochastic order captures the economic notion of increasing sorting, and then answer the typical comparative static economist

[23]Lindenlaub (2017) finds that sorting increases on cognitive traits with an increase in the relative weight on cognitive (vs. manual) complementarities with bi-linear production and Gaussian types.

[24]If family size were known to the buyers at the time of purchase, then buyer types would be a vector (x, z) and this would be a multidimensional type matching model as in Lindenlaub (2017). The comparative statics of sorting with multidimensional types is beyond the scope of the current work.

want: what productivity or type distribution shifts increase sorting in the PQD order?

We show that sorting increases if synergy globally increases and synergy crosses everywhere from negative to positive as types rise, and obey a cross-sectional assumption — e.g. it changes sign at most once on rectangles as the rectangle shifts northeast, or if a purely local condition called proportionate upcrossing is met.

We revisit the matching literature since 1990, quickly deriving and strengthening their findings, using our theory. Our paper offers a tractable foundation for future theoretical and empirical analysis of matching. A subtle and valuable direction for future work is a multidimensional extension of our theory (Lindenlaub, 2017).

A Match Output Reformulation

Lemma 2 (n Types). $\sum_{i,j=1}^{n} f_{ij}m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left([f_{n,j+1} - f_{nj}]\, j - \sum_{i=1}^{n-1} s_{ij}M_{ij}\right)$

Proof: Summing $\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}m_{ij}$ by parts in j and then i yields:

$$
\begin{aligned}
\textstyle\sum_{i=1}^{n}\left[\sum_{j=1}^{n} f_{ij}m_{ij}\right] &= \sum_{i=1}^{n}\left[f_{in}\sum_{j=1}^{n} m_{ij} - \sum_{j=1}^{n-1}[f_{i,j+1} - f_{ij}]\sum_{k=1}^{j} m_{ik}\right] \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\sum_{i=1}^{n}[f_{i,j+1} - f_{ij}]\sum_{k=1}^{j} m_{ik} \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left([f_{n,j+1} - f_{n,j}]\sum_{\ell=1}^{n}\sum_{k=1}^{j} m_{\ell k} - \sum_{i=1}^{n-1} s_{ij}\sum_{\ell=1}^{i}\sum_{k=1}^{j} m_{\ell k}\right) \\
&= \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left([f_{n,j+1} - f_{nj}]\, j - \sum_{i=1}^{n-1} s_{ij}M_{ij}\right)
\end{aligned}
$$

B Common Sorting Measures Easily Conflict

Lemma 1 established that a PQD increase implies commonly used sorting statistics increase. We now show that sorting predictions based on these common statistics depend on the scaling of types, and can easily move in *opposite* directions when the matching changes. This highlights why we use the stronger ordinal PQD sorting order.

Note that the *covariance* and correlation coefficient of matching partner types, and the linear regression coefficient of y on partner type x are co-monotone for matching changes, since each statistic is an increasing function of the other. So we consider the covariance sorting statistics and the average *distance* between match partner types.

Figure 11: **Sorting Statistic Disagreement.** These 4-type matchings are not PQD comparable: the bullet matching has a pairing among the lowest two x and y types; the circle matching does not, but matches the lowest x type to a strictly lower type.

We show that: (i) *the sign of a sorting statistic — either covariance or distance — depends on the type scaling*, and (ii) these statistics may move in opposite directions as the matching changes — even if we correct each statistic using skill quantiles.

Assume three types, and consider a non-PQD comparable NAM1 to NAM3 change. If $x \in \{1, 2, 3\}$ and $y \in \{0.5, 1.8, 3\}$, then the covariance between matched types and average distance between partners both fall, i.e. sorting falls if measured by type correlation, but rises if measured by average distance between matched types. On the other hand, if $y \in \{0.5, 2.5, 5\}$, match type correlation rises, and average distance between matched types falls. Both sorting measures fall when $y \in \{0.5, 2.5, 3\}$ and both rise when $y \in \{0.5, 2.5, 3\}$. So *any sign pattern is consistent with a NAM1 to NAM3 shift*.

If we convert to *quantile space* and restrict to three types, then the covariance and the average distance ranking coincides for (NAM1,NAM3) and (PAM2,PAM4). But this equivalence breaks down with four types, and again the sorting conclusions depend on the specific statistic chosen. To see this, consider a uniform distribution over types $\{1, 2, 3, 4\}$ in Figure 11. The covariance between matched pairs is higher at right, but the average distance between match partners is lower at left. Thus, covariance -based sorting statistics (e.g. correlation coefficient) deems the right matching more sorted, while the left matching is more sorted by the average distance between partners.

C Integral Preservation of Upcrossing Properties

C.1 Integral Preservation of Upcrossing Functions on Lattices

Given a real or integer lattice[25] $Z \subseteq \mathbb{R}^N$ and poset $(\mathcal{T}, \succeq)$, the function $\sigma : Z \times \mathcal{T} \to \mathbb{R}$ is proportionately upcrossing if $\forall z, z' \in Z$ and $t' \succeq t$.[26]

$$\sigma^-(z \wedge z', t)\sigma^+(z \vee z', t') \geq \sigma^-(z, t')\sigma^+(z', t) \tag{15}$$

[25]We prove a stronger than needed result, as it applies to general lattices; we just need it for $\mathbb{R}^2$.

[26]This result is related to Theorem 2 in Quah and Strulovici (2012). They do not assume (15). Rather, they assume σ is upcrossing in (z, θ), and a time a series condition: signed ratio monotonicity. Our results are independent, but overlap more closely for our smoothly LSMP condition in §C.2.

Theorem 1. *Let $\sigma(z,t)$ be proportionately upcrossing. Then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is weakly upcrossing in t,*[27] *and upcrossing in t if $\sigma(z,t)$ is upcrossing in t.*[28]

This generalizes a key information economics result by Karlin and Rubin (1956): *If $\sigma_0(z)$ is upcrossing in $z \in \mathbb{R}$, and $\sigma_1 \geq 0$ is LSPM, then $\int \sigma_0(z)\sigma_1(z,t)d\lambda(z)$ is upcrossing.* Our result subsumes theirs when $n = 1$ and $\sigma = \sigma_0\sigma_1$ is proportional upcrossing.

Proof: Karlin and Rinott (1980) prove the following: *If functions $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ obey $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for $z \in Z \subseteq R^N$, then for all positive measures λ:*

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z) \tag{16}$$

Now, if $t' \succeq t$, then (15) reduces to $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for the functions:

$$\xi_1(z) \equiv \sigma^+(z,t), \quad \xi_2(z) \equiv \sigma^-(z,t'), \quad \xi_3(z) \equiv \sigma^+(z,t'), \quad \xi_4(z) \equiv \sigma^-(z,t)$$

Thus, by (16):

$$\int \sigma^+(z,t')d\lambda(z) \int \sigma^-(z,t)d\lambda(z) \geq \int \sigma^+(z,t)d\lambda(z) \int \sigma^-(z,t')d\lambda(z) \tag{17}$$

This precludes $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$ and $\int \sigma^+(z,t')d\lambda(z) < \int \sigma^-(z,t')d\lambda(z)$, simultaneously. And thus, $\Sigma(t) > 0$ implies $\Sigma(t') \geq 0$, proving weakly upcrossing.

We now argue Σ upcrossing. First assume $\Sigma(t) > 0$. Then $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$. By (17), either $\int \sigma^+(z,t')d\lambda(z) > \int \sigma^-(z,t')d\lambda(z)$, or $\int \sigma^+(z,t')d\lambda(z) = \int \sigma^-(z,t')d\lambda(z) = 0$. But the latter is impossible, since $\int \sigma^+(z,t')d\lambda(z) = 0$ implies $\int \sigma^+(z,t)d\lambda(z) = 0$, as $\sigma(z,t)$ is upcrossing in t — contradicting $\Sigma(t) > 0$. So $\Sigma(t') > 0$.

Next, posit $\Sigma(t) = 0$, then $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z)$. By (17), either $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, and so $\Sigma(t') \geq 0$. Or, we have $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z) = 0$, whereupon $\int \sigma^-(z,t')d\lambda(z) = 0$ — as $\sigma(z,t)$ is upcrossing in t, and so $\sigma^-(z,t)$ is downcrossing. Thus, $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, or $\Sigma(t') \geq 0$. □

C.2 Proportionately Upcrossing and Log-supermodularity

We now introduce a sufficient condition for (15) that emphasizes the link between log-complementarity and proportional upcrossing. Let $\theta \in \mathbb{R}$, and call $\sigma(z,\theta)$ *smoothly signed log-supermodular (LSPM)* if its derivatives obey the inequality $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$.

[27] Proportionately upcrossing implies *weakly upcrossing*; namely, $\sigma(z,t) > 0$ implies $\sigma(z',t') \geq 0$ for all $(z',t') \succeq (z,t)$. Proof: Fix $t = t'$ and suppress t. If $z' \succeq z$, inequality (15) is an identity. If $z \succ z'$, inequality (15) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

[28] The proof for the integer lattice requires that λ be a counting measure. Also true: if λ does not place all mass on zeros of σ, then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is upcrossing in t.

Theorem 2. *If $\sigma(z,\theta)$ is upcrossing and smoothly signed LSPM, then σ obeys* (15)*.*

STEP 1: RATIO ORDERING. Abbreviate $w = (z,\theta) \in \mathbb{R}^{N+1}$. Assume $\hat{w} \geq w$, sharing the i coordinate $w_i = \hat{w}_i$, with $\sigma(\bar{x}, w_{-i}) < 0 < \sigma(\hat{w})$ for some $\bar{x} > w_i$. Then we prove:

$$\sigma_i(x, w_{-i})\sigma(x, \hat{w}_{-i}) \geq \sigma_i(x, \hat{w}_{-i})\sigma(x, w_{-i}) \quad \forall\, x \in [w_i, \bar{x}] \tag{18}$$

Since σ is upcrossing, $\sigma(x, w_{-i}) < 0 < \sigma(x, \hat{w}_{-i})$ for all $x \in [w_i, \bar{x}]$. If (18) fails, then for some $x' \in [w_i, \bar{x}]$:

$$\frac{\sigma_i(x', w_{-i})}{\sigma(x', w_{-i})} > \frac{\sigma_i(x', \hat{w}_{-i})}{\sigma(x', \hat{w}_{-i})}$$

This contradicts smoothly LSPM, as $(\sigma_i/\sigma)_j \geq 0$ for all $\sigma \neq 0$ and $i \neq j$. So (18) holds. Given $\sigma(x, \hat{w}_{-i}) \neq 0$, the ratio $\sigma(x, w_{-i})/\sigma(x, \hat{w}_{-i})$ is non-decreasing in x on $[w_i, \bar{x}]$, so that:

$$\frac{\sigma(w)}{\sigma(\hat{w})} \leq \frac{\sigma(\bar{x}, w_{-i})}{\sigma(\bar{x}, \hat{w}_{-i})} \tag{19}$$

STEP 2: σ OBEYS (15). By assumption $\theta' \geq \theta$ (now a real). So if $(z,\theta') \leq (z \wedge z', \theta)$, we have $z \leq z'$ and $\theta' = \theta$, in which case (15) is an identity. If not $(z,\theta') \leq (z \wedge z', \theta)$, then let $i_1 < \cdots < i_K$ be the indices with $(z,\theta')_{i_k} > (z \wedge z', \theta)_{i_k}$ for $k = 1, \ldots, K$. Let's change $w^0 \equiv (z \wedge z', \theta)$ into $w^K \equiv (z, \theta')$ in K steps, $w^0, \ldots, w^K$, one coordinate at a time, and likewise $\hat{w}^0 \equiv (z', \theta)$ into $\hat{w}^K \equiv (z \vee z', \theta')$, changing coordinates in the same order. Notice that $w^{k-1}_{i_k} = \hat{w}^{k-1}_{i_k} = (z', \theta)_{i_k} < (z, \theta')_{i_k}$ and $\hat{w}^k \geq w^k$ for all k.

Now, inequality (15) holds if its RHS vanishes. Assume instead the RHS of (15) is positive for some $\theta' \geq \theta$, so that $\sigma(z, \theta') < 0 < \sigma(z', \theta)$; and so, replacing $\hat{w}^0 = (z', \theta)$ and $w^K = (z, \theta')$, we get $\sigma(w^K) < 0 < \sigma(\hat{w}^0)$. But then since the sequences $\{w^k\}$ and $\{\hat{w}^k\}$ are increasing and σ is upcrossing, we have $\sigma(w^k) < 0 < \sigma(\hat{w}^{k-1})$ for all k. Altogether, we may repeatedly apply inequality (19) to get:

$$\frac{\sigma(z \wedge z', \theta)}{\sigma(z', \theta)} \equiv \frac{\sigma(w^0)}{\sigma(\hat{w}^0)} \leq \frac{\sigma(w^k)}{\sigma(\hat{w}^k)} \leq \cdots \leq \frac{\sigma(w^K)}{\sigma(\hat{w}^K)} \equiv \frac{\sigma(z, \theta')}{\sigma(z \vee z', \theta')}$$

So given $\sigma(z \wedge z', \theta), \sigma(z, \theta') < 0 < \sigma(z', \theta), \sigma(z \vee z', \theta')$, inequality (15) follows from:

$$\frac{\sigma^-(z \wedge z', \theta)}{\sigma^+(z', \theta)} \geq \frac{\sigma^-(z, \theta')}{\sigma^+(z \vee z', \theta')} \qquad \square$$

D Omitted Proofs

D.1 One Crossing Weighted Synergy via Linear Synergy

Claim 1. *If synergy is linear in a parameter θ, say $\phi_{12}(x,y|\theta) = A(x,y) + \theta B(x,y)$ or $s_{ij}(\theta) = A_{ij} + \theta B_{ij}$, and A is globally positive (negative), then weighted synergy is strictly downcrossing (upcrossing) in θ, and so also is summed rectangular synergy.*

Proof: Assume the case with $A > 0$ globally. Then for all $\theta'' > \theta' \geq 0$ and $\lambda \geq 0$:

$$\int A(x,y)\lambda(x,y) + \theta' \int B(x,y)\lambda(x,y) \leq 0 \;\Rightarrow\; \int A(x,y)\lambda(x,y) + \theta'' \int B(x,y)\lambda(x,y) < 0$$

as $A > 0$ and $\theta > 0$, together imply $\int B(x,y)\lambda(x,y) < 0$. Symmetric logic establishes the finite type case and weighted synergy strictly upcrossing in θ when $A < 0$. □

D.2 Proof of Proposition 1: Increasing Sorting for Finite Types

Lemma 3. *An optimal matching is generically unique and pure for finite types.*

Proof: The optimal matching is generically unique, by Koopmans and Beckmann (1957). A non-pure matching M is a mixture $M = \sum_{\ell=1}^{L} \lambda_\ell M_k$ over $L \leq n+1$ pure matchings $M_1, \ldots, M_n$, with $\lambda_\ell > 0$ and $\sum_\ell \lambda_\ell = 1$.[29] As the objective function (3) is linear, if the non-pure matching M is optimal, so is each pure matching M_ℓ. □

A. Big Picture of the Proof. We show, for all n, that matching models in some domain $\hat{\mathcal{D}}_n$ obey our sorting conclusion. Our induction proof argues the stronger claim that it holds on a larger recursively convenient domain $\mathcal{D}_n^* \supset \hat{\mathcal{D}}_n$. It chases down failures of the implication to a possible shift from NAM to the n-type version of NAM3.

B. Critical Proof Ingredients.

(a) Consider the generic case with unique optimal pure matchings μ, described by men partners $(\mu_1, \ldots, \mu_n)$ of women, or women partners $\omega = (\omega_1, \ldots, \omega_n)$ of men.

(b) To emphasis the dependence on the number of types n, write rectangular synergy as $S^n(r|\theta)$, and the *summed rectangular synergy* as $\mathcal{S}^n(K|\theta) = \sum_{k=1} S^n(r_k|\theta)$ for any finite set of non-overlapping rectangles $K = \{r_k\}$.

(c) We consider the *summed rectangular synergy dyad* $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ for generic $\theta'' \succeq \theta'$. Let domain $\mathcal{D}_n$ be the space of summed rectangular synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ that are each upcrossing in K on rectangles $\mathcal{R}$ and upcrossing

[29] This follows from Carathéodory's Theorem. It says that non-empty convex compact subset $\mathcal{X} \subset \mathbb{R}^n$ are weighted averages of extreme points of $\mathcal{X}$. The extreme points here are the pure matchings.

in θ on $\{\theta', \theta''\}$ for any $K \in \mathcal{R}$. The domain $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n$ further insists that they be upcrossing in θ for finite sets of non-overlapping rectangles K. Proposition 1 assumes that summed rectangular synergy dyads are in $\hat{\mathcal{D}}_n$ for all n.

(d) *Removing couple* (i, j) from an n-type market *induces rectangular synergy* S_{ij}^{n-1} among the remaining $n-1$ types, satisfying the formula:

$$S_{ij}^{n-1}(r|\theta) \equiv S^n(r + \mathcal{I}_{ij}(r)|\theta) \quad \text{for} \quad \mathcal{I}_{ij}(r) = (\mathbb{1}_{r_1 \geq i}, \mathbb{1}_{r_2 \geq j}, \mathbb{1}_{r_3 \geq i}, \mathbb{1}_{r_4 \geq j}) \tag{20}$$

where $\mathcal{I}_{ij}(r)$ increments by one the index of the women $i' \geq i$ and men $j' \geq j$, *where the type indices refer to the original model whenever removing types* henceforth.

(e) To avoid ambiguity when changing the number n of types, we denote by (i_n, j_n) the ith highest woman and the jth highest man. Now, consider the sequence models with $\kappa = n + k, n + k - 1, \ldots, n$ types induced by removing couple (i'_κ, j'_κ) at θ' and (i''_κ, j''_κ) at θ'' from the κ type model. We say the sequence of couples has *higher partners at* θ' *than* θ'' if $(i'_\kappa, j'_\kappa) \geq (i''_\kappa, j''_\kappa)$ and $i'_\kappa = i''_\kappa$ or $j'_\kappa = j''_\kappa$.

(f) Domain $\mathcal{D}_n^*$ is the set of summed rectangular synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ induced by sequentially removing k optimally matched couples with higher partners at θ' than θ'' from dyads $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$, for some $k \in \{0, 1, \ldots\}$.

C. Key Properties of our Domains and Pure Matchings.

Fact 1. *Fix a summed rectangular synergy dyad in* $\mathcal{D}_{n+1}^*$*. Removing couple* (i', j') *at* θ' *and* (i'', j'') *at* θ'' *induces such a dyad in* $\mathcal{D}_n^*$ *if* $(i', j') \geq (i'', j'')$ *and* $i' = i''$ *or* $j' = j''$.

Fact 2. *Given a summed rectangular synergy dyad in* $\mathcal{D}_{n+1}$*, removing couple* (i', j') *at* θ' *and* (i'', j'') *at* θ'' *induces a summed rectangular synergy dyad in* $\mathcal{D}_n$ *if* $\langle i' = i''$ *and* $j' \geq j'' \rangle$ *or* $\langle j' = j''$ *and* $i' \geq i'' \rangle$.

Proof: We prove this for $i' = i''$ and $j' \geq j''$. For any θ, rectangular synergy $S_{ij}^n(r|\theta)$ is upcrossing in r, needing fewer inequalities. To see that summed rectangular synergy is upcrossing in θ on rectangular sets in $\mathbb{Z}_{n-1}^2$, assume $S_{ij'}^n(r|\theta') \geq (>) 0$ for some r. Then

$$\begin{aligned} S^{n+1}(r + \mathcal{I}_{ij'}(r)|\theta') \geq (>) 0 \quad &\Rightarrow \quad S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta') \geq (>)\, 0 \\ &\Rightarrow \quad S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta'') \geq (>)\, 0 \\ &\Rightarrow \quad S_{ij''}^n(r|\theta'') \geq (>)\, 0 \end{aligned}$$

respectively, as (i) $S^{n+1}(r|\theta)$ is upcrossing for rectangles r, non-increasing $\mathcal{I}_{ij}$ in j, and $j'' \leq j'$, and (ii) $S^{n+1}(r|\theta)$ is upcrossing in θ for rectangles r, and (iii) by (20). $\square$

Fact 3. *The domains are nested:* $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n^* \subseteq \mathcal{D}_n$.

Proof: Trivially, $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n^*$, since we may set $k = 0$ in the definition of $\mathcal{D}_n^*$.

To get $\mathcal{D}_n^* \subseteq \mathcal{D}_n$, pick $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta'')) \in \mathcal{D}_n^*$. This dyad is induced by removing k optimally matched couples with higher partners at θ' than θ'' from a dyad $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k} \subseteq \mathcal{D}_{n+k}$, where $k \geq 0$. For $\ell = 1, \ldots, k$, induce dyads $(\mathcal{S}^{n+k-\ell}(K|\theta'), \mathcal{S}^{n+k-\ell}(K|\theta''))$, sequentially removing optimally matched couples. So $(\mathcal{S}^{n+k-\ell}(K|\theta'), \mathcal{S}^{n+k-\ell}(K|\theta'')) \in \mathcal{D}_{n+k-\ell}$ for $\ell = 1, \ldots, k$, as removed couples are ordered, as Fact 2 needs. So $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta'')) \in \mathcal{D}_n$. □

Fact 4. *If $M \neq \hat{M}$ are pure n-type matchings, $\hat{\mu}_i > \mu_i$ at some i and $\hat{\omega}_j > \omega_j$ at some j.*

Proof: Since $M \neq \hat{M}$, there is a highest type man j matched with woman $\hat{\omega}_j > \omega_j$. Logically then, woman $i = \hat{\omega}_j$ is matched to a lower man under M, i.e. $j = \hat{\mu}_i > \mu_i$. □

Adding a couple (i_0, j_0) to a matching μ creates a new matching $\hat{\mu}$ with indices of women $i \geq i_0$ and men $j \geq j_0$ renamed $i + 1$ and $j + 1$, respectively. Equivalently, this means inserting a row i and column j into the matching matrix m — with all 0's except 1 at position (i, j) — and shifting later rows and columns up one.

Fact 5. *Adding respective couples $(1, \hat{m}) \leq (1, m)$, or $(\hat{w}, 1) \leq (w, 1)$, to the n-type matchings $\hat{\mu} \succeq_{PQD} \mu$ preserves the PQD order for the resulting $n + 1$ type matchings.*

Proof: We just consider adding couples $(1, \hat{m}) \leq (1, m)$, as the analysis for $(\hat{w}, 1) \leq (w, 1)$ is similar. For pure matchings μ, let $C^\mu(i_0, j_0)$ count matches by women $i \leq i_0$ with men $j \leq j_0$, and so call $C^\mu(0, j) = C^\mu(i, 0) = 0$. So $\hat{\mu} \succeq_{PQD} \mu$ *iff* $C^{\hat{\mu}} \geq C^\mu$.

By adding a couple $(1, m)$, the new count is:

$$\mathcal{C}_m^\mu(i, j) \equiv C^\mu\left(i - 1, j - \mathbb{1}_{j \geq m}\right) + \mathbb{1}_{j \geq m} \quad \text{for all } i, j \in \{1, 2, \ldots, n + 1\}$$

To prove the step, we must show that if $\hat{\mu} \succeq_{PQD} \mu$, then $\mathcal{C}_{\hat{m}}^{\hat{\mu}} \geq \mathcal{C}_m^\mu$ for all $\hat{m} \leq m$.

By assumption $\hat{\mu} \succeq_{PQD} \mu$ and thus, $C^{\hat{\mu}} \geq C^\mu$. So since $\hat{m} \leq m$:

$$.\mathcal{C}_{\hat{m}}^{\hat{\mu}}(i, j) - \mathcal{C}_m^\mu(i, j) = \begin{cases} C^{\hat{\mu}}(i - 1, j) - C^\mu(i - 1, j) & \geq \;\; 0 \quad \text{for } j < \hat{m} \\ C^{\hat{\mu}}(i - 1, j - 1) + 1 - C^\mu(i - 1, j) & \geq \;\; 0 \quad \text{for } \hat{m} \leq j < m \\ C^{\hat{\mu}}(i - 1, j - 1) - C^\mu(i - 1, j - 1) & \geq \;\; 0 \quad \text{for } j \geq m \end{cases}$$

To understand the middle line, note that this match count can be written as

$$C^{\hat{\mu}}(i - 1, j - 1) - C^\mu(i - 1, j - 1) - [C^\mu(i - 1, j) - C^\mu(i - 1, j - 1) - 1]$$

As $C^\mu(i-1, j) – C^\mu(i-1, j-1) \leq 1$, this is at least $C^{\hat{\mu}}(i-1, j-1) – C^\mu(i-1, j-1) \geq 0$. □

D. The Induction Proof: Detailed Steps. Let M'_n and M''_n be uniquely optimal n type matchings at θ' and θ''. Proposition 1 assumes summed rectangular synergy dyads in $\hat{\mathcal{D}}_n$. Until Step 8, we work on the larger domain $\mathcal{D}^*_n$.

Premise $\mathcal{P}_n$: *Summed rectangular synergy dyad is in* $\mathcal{D}^*_n \Rightarrow M''_n \succeq_{PQD} M'_n$.

Step 1. *Base Case* $\mathcal{P}_2$: *Summed rectangular synergy dyad is in* $\mathcal{D}^*_2 \Rightarrow M''_2 \succeq_{PQD} M'_2$.

Proof: If not, then NAM is uniquely optimal at θ'' and PAM at θ'. Since $\mathcal{D}^*_2 \subseteq \mathcal{D}_2$ by Fact 3, rectangular synergy is upcrossing in θ. This precludes negative rectangular synergy at θ'' (NAM) and positive rectangular synergy at θ' (PAM). □

- A *pair* refers to two *couples*, such as (i_1, j_1) and (i_2, j_2).
- A pair is a *PAM pair* is $(i_1, j_1) < (i_2, j_2)$, and a *NAM pair* is $i_1 < i_2$ and $j_1 > j_2$.

Step 2. *If the summed rectangular synergy dyad is in* $\mathcal{D}^*_{n+1}$*, then neither* M'_{n+1} *nor* M''_{n+1} *includes a matched NAM pair that exceeds a matched PAM pair.*

Proof: By Fact 3, $\mathcal{D}^*_{n+1} \subseteq \mathcal{D}_{n+1}$. So $S^{n+1}(r|\theta)$ is upcrossing in rectangles r for θ' and θ''. Also, PAM (NAM) is optimal for a pair *iff* $S^{n+1}(r|\theta) \geq (\leq) 0$ on rectangle r. As the optimal matching is unique, $S^{n+1}(r|\theta) \neq 0$ for all optimally matched pairs. □

Steps 3–8 impose premises $\mathcal{P}_2, \ldots, \mathcal{P}_n$, but not $\mathcal{P}_{n+1}$, and arrive at a contradiction:

(‡‡)**:** *In a model with summed rectangular synergy dyads in* $\mathcal{D}^*_{n+1}$*, the uniquely optimal matchings at* $\theta'' \succ \theta'$ *are not ranked* $\mu'' \succeq_{PQD} \mu'$ *(*$\omega'' \succeq_{PQD} \omega'$*).*

Step 3. *At states* θ' *and* θ''*, the matchings obey* $\mu''_1 = \mu'_1 + 1 \geq 2$ *and* $\omega''_1 = \omega'_1 + 1 \geq 2$.

We establish the first relationship. Symmetric steps would prove the second.

Proof of $\mu''_1 > \mu'_1$*:* If not, then $\mu''_1 \leq \mu'_1$. In this case, remove couple $(1, \mu'_1)$ at θ', and couple $(1, \mu''_1)$ at θ''. The remaining matching is PQD higher at θ'', by Induction Premise $\mathcal{P}_n$ and Fact 1. By Fact 5, if we add back the optimally matched pairs $(1, \mu'_1)$ and $(1, \mu''_1)$, then the PQD ranking still holds with $n+1$ types, given $\mu''_1 \leq \mu'_1$, namely $\mu'' \succeq_{PQD} \mu'$. This contradiction to (‡‡) proves that $\mu''_1 > \mu'_1$. □

Proof of $\mu''_1 < \mu'_1 + 2$. If not, then $\mu''_1 \geq \mu'_1 + 2$. By Fact 4, choose a woman $i > 1$ with $\mu''_i < \mu'_i$. Remove couples (i, μ'_i) at θ', and (i, μ''_i) at θ''. Since $\mu''_i < \mu'_i$, the resulting matching is PQD higher at θ'' than θ', by Fact 1 and Premise $\mathcal{P}_n$. In the resulting model, woman 1 is not matched to a higher man at θ'' than θ'. This is impossible if $\mu''_1 \geq \mu'_1 + 2$, as $\mu''_1 - \mu'_1$ falls by at most 1 when removing man μ_i at θ' and μ''_i at θ''. □

Step 4. *The couple* (ω''_1, μ''_1) *is matched at* θ'*, namely,* $\mu'_{\omega''_1} = \mu''_1$ *and* $\omega'_{\mu''_1} = \omega''_1$.

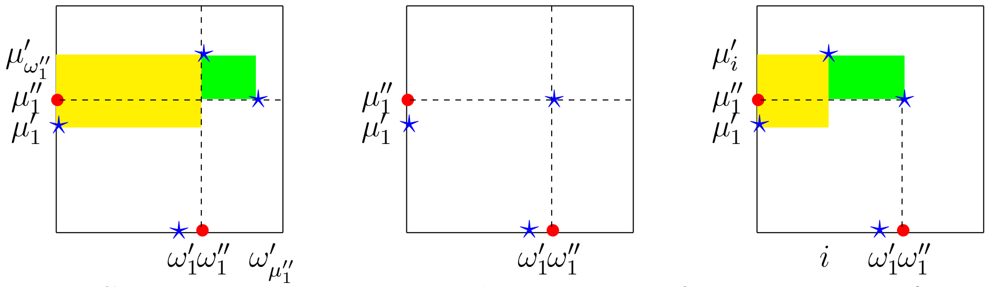

Figure 12: **Steps 4 and 5 in the Induction Proof.** In the counterfactual logic in Step 4 and 5, stars and dots denote respective proposed matched pairs at θ' and θ''. The left panel depicts the NAM pair (green) above the PAM pair (yellow) in Step 4. The middle panel depicts the conclusion of Step 4: man μ''_1 and woman ω''_1 must match under θ'. The right panel depicts the NAM pair above the PAM pair in Step 5-(a).

Proof of $\mu'_{\omega''_1} \geq \mu''_1$ and $\omega'_{\mu''_1} \geq \omega''_1$: We prove the first inequality. If not, then $\mu'_{\omega''_1} < \mu''_1$. As man $\mu'_1 = \mu''_1 - 1$ is matched at θ' by Step 3, $\mu'_{\omega''_1} < \mu''_1 - 1 = \mu'_1$. Remove couple $(\omega''_1, \mu'_{\omega''_1})$ at θ' and $(\omega''_1, 1)$ at θ''. This new matching is PQD higher at θ'', by $\mathcal{P}_n$ and Fact 1. As man $\mu'_{\omega''_1}$ removed at θ' and man 1 removed at θ'' are below $\mu'_1 = \mu''_1 - 1$, the match count weakly below $(1, \mu'_1)$ is unchanged at θ'' and θ'. By Step 3, this count is higher at θ' than θ'', contradicting the n type matching PQD higher at θ''. □

Proof of $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$: Just one strict inequality in part (a) is impossible, as it overmatches some type: $\omega'_{\mu''_1} > \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$ or $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} > \mu''_1$. Next assume two strict inequalities in part (a). As $\mu'_{\omega''_1} > \mu''_1$, the θ' matching includes the PAM pair $(1, \mu'_1) < (\omega''_1, \mu'_{\omega''_1})$ — by Step 3 — and the higher NAM pair $(\omega''_1, \mu'_{\omega''_1})$ and $(\omega'_{\mu''_1}, \mu''_1)$. NAM pairs above PAM pairs violate Step 2 (left panel of Figure 12). □

The middle panel of Figure 12 depicts the takeout of Steps 3–4. We iteratively use this matching patter to show how (‡‡) greatly restricts the matching at θ' and θ''.

Step 5. $\mu'_1 \geq \mu'_i = \mu''_i - 1$ *for* $i = 1, \ldots, \omega'_1$ *and* $\omega'_1 \geq \omega'_j = \omega''_j - 1$ *for* $j = 1, \ldots, \mu'_1$.

Proof: We proved this for $i = 1$ and $j = 1$, and now prove the claimed ordering $\mu'_1 \geq \mu'_i = \mu''_i - 1$ for $i = 2, \ldots, \omega'_1$. By symmetry, $\omega'_1 \geq \omega'_j = \omega''_j - 1$ for $j = 2, \ldots, \omega'_1$.

Part (a): $\mu'_i < \mu'_1$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu'_1$ for some $2 \leq i \leq \omega'_1$. And since $\mu'_i = \mu'_1$ entails overmatching, we have $\mu_i > \mu_1$ for $i = 2, \ldots, \omega'_1$. Thus, μ' involves a PAM pair $(1, \mu'_1) < (i, \mu'_i)$. We claim that (i, μ'_i) and (ω''_1, μ''_1) constitutes a higher NAM pair, violating the upcrossing of $S(r|\theta)$ in r, by Step 2. Indeed, $i \leq \omega'_1 < \omega''_1$ (by the premise above, and Step 3, respectively). Also, $\mu'_i > \mu''_1$, since we have assumed $\mu'_i > \mu'_1$, and deduced $\mu'_1 = \mu''_1 - 1$ in Step 3, and, in Step 4, that μ''_1 is matched to ω''_1 at θ', and we just showed $\omega''_1 > i$. (See the right panel of Figure 12.) □

Part (b): $\mu'_i < \mu''_i$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu''_i$ for some $2 \leq i \leq \omega'_1$. Since $\mu'_i \geq \mu''_i$, if we remove couple (i, μ'_i) at θ' and couple (i, μ''_i) at θ'', then the resulting matching is PQD higher at θ'', by Fact 1 and $\mathcal{P}_n$. In the resulting matching, woman 1's partner is thus not higher at θ'' than θ'. But $\mu_1'' = \mu_1' + 1$ by Step 3, and $\mu_1' > \mu_i' \geq \mu_i''$ by part (a) and the premise of (b). Both removed men μ_i' and μ_i'' are then strictly below μ_1'. So, woman 1's partner is still 1 higher at θ'' than θ'. Contradiction. □

Part (c): $\mu_i' \geq \mu_i'' - 1$ for $i = 2, \ldots, \omega_1'$. If not, then $\mu_{i^*}' < \mu_{i^*}'' - 1$ for some $2 \leq i^* \leq \omega_1'$. Remove couple (ω_1'', μ_1'') at θ' (matched, by Step 4), and the couple $(\omega_1'', 1)$ at θ''. By Fact 1 and Assumption $\mathcal{P}_n$, the resulting matching is PQD higher at θ''.

But since $\omega_1'' > \omega_1'$ by Step 3, all women $i = 1, \ldots, \omega_1'$ remain. Each has a weakly lower partner at θ' than θ'', since we started with $\mu_i' < \mu_i''$ for $i = 1, \ldots, \omega_1'$ by Step 3 for $i = 1$, and part (b) for $i > 1$. Also, woman $i^* \leq \omega_1'$ has a strictly lower partner, as $\mu_{i^*}' < \mu_{i^*}'' - 1$. The resulting matching cannot be PQD higher at θ''. Contradiction. □

Step 6. *The matching μ'' is NAM among men and women at most $\omega_1'' = \mu_1'' \geq 2$.*

Proof of $\omega_1'' = \mu_1''$. By Steps 3 and 5, we get $\mu_1'' = \mu_1' + 1 \geq \mu_i''$ for $i = 1, \ldots, w_1' = \omega_1'' - 1$ and $\mu_1'' \geq 2 > 1 = \mu_{w_1''}''$. So in matching μ'', women $i \leq \omega_1''$ match with men $j \leq \mu_1''$. Hence, $\mu_1'' \geq \omega_1''$. Ditto, by Steps 3 and 5, $\omega_1'' \geq \omega_j''$ for $j = 1, \ldots, \mu_1''$, and in matching ω'', men $j \leq \mu_1''$ match with women $i \leq \omega_1''$. Hence, $\mu_1'' \leq \omega_1''$. Thus, $\mu_1'' = \omega_1'' \geq 2$. □

Proof of $\mu_i'' = \mu_1'' - i + 1$ for $1, \ldots, \omega_1''$. This is an identity at $i = 1$ and true at $i = \omega_1''$ by $\omega_1'' = \mu_1''$ (just proven) and $\mu_{\omega_1''}'' = 1$. So, henceforth assume $i \in \{2, \ldots, \omega_1'' - 1\}$. We claim that for all such i, $\mu_1' \geq \mu_i''$. Indeed, by Steps 3 and 5, $\mu_1'' = \mu_1' + 1 \geq \mu_i''$; and since we do not over match, $\mu_1'' \neq \mu_i''$ for $i \neq 1$. Since $\mu_1' \geq \mu_i''$, Step 5 yields equality $\omega_j' = \omega_j'' - 1$ at $j = \mu_i''$, and so $\omega_{\mu_i''}' = \omega_{\mu_i''}'' - 1 = i - 1$. But then since $\omega_{\mu_{i-1}'}' = i - 1$ and each woman has a unique partner, $\omega_{\mu_i''}' = i - 1$ implies $\mu_i'' = \mu_{i-1}'$. As $\mu_{i-1}' = \mu_{i-1}'' - 1$ by Step 5 and $i \leq \omega_1'' - 1 = \omega_1'$ (by our premise and Step 3), we have $\mu_i'' = \mu_{i-1}'' - 1$. □

An n-type pure matching μ is NAM* if $\mu_n = n$ and $\mu_i = n - i$ for $i = 1, \ldots, n-1$, i.e. NAM among types $1, \ldots, n-1$, so that NAM* = NAM3 when $n = 3$.

Step 7. *The matching μ' is NAM* among men and women at most $\omega_1'' = \mu_1'' \geq 2$.*

Proof: Steps 3, 5 and 6 imply $\mu_i' = \mu_i'' - 1 = \mu_1'' - i$ for $i = 1, \ldots, \omega_1' = \omega_1'' - 1$. Couple (ω_1'', μ_1'') matches under μ', by Step 4. So μ' is NAM* for types $1, \ldots, \mu_1'' = \omega_1''$. □

By Steps 6–7, μ'' is NAM and μ' is NAM* on types $1, \ldots, \omega_1'' = \mu_1'' \equiv k \geq 2$. Since NAM* $\succ_{PQD}$ NAM, if $k < n + 1$ then Premise $\mathcal{P}_k$ fails. Step 8 finishes the proof by showing that NAM at θ'' and NAM* at θ' is also impossible for $k = n + 1$ types.

NAM for men $\{i_1, \ldots, i_\ell\}$ and women $\{j_1, \ldots, j_\ell\}$ is $\{(i_1, j_\ell), (i_2, j_{\ell-1}), \ldots, (i_\ell, j_1)\}$. Rematching to NAM*, $\{(i_1, j_{\ell-1}), (i_2, j_{\ell-2}), \ldots, (i_\ell, j_\ell)\}$ changes payoffs by

$$\textstyle\sum_{u=1}^{\ell-1}(f_{i_u,j_{\ell-u}} - f_{i_u,j_{\ell+1-u}}) + f_{i_\ell,j_\ell} - f_{i_\ell,1} = \sum_{u=1}^{\ell-1}\left[(f_{i_\ell,j_{\ell+1-u}} - f_{i_\ell,j_{\ell-u}}) - (f_{i_u,j_{\ell+1-u}} - f_{i_u,j_{\ell-u}})\right]$$

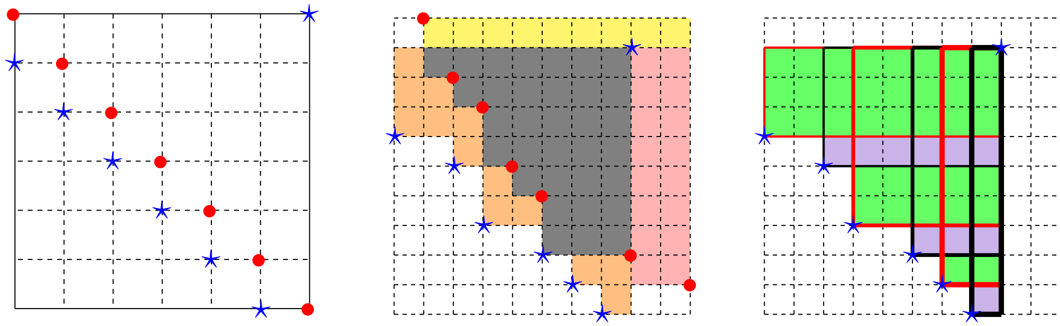

Figure 13: **Step 8 of Induction Proof.** We rule out NAM for θ'' (dots) and NAM* for θ' (stars) with $n+1$ types (left). Middle: These matches with $n+k > n+1$ types, after adding couples weakly higher at θ' than θ''. Let K^G, K^O, K^P, K^Y be the grey, orange, pink, and yellow regions. By (21), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^O|\theta') > 0$, as NAM* is optimal for θ'. But $\mathcal{S}^{n+k}(K^O|\theta') < 0$, as K^O is the union of of rectangles, each below a NAM pair for θ''. So $\mathcal{S}^{n+k}(K^G|\theta') > 0$. By (21), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^P \cup K^Y|\theta'') < 0$, negative by NAM optimal for θ''. Finally, $\mathcal{S}^{n+k}(K^Y|\theta'), \mathcal{S}^{n+k}(K^P|\theta') > 0$, as the yellow and pink rectangles are each above a PAM pair for θ'. So $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. But since $\mathcal{S}^{n+k}(K^G|\theta') > 0$, this contradicts upcrossing summed rectangular synergy in θ. The right panel illustrates Step 8(c).

So *the payoff of NAM* less that of NAM on any subset of ℓ types equals* (suppressing the superscript on S)

$$\textstyle\sum_{u=1}^{\ell-1} S(i_u, j_{\ell-u}, i_\ell, j_{\ell+1-u}) \tag{21}$$

Step 8. *NAM at $\theta'' \Rightarrow \nexists$ NAM* at θ' for summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$.*

Part (a): Contradiction Assumption. For $n+1$ types, posit NAM* and NAM uniquely optimal at θ' and θ'' (Figure 13, panel 1). Induce summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$ by removing $k-1 \geq 0$ optimally matched couples with higher partners at θ' than θ'' (building block (f)) from a summed rectangular synergy dyad $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$. The θ' matching here is NAM* for men $\boldsymbol{i}' = (i'_1, \ldots, i'_{n+1})$ and women $\boldsymbol{j}' = (j'_1, \ldots, j'_{n+1})$, while the θ'' matching with these $n+k$ types is NAM for men $\boldsymbol{i}'' = (i''_1, \ldots, i''_{n+1})$ and women $\boldsymbol{j}'' = (j''_1, \ldots, j''_{n+1})$, with $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (Figure 13, panel 2).

Part (b): couple sets U', U'' with $\mathcal{S}^{n+k}(U''|\theta'') < 0 < \mathcal{S}^{n+k}(U'|\theta')$. For rectangles $r'_u \equiv (i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u})$ and $r''_u \equiv (i''_u, j''_{n+1-u}, i''_{n+1}, j''_{n+2-u})$ define "upper sets":

- $U' \equiv \cup_{u=1}^n r'_u$, the union of the grey and orange rectangles in panel 2 of Figure 13
- $U'' \equiv \cup_{u=1}^n r''_u$, the union of the grey, yellow, and pink regions

As NAM* is uniquely optimal for the subsets of men $\boldsymbol{i}'$ and women $\boldsymbol{j}'$ at θ', it payoff-dominates NAM. Given linearity of summed rectangular synergy at $\ell = n+1$ in (21),

$$\mathcal{S}^{n+k}(U'|\theta') = \sum_{u=1}^{n+1} S^{n+k}(r'_u|\theta') = \sum_{u=1}^{n+1} S^{n+k}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u}|\theta') > 0$$

Likewise, NAM uniquely optimal for subsets $\boldsymbol{i}''$ and $\boldsymbol{j}''$ at θ'' implies $\mathcal{S}^{n+k}(U''|\theta'') < 0$.

Part (c): $\mathcal{S}^{n+k}(K^G|\theta') > 0$ for $K^G \equiv U' \cap U''$. First, $U' = \cup_{u=1}^{n}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+1})$, i.e., a union of rectangles with fixed northeast (Figure 13, panel 3). Likewise, we have $U'' \equiv \cup_{u=1}^{n} r''_u$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), if $(i,j) \in U' \setminus U'' = U' \setminus K^G$ (orange shade, Figure 13, panel 2), then $(i'_{u^*}, j'_{n+1-u^*}) \leq (i,j)$, and $i \leq i''_{u^*}$ or $j \leq j''_{n+1-u^*}$, with at least one strict, at some u^*. So couple (i,j) is below the meet of the θ'' matched NAM pair $(i''_{u^*}, j''_{n+2-u^*})$ and $(i''_{u^*+1}, j''_{n+1-u^*})$. As rectangular synergy is upcrossing in types, $s_{ij}(\theta'') < 0$. Then $s_{ij}(\theta') < 0$, as synergy is upcrossing in θ. Then $\mathcal{S}^{n+k}(U' \setminus K^G|\theta') < 0$, as this holds for all $(i,j) \in U' \setminus K^G$. As summed rectangular synergy is additive and $\mathcal{S}^{n+k}(U'|\theta') > 0$ (part (b)), $\mathcal{S}^{n+k}(K^G|\theta') = \mathcal{S}^{n+k}(U'|\theta') - \mathcal{S}^{n+k}(U' \setminus K^G|\theta') > 0$.

Part (d): $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), define rectangles $K^Y \equiv (i''_1, j'_{n+1}, i'_{n+1}, j''_{n+1})$ and $K^P \equiv (i'_{n+1}, j''_1, i''_{n+1}, j'_{n+1})$ (resp., yellow and pink regions, Figure 13, panel 2). Then $U'' \setminus K^G = K^Y \cup K^P$. As summed rectangular synergy is linear:

$$\mathcal{S}^{n+k}(K^G|\theta) = \mathcal{S}^{n+k}(U''|\theta) - \mathcal{S}^{n+k}(K^Y|\theta) - \mathcal{S}^{n+k}(K^P|\theta) \tag{22}$$

Rectangle K^Y is above the rectangle defined by the θ' PAM pair (i'_1, j'_n) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^Y|\theta'') > 0$, as summed rectangular synergy is upcrossing on rectangles and θ. Likewise, K^P is above the rectangle defined by the θ' PAM pair (i'_n, j'_1) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^P|\theta'') > 0$. Then $\mathcal{S}^{n+k}(K^G|\theta'') < 0$, as $\mathcal{S}^{n+k}(U''|\theta'') < 0$ (part (b)) and (22).

Since $\mathcal{S}^{n+k}(K^G|\theta') > 0$ (part (c)), we cannot have $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$; and thus, by part ($a$) we have contradicted dyads $(\mathcal{S}^{n+1}(K|\theta'), \mathcal{S}^{n+1}(K|\theta'')) \in \mathcal{D}^*_{n+1}$, and thus conclude that NAM at θ'' and NAM* at θ' is impossible.[30] □

D.3 Proof of Proposition 1 for a Continuum of Types

Step 1. *Uniquely optimal finite type matchings exist for a payoff perturbation with summed rectangular synergy upcrossing in θ.*

Proof: Let $\mathcal{X}^n = \{x_1^n, \ldots, x_n^n\}$ and $\mathcal{Y}^n = \{y_1^n, \ldots, y_n^n\}$ be equal quantile increments, with $G(x_1^n) = H(y_1^n) = 1/n$ and $G(x_i^n) = G(x_{i-1}^n) + 1/n$ and $H(y_j^n) = H(y_{j-1}^n) + 1/n$. Let G^n and H^n be cdfs on $[0,1]$, stepping by $1/n$ at $\mathcal{X}^n$ and $\mathcal{Y}^n$ (resp.). Put $f_{ij}^n(\theta) = \phi(x_i^n, y_j^n|\theta)$. The set $\mathcal{M}^n(\theta)$ of pure optimal matchings is non-empty, by Lemma 3.

[30] This last step assumes upcrossing synergy sums on connected *join semi-lattices* (sets that contain the join of any pair of elements). All of our results only require this weaker time series assumption.

Since unique optimal matchings are pure, we restrict to pure matchings. These are uniquely defined by the male partner vector $\mu = (\mu_1, \ldots, \mu_n)$. Call the pure matching $\hat{M}$ *lexicographically higher* than M *iff* its male partner vector $\hat{\mu}$ lexicographically dominates μ. Let $\bar{M}^n(\theta)$ (resp. $\bar{\mu}^n(\theta)$) be the optimal pure matching highest in the lexicographic order, and $\underline{M}^n(\theta)$ (resp. $\underline{\mu}^n(\theta)$) the lowest. Easily, each is well-defined.

Fix $\theta'' \succ \theta'$. Let $\iota(j) = \bar{\mu}^n_j(\theta') - 1$ and pick $\varepsilon > 0$. Perturb synergy down at θ':

$$s^{n\varepsilon}_{ij}(\theta') \equiv s_{ij}(\theta') - \varepsilon^j \mathbb{1}_{(i,j)=(\iota(j),j)} \tag{23}$$

We prove that $\bar{M}^n(\theta')$ is uniquely optimal at θ' for any production function with ε-perturbed synergy (23), for all small $\varepsilon > 0$. Similar logic will prove that $\underline{M}^n(\theta'')$ is uniquely optimal at θ'' with $s^{n\varepsilon}_{ij}(\theta'') \equiv s_{ij}(\theta'') + \varepsilon^j \mathbb{1}_{(i,j)=(\underline{\mu}^n_j(\theta''),j)}$ for all small $\varepsilon > 0$.

Pick a matching M that is not optimal at $\varepsilon = 0$. Since $\bar{M}^n(\theta')$ is optimal at $\varepsilon = 0$, $\bar{M}^n(\theta')$ yields a higher payoff than M for all small $\varepsilon > 0$.

As $\bar{\mu}^n(\theta')$ is the lexicographically highest optimal matching at θ', another optimal μ obeys $(\bar{\mu}^n_1(\theta'), \ldots, \bar{\mu}^n_{\ell-1}(\theta')) = (\mu_1, \ldots, \mu_{\ell-1})$, and first diverges at $\bar{\mu}^n_\ell(\theta') > \mu_\ell$, for some woman $\ell < n$. Using $M_{ij} = \sum_{k=1}^{j} \mathbb{1}_{\mu_k \le i}$, equation (4), and (23), the payoff $\bar{M}^n(\theta')$ exceeds that of $M \in \mathcal{M}^n(\theta')$ by $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s^{n\varepsilon}_{ij}(\theta')\left[\bar{M}^n_{ij}(\theta') - M_{ij}\right]$. This expands to:

$$\sum_{j=1}^{n-1} \varepsilon^j \left[M_{\iota(j)j} - \bar{M}^n_{\iota(j)j}(\theta')\right] \;=\; \varepsilon^\ell + \sum_{j=\ell+1}^{n-1} \varepsilon^j \sum_{k=\ell+1}^{j} \left[\mathbb{1}_{\mu_k \le \iota(j)} - \mathbb{1}_{\bar{\mu}^n_k \le \iota(j)}\right]$$

Altogether, $\lim_{\varepsilon \to 0} \varepsilon^{-\ell} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s^{n\varepsilon}_{ij}(\theta')\left[\bar{M}^n_{ij}(\theta') - M_{ij}\right] = 1 > 0$. □

Step 2. *If $\theta'' \succ \theta'$, then $\bar{M}^n(\theta'') \succeq_{PQD} \underline{M}^n(\theta')$ for all n.*

Proof: Since $S^{n\varepsilon}(r|\theta)$ is continuous in ε, there exists $\hat{\varepsilon}_n > 0$ such that, for all $r = (i_1, j_1, i_2, j_2)$ and $0 \le \varepsilon < \hat{\varepsilon}_n$, if $S^{n0}(r|\theta) \lesseqgtr 0$ then $S^{n\varepsilon}(r|\theta) \lesseqgtr 0$. By the contrapositives:

$$S^{n\varepsilon}(r|\theta) \ge 0 \;\Rightarrow\; S^{n0}(r|\theta) \ge 0 \quad \text{and} \quad S^{n\varepsilon}(r|\theta) \le 0 \;\Rightarrow\; S^{n0}(r|\theta) \le 0. \tag{24}$$

We claim that $S^{n\varepsilon}(r|\theta)$ *is strictly upcrossing in r for all* $0 < \varepsilon < \hat{\varepsilon}_n$. For if not, then $S^{n\varepsilon}(r''|\theta) \le 0 \le S^{n\varepsilon}(r'|\theta)$ for some $r'' \succ_{NE} r'$. But then $S^{n0}(r''|\theta) \le 0 \le S^{n0}(r'|\theta)$ by (24), contradicting $S^{n0}(r|\theta)$ strictly upcrossing in r, as follows from Step 1.

Continuum summed rectangular synergy is upcrossing in θ by assumption; and thus, finite summed rectangular synergy $\sum_{k=1} S^{n0}(r_k|\theta)$ for all finite approximations. Then, perturbed summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k|\theta)$ is upcrossing in θ, since synergy $s^{n\varepsilon}_{ij}(\theta')$ is non-increasing in ε and $s^{n\varepsilon}_{ij}(\theta'')$ is non-decreasing in ε by construction (23).

So for $\varepsilon \in (0, \hat{\varepsilon}_n)$, rectangular synergy $S^{n\varepsilon}(r|\theta)$ is strictly upcrossing in r and summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k|\theta)$ upcrossing in θ, for couple sets $K \subseteq \mathbb{Z}^2_n$. Given $\bar{M}^n(\theta'), \underline{M}^n(\theta'')$ uniquely optimal, $\underline{M}^n(\theta'') \succeq_{PQD} \bar{M}^n(\theta')$ $\forall n$, by Proposition 1. □

Step 3. *There exists a subsequence of matchings $\{M^{n_k}(\theta)\}$ that converges to an optimal matching in the continuum model.*

Proof: Define a step function $\phi^n(x,y|\theta) = f^{n\varepsilon_n}_{ij}(\theta)$ for $(x,y) \in [x^n_{i-1}, x^n_i) \times [y^n_{j-1}, y^n_j)$, where $\varepsilon_n = \hat{\varepsilon}_n/n$. Then $\{G^n\}$ and $\{H^n\}$ weakly converge to G and H as $n \to \infty$, while ϕ^n uniformly converges to ϕ. By Theorem 5.20 in Villani (2008), their optimal matching cdfs have a convergent subsequence $\{M^{n_k}(\theta)\}$ with limit point $M^\infty(\theta)$ optimal in the continuum model.[31] □

Step 4. $M^\infty(\theta'') \succeq_{PQD} M^\infty(\theta')$ *for all* $\theta'' \succeq \theta'$

Proof: Fix $\theta'' \succeq \theta'$, and let $\{n_k\}$ be a subsequence along which the sequence of finite type matchings $\{M^{n_k}(\theta')\}$ converges to $M^\infty(\theta')$, as defined in Step 3. Now, since cdfs $\{G^{n_k}\}$ and $\{H^{n_k}\}$ weakly converge to G and H, and $\phi^{n_k}(x,y|\theta'')$ converges uniformly to $\phi(x,y|\theta'')$, there exists a subsequence $\{n_{k_\ell}\}$ of $\{n_k\}$, along which the sequence of finite type matchings $\{M^{n_{k_\ell}}(\theta'')\}$ converges to $M^\infty(\theta'')$ by Theorem 5.20 in Villani (2008). Further, by Step 2, $M^{n_{k_\ell}}(\theta'') \succeq_{PQD} M^{n_{k_\ell}}(\theta')$. But then, the limits must be ordered $M^\infty(\theta'') \succeq_{PQD} M^\infty(\theta')$ by Theorem 9.A.2.a in Shaked and Shanthikumar (2007). □

D.4 Marginal Rectangular Synergy: Proof of Proposition 2

We assume marginal rectangular synergy is upcrossing in types. The steps for downcrossing marginal rectangular synergy are symmetric. We use the relationship:

$$S(x_1,x_2,y_1,y_2|\theta) = \int_{x_1}^{x_2} \Delta_x(x|y_1,y_2,\theta)dx = \int_0^1 \Delta_x(x|y_1,y_2,\theta)\mathbb{1}_{x\in[x_1,x_2]}dx \qquad (25)$$

Step 1. *(Strictly) upcrossing marginal rectangular synergy ⇒ (strictly) upcrossing rectangular synergy.*

Proof: We prove the continuum case, which implies the finite type result. Any indicator function $\mathbb{1}_{x\in[x_1,x_2]}$ is log-supermodular function in (x,x_1) and (x,x_2).[32] By Karlin and Rubin's classic 1956 result, if $\Delta_x(x|y_1,y_2,\theta)$ is upcrossing in x, then the last integral in (25) is upcrossing in x_1 and x_2, and so in (x_1,x_2). Symmetrically, rectangular synergy is upcrossing in (y_1,y_2) when the y-marginal rectangular synergy is upcrossing in y. Altogether, rectangular synergy S is upcrossing in types if both MPIs are upcrossing.

[31] Namely: Fix a sequence $\{\phi_k\}$ of continuous and uniformly bounded production functions converging uniformly to ϕ. Let $\{G_k\}$ and $\{H_k\}$ be cdf sequences and M_k an optimal matching for ϕ, given G_k and H_k. If G_k and H_k weakly converge to G and H, then some subsequence of $\{M_k\}$ weakly converges to a matching M^* optimal for ϕ, G, and H.

[32] $\phi(x,y) \geq 0$ is *log-supermodular* (LSPM) if $\phi(x',y')\phi(x'',y'') \geq \phi(x',y'')\phi(x'',y')$ for all $x' \leq x''$ and $y' \leq y''$. Easily, we can check that the indicator is LSPM: If $x \in [x_1,x_2]$ and $x' \in [x'_1,x'_2]$ then $\max(x,x') \in [\max(x_1,x'_1), \max(x_2,x'_2)]$ and $\min(x,x') \in [\min(x_1,x'_1), \min(x_2,x'_2)]$.

Now assume $\Delta_x(x|y_1, y_2)$ is strictly upcrossing; and so, if $S(x_1', y_1, x_2', y_2) = 0$ then $\Delta_x(x_1'|y_1, y_2) < 0 < \Delta_x(x_2'|y_1, y_2)$. So $S_{x_1}(x_1', y_1, x_2', y_2) = -\Delta_x(x_1'|y_1, y_2) > 0$ and $S_{x_2}(x_1', y_1, x_2', y_2) = \Delta_x(x_2'|y_1, y_2) > 0$. Then $S(x_1'', y_1, x_2'', y_2) > 0$ for all $(x_1'', x_2'') > (x_1', x_2')$. By symmetric reasoning, S strictly upcrosses in (y_1, y_2). □

Step 2. *The optimal matching is unique in the continuum type model.*

Proof: By Theorem 5.1 in Ahmad, Kim, and McCann (2011), there is a unique optimal matching when: (i) G is absolutely continuous, (ii) ϕ is C^2, and (iii) the critical points of (their "twist difference") $\phi(x, y_2) - \phi(x, y_1)$ include at most one local max and one local min, for all y_1, y_2. Our continuum types model imposes (i) and (ii). We claim that (iii) follows from marginal rectangular synergy $\Delta_x(x|y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x, for $y_2 > y_1$. In particular, if $y_2 > y_1$, then $\Delta_x(x|y_1, y_2)$ is upcrossing in x, and any critical point of the twist difference is a global minimum. Similarly, then any critical point is a global maximum if $y_2 < y_1$. □

Step 3. *Sorting increases in θ.*

Proof: Proposition 1 and Proposition 2 share the same time series assumption. Step 1 establishes that the cross-sectional premise of Proposition 2 implies the cross-sectional premise of Proposition 1. Finally, the optimal matching is generically unique for any finite type model and is unique for continuum type models by Step 2. Altogether, all assumptions in Proposition 1 are met; and thus, sorting increasing in θ. □

D.5 Increasing Sorting: Proof of Proposition 3

FINITE TYPES PROOF. We verify the premise of Proposition 1. First, by Theorem 1, total synergy $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in Z}$ on any set of couples $Z \subseteq \mathbb{Z}_n^2$ is upcrossing in the parameter $t = \theta$. Thus, summed rectangular synergy $\sum_k S(r_k|\theta)$ is upcrossing in θ for any non-overlapping set of rectangles $\{r_k\}$. Next, rectangular synergy $S(r|\theta) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ is upcrossing in r by Theorem 1 with $t = r \in \mathbb{R}^4$. By a similar proof to footnote 32, the indicator function $\mathbb{1}_{(i,j)\in r}$ is LSPM in (i, j, r), since a rectangle r is a sublattice.[33] Then $s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ obeys inequality (7) in $z = (i, j)$ and r, since $s_{ij}(\theta)$ obeys (7) for fixed θ. Rectangular synergy upcrosses in r by Theorem 1. □

CONTINUUM OF TYPES PROOF. We apply Proposition 2. By Theorem 1, total synergy $\int_Z \phi_{12}(x, y|\theta)dxdy$ is upcrossing in $t = \theta$ for any measurable set $Z \subseteq [0, 1]^2$. Thus, summed rectangular synergy $\sum_k S(R_k|\theta)$ is upcrossing in θ for any non-overlapping

[33] Theorem 1 assumes $t \in \mathcal{T}$, a poset. Here we exploit the fact that the space of rectangular *sets* of couples is a sublattice of $\mathbb{Z}^2$, even though the PQD order on *distributions* over couples is not a lattice.

set of rectangles $\{R_k\}$. Next, the x-marginal rectangular synergy $\int \phi_{12}(x,y)\mathbb{1}_{y\in[y_1,y_2]}dy$ is strictly upcrossing in x. Let $x''>x'$. Posit for a contradiction:

$$\int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y_2]}dy \leq 0 \leq \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y_2]}dy \tag{26}$$

As synergy $\phi_{12}(x,y)$ is strictly upcrossing in x and y, by (26), there exist zeros $y',y'' \in (y_1,y_2)$ such that $\phi_{12}(x',y) \lesseqgtr 0$ for $y \lesseqgtr y'$ and $\phi_{12}(x'',y) \lesseqgtr 0$ for $y \lesseqgtr y''$. Easily, these zeros are ordered $y''<y'$. But then inequalities in (26) are simultaneously impossible, for:

$$\begin{aligned} 0 &\leq \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y_2]}dy < \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy \\ \Rightarrow\ 0 &< \int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy < \int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y_2]}dy \end{aligned}$$

by Theorem 1, since $\int \phi_{12}(x,y)\lambda(y)dy$ is upcrossing in $t=x$ for any non-negative $\lambda(y)$ — because $\phi_{12}(x,y)$ is proportionately upcrossing in types and upcrossing in y. □

D.6 Type Distribution Shifts: Proof of Corollary 2

Proof: Throughout, we WLOG assume types shift up in the parameter θ.

Summed Rectangular Quantile Synergy is Upcrossing in θ. In fact, we make the stronger claim that total quantile synergy (9) is upcrossing in θ on any measurable set of quantile pairs $Z \subseteq [0,1]^2$. In the continuum type model:

$$\Upsilon(\theta) \equiv \int\int \varphi_{12}(p,q|\theta)\mathbb{1}_{(p,q)\in Z}dpdq = \int\int \phi_{12}(x,y)\mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy$$

by the change of variables $x=X(p,\theta)$ and $y=Y(q,\theta)$ (equivalently, $p=G(x|\theta)$ and $q=H(y|\theta)$); and thus, $dx=X_p dp$ and $dy=Y_q dq$. Since distributions G and H fall in θ, the cdf associated with pdf $\lambda(x,y|\theta) \equiv \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}/[\int\int \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy]$ is stochastically increasing in θ. And thus, since $\phi_{12}(x,y)$ is strictly increasing:

$$0 \leq \Upsilon(\theta) \Rightarrow 0 \leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta)dxdy \leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta')dxdy \Rightarrow 0 \leq \Upsilon(\theta')$$

Identical yields total synergy upcrossing in θ on any set of couples with finite types.

Case (a): Quantile Rectangular Synergy is Upcrossing. Because types $X(p,\theta)$ and $Y(q,\theta)$ are non-decreasing in the quantiles p and q, *quantile rectangular synergy* $S(X(p_1,\theta),Y(q_1,\theta),X(p_2,\theta),Y(q_2,\theta))$ upcrosses in (p_1,q_1,p_2,q_2). Hence, quantile sorting increases in θ by Proposition 1.

Case (b): Quantile Marginal Rectangular Synergy Strictly Upcrosses. Non-decreasing synergy is proportionately upcrossing; and thus $\Delta_x(x|y_1,y_2)$ strictly upcrosses in x as shown in §D.5. Given $G(x|\theta)$ absolutely continuous $X_p > 0$; and so,

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(X(p, \theta)|Y(q_1, \theta), Y(q_2, \theta))X_p(p, \theta)$$

is strictly upcrossing in p. Similarly, $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in q. All told, we've seen that quantile sorting increases in θ, by Step 1 and Proposition 2. □

E Omitted Analysis for Economic Applications

1. Diminishing Returns: Let $\phi(x, y|\theta) = \psi(xy|\theta)$ with $\psi'' < 0 < \psi'$ and $R(z, \theta) \equiv -z\psi''(z|\theta)/\psi'(z|\theta)$ falls in z and θ, then synergy $\psi'(xy|\theta)(1-R(xy, \theta))$ strictly upcrosses in x, y, and θ. Easily, $\psi'(xy|\theta)$ is smoothly LSPM in (x, y, θ) when $R(z, \theta)$ decreases in z and θ. A product of a positive smoothly LSPM function and an increasing function, synergy is proportionately upcrossing. So sorting increases in θ by Proposition 3.

2. Convex Technologies: We verify the premise of Proposition 2 to prove that sorting sorting increases in ρ for $\phi(x, y) = \psi(q(x, y))$ as in §6.2. Symmetric steps generalize this result for any $\psi'' < 0 < \psi'$, obeying $2\psi''(q) + q\psi'''(q) \leq 0$.

$$\phi_{12}(x, y) = \frac{q_1(x, y)q_2(x, y)}{q(x, y)} \left[(1 + \rho)(\alpha - 2\beta q(x, y) - 2\beta q(x, y)\right] \tag{27}$$

Step 1. *Marginal rectangular synergy is strictly downcrossing in types.*

Proof: Since $q(x, y)$ increases in (x, y) and falls in ρ, the bracketed term in (27) falls in (x, y) and rises in ρ. Thus, synergy (27) is upcrossing in ρ and is strictly downcrossing in (x, y). Further, since $q_1(x, y)q_2(x, y)/q(x, y)$ is LSPM in (x, y) when $\rho \geq 0$, synergy is proportionately downcrossing in (x, y). So, marginal rectangular synergy is downcrossing in types, by Theorem 1. Finally, marginal rectangular synergy is strictly downcrossing in (x, y) by the proof logic after inequality (26) in Appendix D.5. □

Step 2. *Summed rectangular synergy is upcrossing in ρ.*

Proof: Since $\phi_{12}(x, y) = \phi_{12}(y, x)$, weighted synergy $\int_{[0,1]^2} \phi_{12}\hat{\lambda}$ is upcrossing in ρ for all weighting functions $\hat{\lambda}$, *iff* $\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dxdy$ is upcrossing in ρ for all weighting functions λ. Now use change of variable $y = kx$ to get:

$$\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dydx = 2\int_0^1 \int_0^1 x\phi_{12}(x, kx)\lambda(x, kx)dkdx$$

Let $x\phi_{12}(x, kx) = \sigma_A(k, \rho)\sigma_B(x, k, \rho)$, where $\sigma_A \equiv xq_1(x, kx)q_2(x, kx)/q(x, kx)$ and σ_B is the bracketed term in (27) evaluated at $y = kx$. Routine algebra yields $\sigma_A(k, \rho)$

LSPM in (k,ρ), while $\sigma_B(x,k,\rho)$ is decreasing in (x,k) and increasing in ρ. Altogether, $\sigma_A\sigma_B$ is proportionately upcrossing in (x,k,ρ). As synergy is also upcrossing in ρ by Step 1, so is weighted synergy, by Theorem 1 — as is summed rectangular synergy. □

3. Imperfect Credit Markets. Differentiating production function (11) in x yields:

$$\phi_1(x,y) = \frac{(\pi - c - d)(1-\delta y^2) + 2cy}{(1-\delta x - \delta y + \delta xy)^2} > 0 \tag{28}$$

As $\partial[\phi_1(x,y_2)/\phi_1(x,y_1)]/\partial x < 0$ for all $y_2 > y_1$,[34] the x-marginal rectangular synergy is strictly downcrossing in x — and symmetrically, for the y-marginal rectangular synergy.

Next consider synergy as a function of parameters $\theta = (c,d,\pi)$. Differentiating (28), yields $\phi_{12}(x,y|\theta) = ca(x,y) + (\pi - d)b(x,y)$ for functions $a(x,y) > 0$ and $b(x,y) \in \mathbb{R}$. Synergy is increasing in c; and thus, so is summed rectangular synergy. Synergy is linear in $\pi - d$ with a positive intercept; thus, summed rectangular synergy is downcrossing in $\pi - d$ by Claim 1 in Appendix D.1. Altogether, sorting is increasing in c and decreasing in $\pi - d$, by Proposition 2.

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[34] Indeed, $D_x[\log(\phi_1(x,y_2)/\phi_1(x,y_1))] = \frac{2\delta(1-\delta)(y_1-y_2)}{[1-\delta(x+y_1(1-x))][1-\delta(x+y_2(1-x))]}$. The numerator is negative by $\delta \in (0,1)$ and $y_1 < y_2$, and the denominator is positive since x, y_1, and y_2 are probabilities.

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F Nowhere Decreasing Optimizers

The space of matching cdf's is not a lattice, since the meet and the join are not defined for arbitrary matchings.[35] The matching problem (3) does not have a lattice constraint or an objective function that is quasi-supermodular in the control: standard monotone comparative static results (e.g. Milgrom and Shannon (1994)) do not apply. The next section presents a general comparative result static for single-crossing functions on partially ordered sets (*posets*) without assuming a well-defined meet or join.[36] We then apply this result to our sorting model to get a nowhere decreasing sorting result.

F.1 Nowhere Decreasing Optimizers for Arbitrary Posets

Let Z and Θ be posets. The correspondence $\zeta : \Theta \to Z$ is *nowhere decreasing* if $z_1 \in \zeta(\theta_1)$ and $z_2 \in \zeta(\theta_2)$ with $z_1 \succeq z_2$ and $\theta_2 \succeq \theta_1$ imply $z_2 \in \zeta(\theta_1)$ and $z_1 \in \zeta(\theta_2)$.

Theorem 3 (Nowhere Decreasing Optimizers)**.** *Let $F : Z \times \Theta \mapsto \mathbb{R}$, where Z and Θ are posets, and let $Z' \subseteq Z$. If $\max_{z \in Z'} F(z, \theta)$ exists for all θ and F is single crossing in (z, θ), then $\mathcal{Z}(\theta|Z') \equiv \arg\max_{z \in Z'} F(z, \theta)$ is nowhere decreasing in θ for all Z'. If $\mathcal{Z}(\theta|Z')$ is nowhere decreasing in θ for all $Z' \subseteq Z$, then $F(z, \theta)$ is single crossing.*

($\Rightarrow$): If $\theta_2 \succeq \theta_1$, $z_1 \in \mathcal{Z}(\theta_1)$, $z_2 \in \mathcal{Z}(\theta_2)$, and $z_1 \succeq z_2$, optimality and single crossing give:

$$F(z_1, \theta_1) \geq F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) \geq F(z_2, \theta_2) \quad \Rightarrow \quad z_1 \in \mathcal{Z}(\theta_2)$$

Now assume $z_2 \notin \mathcal{Z}(\theta_1)$. By optimality and single crossing, we get the contradiction:

$$F(z_1, \theta_1) > F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) > F(z_2, \theta_2) \quad \Rightarrow \quad z_2 \notin \mathcal{Z}(\theta_2)$$

($\Leftarrow$): If F is not single crossing, then for some $z_2 \succeq z_1$ and $\theta_2 \succeq \theta_1$, either: (i) $F(z_2, \theta_1) \geq F(z_1, \theta_1)$ and $F(z_2, \theta_2) < F(z_1, \theta_2)$; or, (ii) $F(z_2, \theta_1) > F(z_1, \theta_1)$ and $F(z_2, \theta_2) \leq F(z_1, \theta_2)$. Let $Z' = \{z_1, z_2\}$. In case (i), $z_2 \in \mathcal{Z}(\theta_1|Z')$ and $z_1 = \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_2 \notin \mathcal{Z}(\theta_2|Z')$. In case (ii), $z_2 = \mathcal{Z}(\theta_1|Z')$ and $z_1 \in \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_1|Z')$. □

F.2 Nowhere Decreasing Sorting

Sorting is nowhere decreasing in θ if the matching never falls in the PQD order. So for all $\theta_2 \succeq \theta_1$, if $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked $M_1 \succeq_{PQD} M_2$, then we have

[35] As shown in Proposition 4.12 in Müller and Scarsini (2006): If M dominates PAM2 and PAM4, then $M(2,1) \geq 1/3$ and $M(1,2) \geq 1/3$, but $M(1,1) = 0$ if NAM1 and NAM3 dominate M. So then $M(2,2) = 2/3$, but then NAM1 cannot PQD dominate M.

[36] This may be a known result. We include it for completeness, and as we cannot find any reference.

$M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$. We say that *weighted synergy is upcrossing*[37] in θ if the following is upcrossing in θ:

- $\int \phi_{12}(x,y|\theta)\lambda(x,y)dxdy$ for all nonnegative (measurable)[38] functions λ on $[0,1]^2$
- $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\lambda_{ij}$ for all positive weights $\lambda \in \mathbb{R}_+^{(n-1)^2}$

We first present the continuum analogue of the finite match output formula (4).[39]

Lemma 4 (Continuum Types). *Given type intervals $\mathcal{I} \equiv [0,1]$ and $\mathcal{J} \equiv (0,1]$, then:*

$$\int_{\mathcal{I}^2} \phi(x,y)M(dx,dy) = \int_{\mathcal{I}} \phi(x,1)G(dx) - \int_{\mathcal{J}} \phi_2(1,y)H(y)dy + \int_{\mathcal{J}^2} \phi_{12}(x,y)M(x,y)dxdy$$

PROOF: If ψ is C^1 on $[0,1]$ and Γ is a cdf on $[0,1]$, integration by parts yields:

$$\int_{[0,1]} \psi(z)\Gamma(dz) = \psi(1)\Gamma(1) - \int_{(0,1]} \psi'(z)\Gamma(z)dz \tag{29}$$

where the interval $(0,1]$ accounts for the possibility that Γ may have a mass point at 0. Since $M(dx,y) \equiv M(y|x)G(dx)$ for a conditional matching cdf $M(y|x)$, we have:

$$M(x,y) \equiv \int_{[0,x]} M(y|x')G(dx') \tag{30}$$

By Theorem 34.5 in Billingsley (1995) and then in sequence (29), (30) and Fubini's Theorem, (29), the objective function $\int_{[0,1]^2} \phi(x,y)M(dx,dy)$ in (3) equals:

$$\begin{aligned}
& \int_{[0,1]}\int_{[0,1]} \phi(x,y)M(dy|x)G(dx) \\
= \; & \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]}\int_{(0,1]} \phi_2(x,y)M(y|x)dyG(dx) \\
= \; & \int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]} \left[\phi_2(1,y)M(1,y) - \int_{(0,1]} \phi_{12}(x,y)M(x,y)dx\right] dy
\end{aligned}$$

which easily reduces to the desired expression, using $M(1,y) = H(y)$. □

Theorem 4. *Sorting is nowhere decreasing in θ if weighted synergy is upcrossing in θ, and thus if synergy is nondecreasing in θ. Also, if sorting is nowhere decreasing in θ for all type distributions G, H, then any rectangular synergy is upcrossing in θ.*

[37] Let Z be a partially ordered set. The function $\sigma : Z \mapsto \mathbb{R}$ is *upcrossing* if $\sigma(z) \geq (>)0$ implies $\sigma(z') \geq (>)0$ for $z' \succeq z$, *downcrossing* if $-\sigma$ is upcrossing. Similarly, σ is strictly upcrossing if $\sigma(z) \geq 0$ implies $\sigma(z') > 0$ for all $z' \succ z$, with strictly downcrossing defined analoguously.

[38] To save space, we henceforth assume measurable sets for integrals whenever needed.

[39] Equation (9) in Cambanis, Simons, and Stout (1976) reduces to our formula when output is C^2. We present our simpler proof for the C^2 case for completeness.

Proof of (a): First, $M' \succeq_{PQD} M$ iff $\lambda \equiv M' - M \geq 0$. As weighted synergy upcrosses:

$$\begin{array}{c} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M_{ij}) \geq (>)\ 0 \Rightarrow \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M_{ij}) \geq (>)\ 0 \\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta)(M' - M) \geq (>)\ 0 \ \Rightarrow\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta')(M' - M) \geq (>)\ 0 \end{array} \tag{31}$$

Thus, match output is single crossing in (M, θ) by (4) (for finite types) and Lemma 4 for continuum types. Then the optimal matching $\mathcal{M}^*(\theta)$ (in the space of feasible matchings $\mathcal{M}(G, H)$) is nowhere decreasing in the state θ, by Theorem 3.

Proof of (b): Assume two women (x_1, x_2) and men (y_1, y_2), and that $S(R|\theta)$ is not upcrossing in θ, i.e. for some $\theta'' \succeq \theta'$ and rectangle $R = (x_1, y_1, x_2, y_2)$, we have $S(R|\theta'') \leq 0 \leq S(R|\theta')$ with one inequality strict. These inequalities imply that NAM optimal at θ'' and PAM optimal at θ', and either NAM is uniquely optimal at θ'' or PAM is uniquely optimal at θ'. Either case precludes nowhere decreasing sorting. □

Easily, weighted synergy is upcrossing in θ if synergy is non-decreasing in θ. Thus:

Corollary 3 (Cambanis, Simons, and Stout (1976))**.** *Sorting is nowhere decreasing in θ if synergy is non-decreasing in θ.*

F.3 Nowhere Decreasing Sorting in Kremer and Maskin (1996)

We prove that *sorting is nowhere decreasing in θ and nowhere increasing in ϱ* in (10).

Step 1. *PAM is not optimal if $\varrho > (1-2\theta)^{-1}$, and is uniquely optimal for $\varrho < (1-2\theta)^{-1}$.*

Proof: In a unisex model, PAM is optimal iff the symmetric rectangular synergy $S(x, x, y, y)$ is globally positive. Its sign is constant along any ray $y = kx$, and proportional to:

$$s(k) \equiv 2^{\frac{1-2\theta}{\varrho}}(1 + k) - 2k^{\theta}(1 + k^{\varrho})^{\frac{1-2\theta}{\varrho}} \tag{32}$$

Since $s(1) = s'(1) = 0$, $s''(1) \propto (1 + \varrho(2\theta - 1))$, and $\theta \in [0, 1/2]$, we have $s(k) < 0$ close to $k = 1$ precisely when $\varrho > (1 - 2\theta)^{-1} \geq 1$. In this case, the symmetric rectangular synergy is negative in a cone around the diagonal, and PAM fails.

Conversely, posit $\varrho < (1-2\theta)^{-1}$. Then $s(k) > 0$ for all $k \in [0, 1]$. Since $S(x, x, y, y)$ is symmetric about $y = x$, it is globally positive and PAM is uniquely optimal. □

Step 2. *If $\varrho \geq (1 - 2\theta)^{-1}$ then weighted synergy is upcrossing in θ, downcrossing in ϱ.*

Proof: Change variables $y = kx$. If $\Delta(k) = \int_0^1 \lambda(x, kx)dx$, weighted synergy is

$$\int\int \phi_{12}(x, y)\lambda(x, y)dydx = 2\int_0^1\int_0^1 x\phi_{12}(x, kx)\lambda(x, kx)dkdx = \int_0^1 \sigma(k, \theta, \varrho)\Delta(k)dk$$

where $\sigma = \sigma_A\sigma_B$ for $\sigma_A \equiv 2k^{\theta-1}(1 + k^{\varrho})^{\frac{1-2\theta-2\varrho}{\varrho}}$ and $\sigma_B \equiv \theta(1 - \theta)(1 + k^{2\varrho}) + (1 - \varrho + 2\theta(\theta - 1 + \varrho))k^{\varrho}$. As $\varrho \geq (1 - 2\theta)^{-1}$, $\sigma_A > 0$ is LSPM in (k, θ, ϱ), σ_B is increasing in $(\theta, -k, -\varrho)$ for $k \in [0, 1]$. So $\sigma = \sigma_A\sigma_B$ is proportionately downcrossing in (k, θ) and $(k, -\varrho)$. Weighted synergy is upcrossing in θ, downcrossing in ϱ, by Theorem 1. □

Step 3. *Sorting is nowhere decreasing in θ and nowhere increasing in ϱ.*

Proof: Pick $\theta'' > \theta'$. If $\varrho < (1-2\theta'')^{-1}$, then PAM is uniquely optimal at θ'' (Step 1) and sorting increases from θ' to θ''. If $\varrho \geq (1-2\theta'')^{-1}$, then $\varrho > (1-2\theta')^{-1}$ and weighted synergy is upcrossing on $[\theta', \theta'']$ (Step 2) and sorting is non-decreasing (Proposition 4).

Now pick any θ and $\varrho'' > \varrho'$. If $\varrho' < (1-2\theta)^{-1}$, then PAM is uniquely optimal at ϱ' (Step 1) and sorting is decreasing from ϱ' to ϱ''. If, instead, $\varrho' \geq (1-2\theta)^{-1}$, then, necessarily, $\varrho'' > (1-2\theta)^{-1}$, weighted synergy is downcrossing from ϱ' to ϱ'' (Step 2) and sorting is non-increasing in ϱ, by Proposition 4. □