An Economic Theory Masterclass

Part I: Matching Foundations of Economics

Lones Smith

February 11, 2021

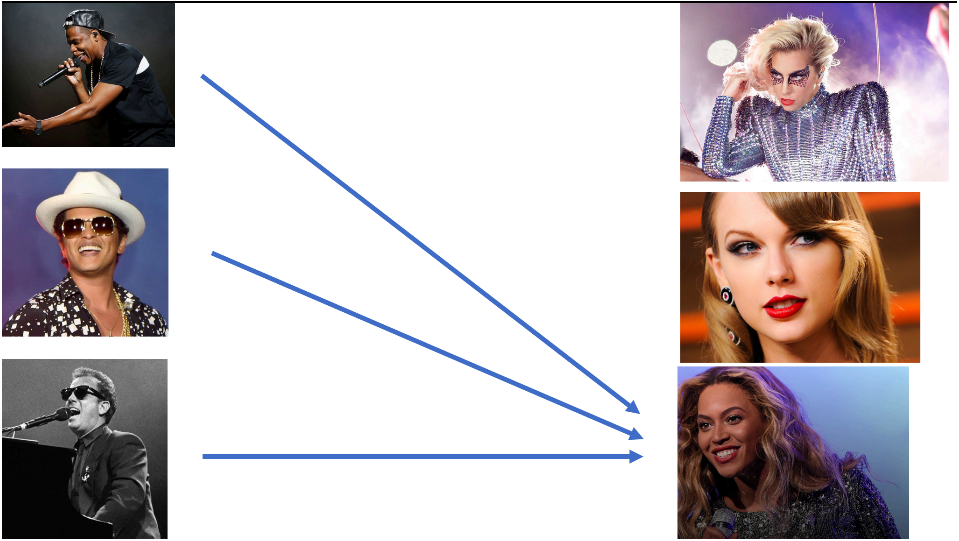
The Matching Paradigm as Metaphor Economic Interaction

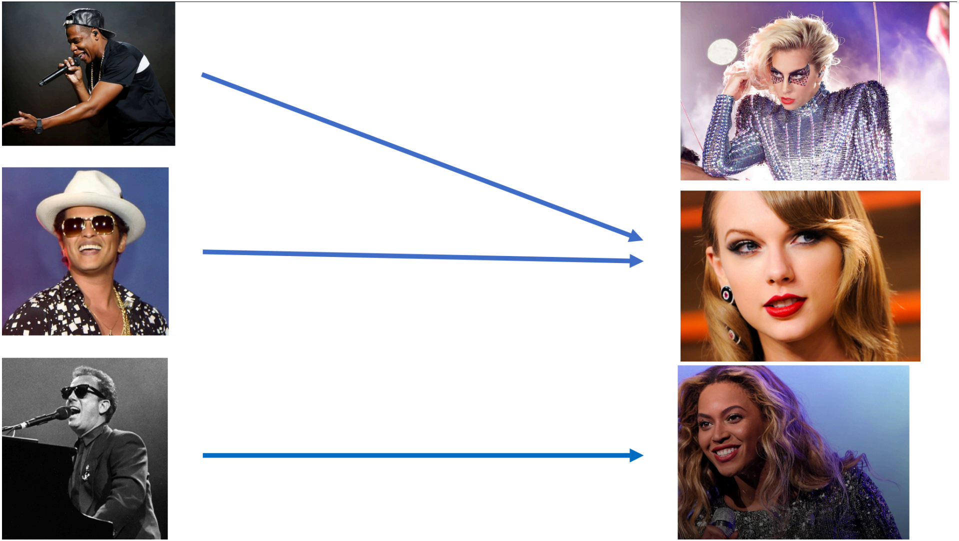
- Buzz for Matching Models:

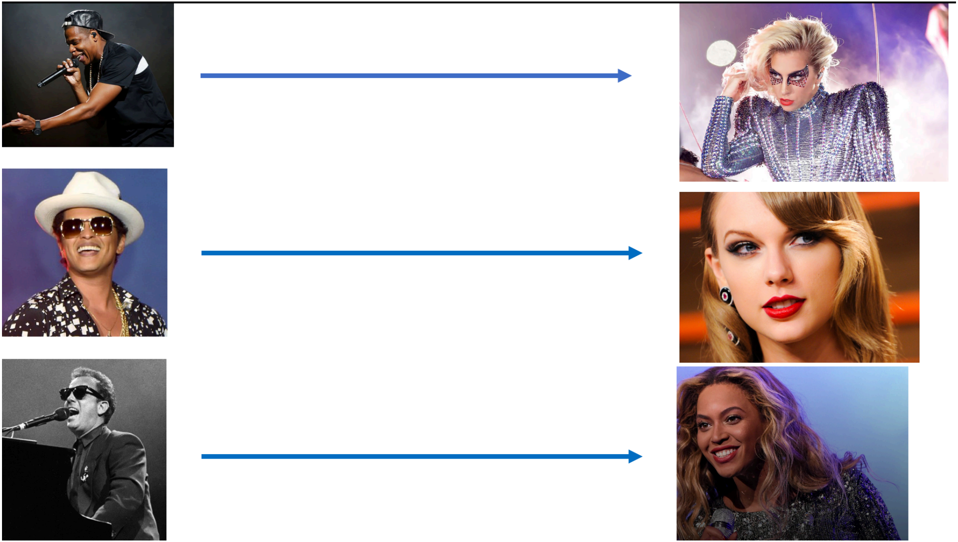
- Pairwise matching models with transferable utility capture in a simplest form the economic structures of many settings:
  - assigning tasks to individuals
  - buyers and sellers trading
  - partnerships, and maybe marriages
  - firms hiring workers
- metaphor: two sides of the market are "men" and "women"
- We wish to understand: Who trades with whom? Who pairs with whom? Who marries whom? Who works with whom?

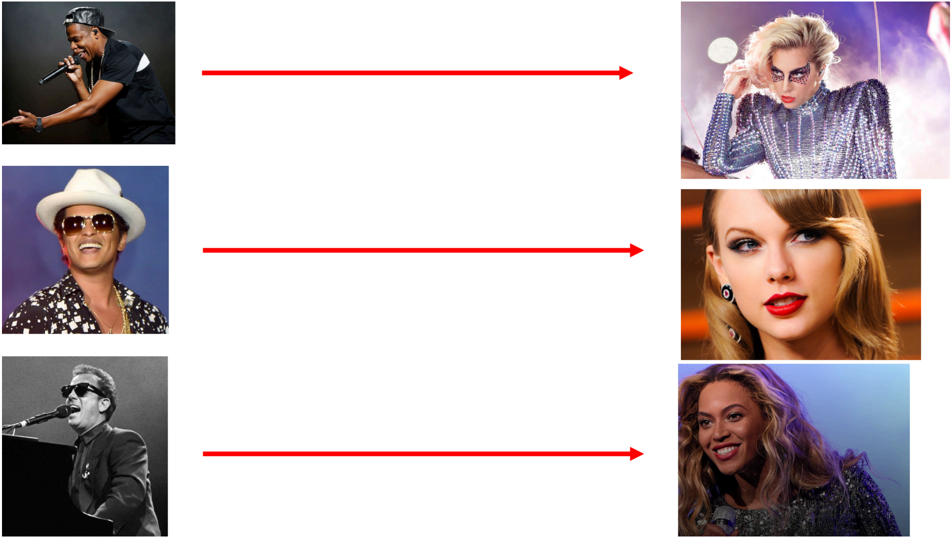
Matching without Transfers: The Girl-Guy Band Contest

- Contest of Beyonce, Taylor Swift, and Lady Gaga to sing a duet with concert with Billy Joel, Bruno Mars, and Jay-Z
- We first only specify **ordinal preferences**
- Men commonly rank: Beyonce $>$ Taylor Swift $>$ Lady Gaga
- Women commonly rank: Billy Joel $>$ Bruno Mars $>$ Jay-Z









Deferred Acceptance Algorithm (DAA)

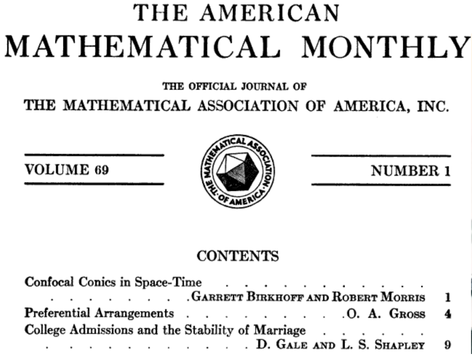
1. All men start unengaged and women start with no suitors.
2. Each unengaged man *proposes* to his most-preferred woman (if any) among those he has not yet proposed to, if he prefers matching to remaining single;
3. Each woman gets *engaged* to the most preferred among all her suitors, including any prior engagements, if she prefers matching with him to remaining single.
4. Rinse and repeat until no more proposals are possible. Engagements become matches.

"I have to go. I'm getting a better call."

Stability

- Matchings should ideally exploit all gains from trade.
- An assignment is *unstable* if there are men, say Alan and Bob, respectively matched to women Alice and Bea, such that Bob prefers Alice to Bea and Alice prefers Bob to Alan
- Say that the matching of Bob and Alice blocks the matching.
- A matching is *stable* if it is not unstable, i.e. $\nexists$ blocking pair.

Gale-Shapley Theorem



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Proposition (Gale & Shapley, American Math Monthly, 1962)

(*a*) *The DAA stops in finite time.*

(*b*) *Given an equal number of men and women, if matching with someone beats remaining single, then everybody matches.*

(*c*) *The DAA matching is stable, i.e. a stable matching exists.*

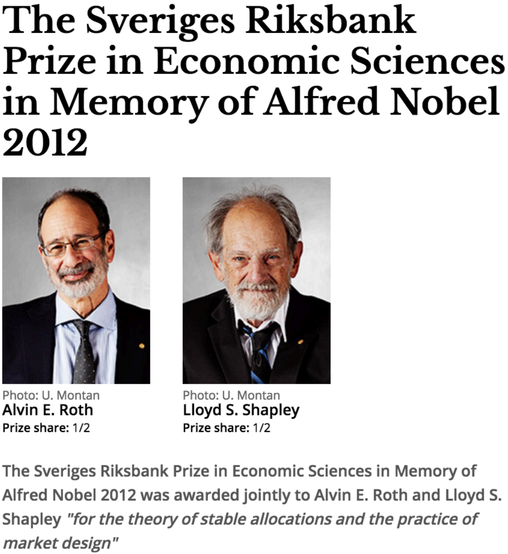
(*d*) *Given strict preferences, the DAA yields a unique matching.*

Proof of Gale-Shapley Theorem

- At each iteration, one man proposes to some new woman
- Let Alice and Bob be married, but not to each other.
- **Claim**: *After the DAA, Alice and Bob cannot prefer each other to their match partners.*
- If Bob prefers Alice to his match partner, then he must have proposed to Alice before his match partner.
- If Alice accepted, yet ends up not married to him, then she must have dumped him for someone she prefers
  $\Rightarrow$ Alice doesn't prefer Bob to her current partner.
- If Alice rejected Bob's proposal, then she was already engaged to someone she prefers to Bob. □

Proof of Gale-Shapley Theorem

- **Claim**: With n men and n women, there are at most n^2 possible ways men can propose.
    - A each stage, one man proposes to someone to whom he has never proposed before
    - With n men and n women, there are n^2 possible events
    - In fact, the maximum number of DAA steps is $n^2 - 2n + 2$.
    - Exercise: Illustrate this for the cases $n = 2$ and $n = 3$. (Solution is in class notes.)
- Al Roth found that the DAA was used to match interns to hospitals. This was a major reason for:



The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Photo: U. Montan

Alvin E. Roth

Prize share: 1/2

Photo: U. Montan

Lloyd S. Shapley

Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

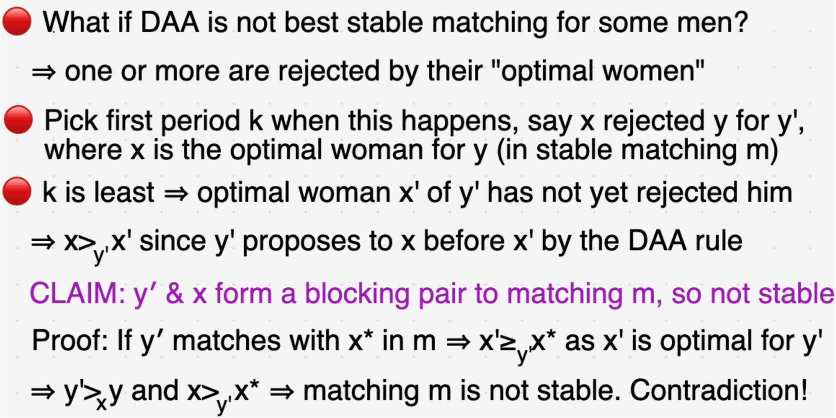
Ranking Stable Matchings

- Assume several stable matchings.
- The set of stable matchings is nonempty.
- x is a **valid partner** of y if they pair in some stable matching.
- In a **male optimal** matching, each man pairs with his best valid partner.
- In a **male pessimal** matching, each man pairs with his worst valid partner.
- Similarly define **woman-optimal** and **woman-pessimal**.

The DAA Yields the Male Optimal Stable Allocation

Proposition (Male Optimality of DAA)

The DAA finds a male-optimal / female-pessimal stable matching.



- What if DAA is not best stable matching for some men?

  ⇒ one or more are rejected by their "optimal women"
- Pick first period k when this happens, say x rejected y for y', where x is the optimal woman for y (in stable matching m)
- k is least ⇒ optimal woman x' of y' has not yet rejected him

  ⇒ $x >_{y'} x'$ since y' proposes to x before x' by the DAA rule

CLAIM: y′ & x form a blocking pair to matching m, so not stable

Proof: If y′ matches with x* in m ⇒ $x' \geq_{y'} x^*$ as x' is optimal for y'

⇒ $y' >_x y$ and $x >_{y'} x^*$ ⇒ matching m is not stable. Contradiction!

Men Optimal Implies Female Pessimal

Unique Stable Outcomes

Corollary (Uniqueness)

The DAA produces the same matching, regardless of which side proposes, if and only if there is a unique stable matching.

- If the stable matching is unique, then the DAA yields the same result regardless of which side proposes.
- If the DAA yields the same result regardless of which side proposes, then it is both optimal and pessimal for both sides, and so is unique. ☐

Three Stable Matchings, but two Outcomes from the DAA

	x_1	x_2	x_3
y_1	5,5	6,2	2,6
y_2	2,6	5,5	6,2
y_3	6,2	2,6	5,5

- When men offer in the DAA, we get the male-optimal and female pessimal matching, where men earn 6 and women 2.
- When women offer in the DAA, we get the female-optimal and male pessimal matching, where women earn 6 and men 2.
- A third stable matching yields payoffs of 5 for everyone.

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(Malibu usually gets what she wants)

Cardinal Preferences

- Start with nontransferable payoffs (all in millions of dollars).
- This might be by organizational rule, eg. NCAA rules forbid payoffs to athletes.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	6,21	12,12	**18,3**
Bruno Mars	4,14	**8,8**	12,2
Jay-Z	**2,7**	4,4	6,1

- Men commonly rank: Beyonce $>$ Taylor Swift $>$ Lady Gaga
- Women commonly rank: Billy Joel $>$ Bruno Mars $>$ Jay-Z
- DAA ends in three periods!

Transferable Utility

- Assume cardinal payoffs (or cardinal utility) is money.
- Every man and woman cares only about total money
- This is a special case of quasi-linear utility, or utility $U(a, z) = u(a) + z$, where a is a real action and z is money
- Quasi-linear utility precludes income effects on the action

Transfers and Bribery

Lady Gaga's Corrupt Thought:

- Gaga schemes to match up with Billy Joel. To do this, she
  - bribes Billy more than his loss of $18 - 6 = 12$ to accept her,
  - pays Beyonce more than her loss of $3 - 1 = 2$, and
  - collects from Jay-Z less than his gain $6 - 2 = 4$ from matching with Billy
- These bribes on net cost as much as $12 + 2 - 4 = 10$. But Lady Gaga gains $21 - 7 = 14$ by matching with Billy Joel.

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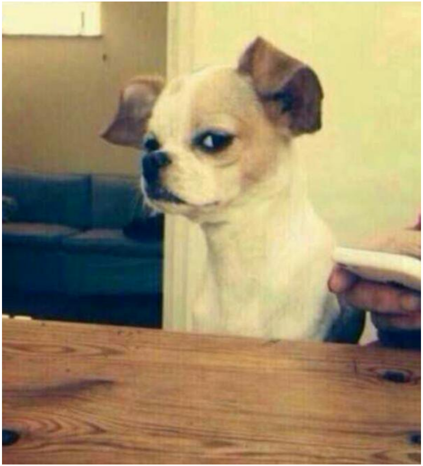
Fido Wonders if Money Helps for Matching Efficiency



Making Matching Immune to Bribery

Only total match payoffs matter in the end with transfers.

♂\♀	Lady Gaga	Taylor Swift	Beyonce
Jay-Z	$6 + 21 = 27$	$12 + 12 = 24$	$18 + 3 = 21$
Bruno Mars	$4 + 14 = 18$	$8+8=16$	$12 + 2 = 14$
Billy Joel	$2 + 7 = 9$	$4 + 4 = 8$	$6 + 1 = 7$

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Bruno Mars	18	16	14
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Making Matching Immune to Bribery

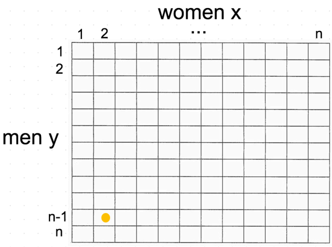
- A matching is **immune to bribes** if there is no set of matched individuals for whom a profitable re-matching exists.
- An **efficient** matching maximizes the sum of payoffs.

Theorem *An efficient matching is immune to bribes.*

♂╲♀	Lady Gaga	Taylor Swift	Beyonce
Billy Joel	**27**	24	21
Bruno Mars	18	**16**	14
Jay-Z	9	8	**7**

Efficient Matching

- Matching Sudoku: Efficiently match n men to n women.
- $=$ Place exactly one dot in every row and column



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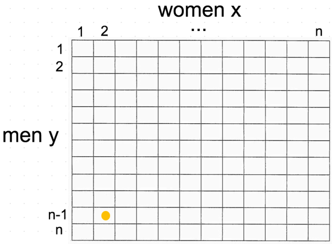
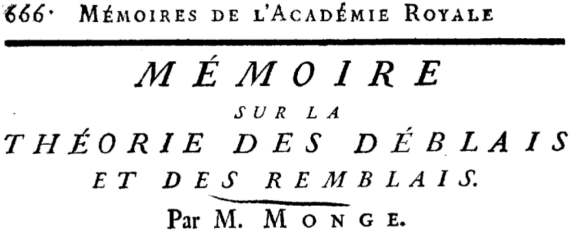


- Obviously, an efficient matching exists. But what is it?
- Problem: There are $n! = 1 \times 2 \times \cdots \times n$ possible allocations.
- E.g. there are 10^{158} pairings of 100 men and 100 women. The number of electrons in the universe is estimated at 10^{80}.

1781 — Transportation Problem: How Best to Move Dirt



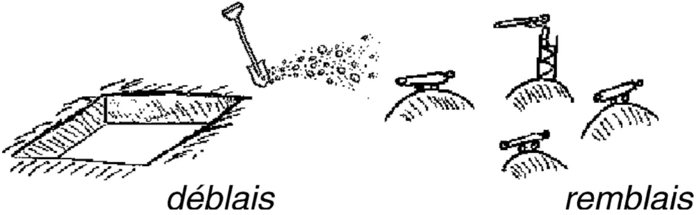
666· MÉMOIRES DE L'ACADÉMIE ROYALE

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.
Par M. MONGE.

1781 — Transportation Problem: How Best to Move Dirt

- Holy Grail of Matching: Solving for the optimal matching in general is too hard — for the transportation problem has been long open, since Gaspard Monge (1781), *Mémoire sur la théorie des déblais et des remblais.* De l'Imprimerie Royale.
- Father of differential geometry
- Assign unit dirt piles $x_i \in \{x_1, ..., x_n\}$ to holes $y_i \in \{y_1, ..., y_n\}$ to minimize the sum of transportation costs $c(x_i, y_j)$?

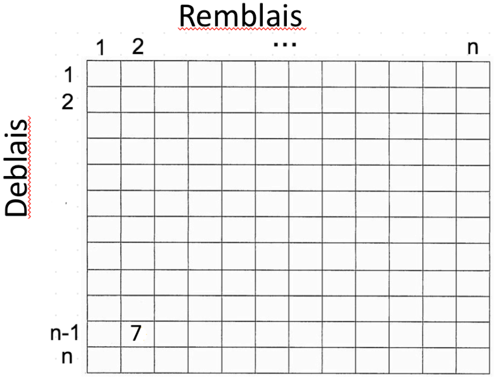
1781 — The Transportation Problem: How to Move Dirt



- Assume the cost of transporting earth from a cut (déblais) to a fill (remblais) depends on the distance, roads, etc.
- $c(x, y) =$ cost of moving dirt from déblais x to remblais y
- What is the cheapest way to transport all the earth from every déblais to some other remblais, while omitting no déblais and overfilling no remblais?
- As formulated, this is an impossible combinatorics exercise.

1781 — The Transportation Problem

- Start with an $n \times n$ matrix of costs $c(x, y)$
- E.g: It costs 7 to move the dirt in déblais $n - 1$ to remblais 2



1957: Transportation Problem as the Assignment Problem

- 160 years passes and linear programming is invented in WWII, by many in USA (e.g. Dantzig) and Kantorovich in Russia

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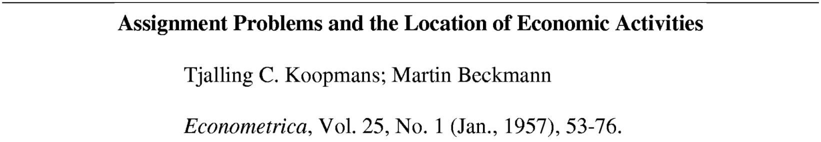

Assignment Problems and the Location of Economic Activities

Tjalling C. Koopmans; Martin Beckmann

Econometrica, Vol. 25, No. 1 (Jan., 1957), 53-76.

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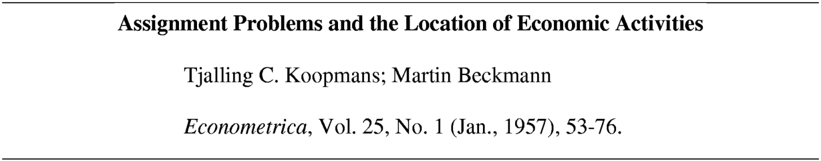

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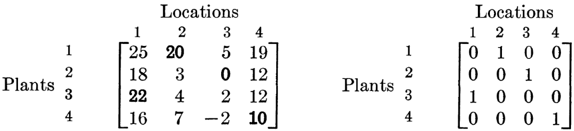

$$\text{Plants}\;\begin{array}{c|cccc} & \multicolumn{4}{c}{\text{Locations}} \\ & 1 & 2 & 3 & 4 \\ 1 & 25 & \mathbf{20} & 5 & 19 \\ 2 & 18 & 3 & \mathbf{0} & 12 \\ 3 & \mathbf{22} & 4 & 2 & 12 \\ 4 & 16 & \mathbf{7} & -2 & \mathbf{10} \end{array} \qquad \text{Plants}\;\begin{array}{c|cccc} & \multicolumn{4}{c}{\text{Locations}} \\ & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

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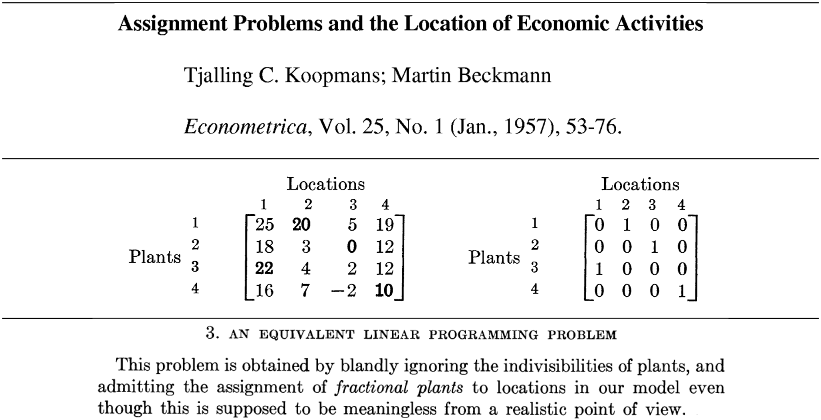

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3. AN EQUIVALENT LINEAR PROGRAMMING PROBLEM

This problem is obtained by blandly ignoring the indivisibilities of plants, and admitting the assignment of *fractional plants* to locations in our model even though this is supposed to be meaningless from a realistic point of view.

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1975
Leonid Vitaliyevich Kantorovich, Tjalling C. Koopmans

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Leonid Vitaliyevich Kantorovich
Prize share: 1/2

Tjalling C. Koopmans
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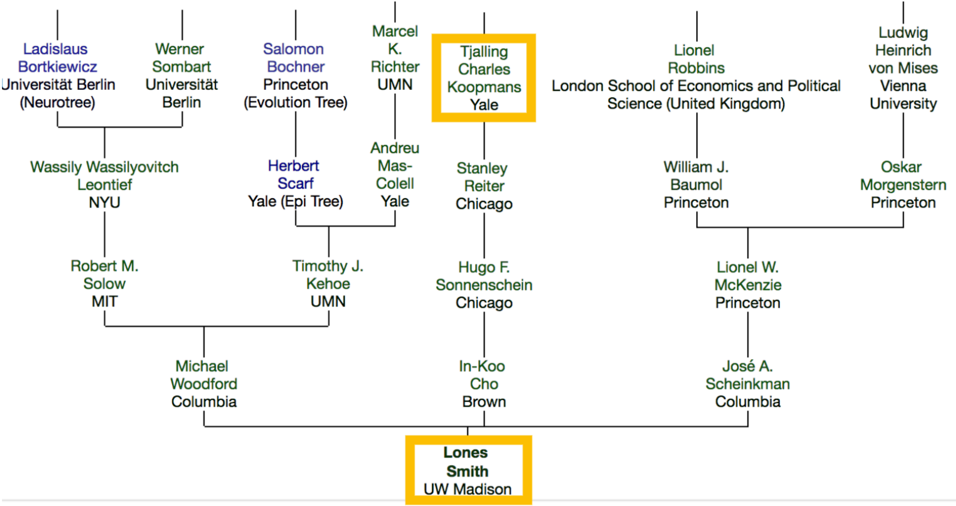

Ladislaus Bortkiewicz
Universität Berlin (Neurotree)

Werner Sombart
Universität Berlin

Salomon Bochner
Princeton (Evolution Tree)

Marcel K. Richter
UMN

Tjalling Charles Koopmans
Yale

Lionel Robbins
London School of Economics and Political Science (United Kingdom)

Ludwig Heinrich von Mises
Vienna University

Wassily Wassilyovitch Leontief
NYU

Herbert Scarf
Yale (Epi Tree)

Andreu Mas-Colell
Yale

Stanley Reiter
Chicago

William J. Baumol
Princeton

Oskar Morgenstern
Princeton

Robert M. Solow
MIT

Timothy J. Kehoe
UMN

Hugo F. Sonnenschein
Chicago

Lionel W. McKenzie
Princeton

Michael Woodford
Columbia

In-Koo Cho
Brown

José A. Scheinkman
Columbia

Lones Smith
UW Madison

Socially Efficient Matching

- *Finitely* many women x and men y (from XX and XY)
- $m(x, y) = 1$ if x is matched to man y, and $m(x, y) = 0$ if not.
- The set $\mathcal{M}$ of *feasible matchings* $[m(x, y)]$
  - symmetry: $m(x, y) = m(y, x)$ for all x, y
  - *no overmatching*: for every x, $m(x, y) = 1$ for at most one y.
- So a woman x remains single if $m(x, y) = 0$ for all $y \in Y$.
- Convexify the matching set:
  - A fraction $m(x, y) \geq 0$ of woman x to match with man y
  - The matching is at most a doubly stochastic matrix (nonnegative entries and unit row and columns sums)
- $h(x, y) =$ output of match of man x and woman y (or $h(x, y)$)
- An **efficient matching** $m \in \mathcal{M}$ maximizes the sum of all match outputs $\sum_x \sum_y m(x, y) h(x, y)$ over $\mathcal{M}$

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Proposition

An efficient matching $m \in \mathcal{M}$ exists.

- Proof: By Weierstrass Theorem, the maximum of a continuous function on a compact set exists

Competitive Equilibrium

- Payoffs: **wages** $v(x)$ and $w(y)$ to woman x and man y
- Price Competition Story. match makers compete to offer wages $v(x)$ and $w(y)$ to men and women, and earn *profits* $h(x, y) - v(x) - w(y)$ for any match they intermediate
- Taking actions as given is in the spirit of Nash equilibrium!
- Free entry of match makers $\Rightarrow$ profits ≤ 0 for all matches
- Free exit of match makers $\Rightarrow$ profits ≥ 0 for all actual matches
- A **competitive equilibrium** (m, w, v) satisfies feasibility and:
  - FREE ENTRY: $v(x) + w(y) \geq h(x, y)$ for any (x, y)
  - FREE EXIT: $v(x) + w(y) \leq h(x, y)$ if $m(x, y) > 0$

$$\Rightarrow \quad v(x) + w(y) \begin{cases} \geq h(x, y) & \text{for all women and men } x, y \\ = h(x, y) & \text{if } x, y \text{ are matched} \end{cases}$$

- By contrast, a **free market** allows market power, not insisting on free entry of market makers, and thus matches that obtain might embed rents. (See the lecture notes.)

Competitive Equilibrium is Efficient

Proposition (First Welfare Theorem of Matching)

A competitive equilibrium (m, v, w) yields an efficient matching m.

- Proof: If a competitive equilibrium (m, v, w) is not efficient

$\Rightarrow$ some feasible matching $\hat{m} \in \mathcal{M}$ has a strictly higher payoff:

$$\begin{aligned}
\sum_x v(x) + \sum_y w(y) &\geq \sum_y \sum_x h(x,y)\hat{m}(x,y) \\
&> \sum_y \sum_x h(x,y) m(x,y) \\
&= \sum_y \sum_x [v(x) + w(y)] m(x,y) \\
&= \sum_x v(x) + \sum_y w(y)
\end{aligned}$$

- The first inequality reflects
  - free entry: For $v(x) + w(y) \geq h(x,y)$ for all (x,y)
  - feasibility: $1 \geq \sum_x \hat{m}(x,y)$ $\forall y$ and $1 \geq \sum_y \hat{m}(x,y)$ $\forall x$
- The equality assumes everyone matches. What if not?

Contrast with Stable Matching

	Y_1	Y_2
X_1	2,0	**0,7**
X_2	**0,7**	2,0

	Y_1	Y_2
X_1	2	**7**
X_2	**7**	2

- At left, are the male and female optimal stable outcomes.
- The male optimal one yields higher total payoffs, but stability only reflects ordinal and not cardinal preferences.
- If outside options are zero, wages obey $v_1, v_2, w_1, w_2 \geq 0$ and:

$$\begin{array}{ll} v_1 + w_1 \geq 2 & v_1 + w_2 = 7 \\ v_2 + w_1 = 7 & v_2 + w_2 \geq 2 \end{array}$$

- Crucially, there are many competitive equilibrium wages
- One set of equilibrium wages is $v_1 = 5, v_2 = 0, w_1 = 7, w_2 = 2$
- Prove that any efficient matching is stable, if the wages are fixed as the match payoffs of the individuals.

Trading Houses (Shapley and Shubik, 1971)

- We now explore an equivalent model to the assignment model of Koopmans and Beckman
- $I \geq 1$ sellers (homeowners) and $J \geq 1$ prospective buyers.
- i-th seller values his house at (opportunity cost) $c_i > 0$ dollars
- j-th buyer values i's house at $\xi_{ij} > 0$ dollars.
- If $\xi_{ij} > c_i$, and seller i to sell his house to buyer j for some price p_i dollars, then i's payoff is exactly $p_i - c_i$ and j's payoff is exactly $\xi_{ij} - p_i$ (reflecting the quasilinear utility).
- Since seller i need not sell his house to buyer j, their match payoff is

$$h_{ij} = \max\{0, \xi_{ij} - c_i\}$$

Primal Problem: Maximizing Total Gains from Trade

- Let seller i sell fraction $m_{ij} \geq 0$ of house i to buyer j.
- Example: buying and selling "time shares" on condominiums.
- constraints on $m_{ij} \geq 0$: no house can be oversold, and no buyer can buy more than one house.

$$\max_{(m_{ij})} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} m_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} m_{ij} \leq 1 \ \forall i \in \{1, \dots, I\}$$

$$\text{and} \quad \sum_{i=1}^{I} m_{ij} \leq 1 \ \forall j \in \{1, \dots, J\}$$

Dual Problem

Lemma

The dual problem to the output maximization is the cost minimization:

$$\min_{v_i,w_j}\sum_{i=1}^{I} v_i+\sum_{j=1}^{J} w_j \quad s.t. \quad v_i+w_j \geq h_{ij}\ \forall i,j \quad and \quad v_i, w_j \geq 0\ \forall i,j$$

- So the cheapest way to afford all match output subject to entry and free exit constraints of a competitive equilibrium occurs at the efficient matching.
- two ways of measuring output — corresponding to gross national product and gross national income — coincide at the optimum.

Linear Programming Duality

- **Primal problem**:

$$\max\{pz|Az \leq q, z \geq 0\}$$

- **Dual problem:**

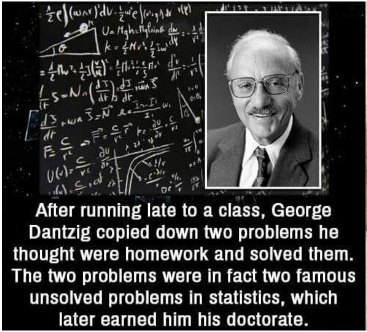
$$\min\{uq|uA \geq p, u \geq 0\}$$

- **Theorem:** These two problems have the same values.
- Primal feasibility $\Rightarrow$ $Az \leq q$ and dual feasibility $\Rightarrow$ $p \leq uA$.
- **weak duality**: $pz \leq uAz \leq uq$ for all $u, z \geq 0$
- So *the value of the primal is at most the value of the dual*.
- The reverse (strong) direction is harder to show.

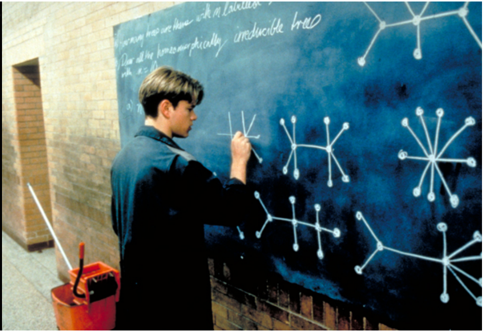
Linear Programming Duality as Deja Vu

- Flashback: von Neumann's Minimax Theorem (Saddle Point)
- George Dantzig, “A Theorem on Linear Inequalities,” 1948 — first formal proof of LP duality
- Air Force *Later Tucker asked me, ”Why didn't you publish it?” I replied, ”Because it was not my result; it was von Neumann's. All I did was to write up, for internal circulation, my own proof of what von Neumann had outlined to me.*
- von Neumann and Dantzig:

Ideal "PhD Conquer the World" Mindset



After running late to a class, George Dantzig copied down two problems he thought were homework and solved them. The two problems were in fact two famous unsolved problems in statistics, which later earned him his doctorate.



Proof of Dual Solution Lemma

- Example with $I = J = 2$ buyers and sellers,

$$q' = (1, 1, 1, 1)$$

$$h' = (h_{11}, h_{12}, h_{21}, h_{22})$$

$$m' = (m_{11}, m_{12}, m_{21}, m_{22})$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Primal: $\sum_i \sum_j h_{ij} m_{ij} = \max_{m \geq 0} \; h' m \quad \text{s.t.} \quad Am \leq q$
- Dual:

$$\min_{w,v \geq 0} \{v_1 + v_2 + w_1 + w_2\} = \min_{v,w \geq 0} (v, w) \cdot q \quad \text{s.t.} \; (v, w) \cdot A \geq h$$

Multipliers and Complementary Slackness Conditions

- *Primal*: $\max\{pz | Az \leq q, z \geq 0\}$
- *Dual:* $\min\{uq | uA \geq p, u \geq 0\}$
- Fictitious zero sum game with payoff $\mathcal{L}(z,u) = pz + uq - uAz$
- By the 1928 Minmax Theorem, this game has saddle point:

$$\max_{z\geq 0} \min_{u\geq 0}[pz + uq - uAz] = \min_{u\geq 0} \max_{z\geq 0}[pz + uq - uAz] \qquad (\bigstar)$$

- A finite saddle point requires $p - uA \leq 0 \leq q - Az$

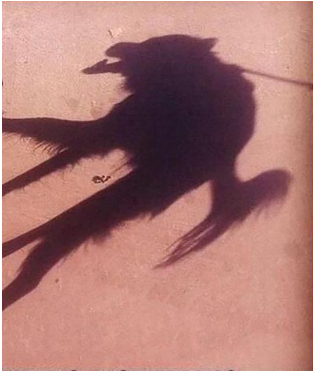
$\Rightarrow$ $z_\ell = 0$ when $p_\ell - (uA)_\ell < 0$, and $u_k = 0$ when $q_k - (Az)_k > 0$.

- Complementary slackness & $(\bigstar) \Rightarrow$ primal value = dual value
- Application of complementary slackness in Shapley-Shubik:

$$v_i + w_j \begin{cases} \geq h_{ij} & \text{for all } i, j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases}$$

Multipliers as Shadow Values

- *Primal*: $\max\{pz | Az \leq q, z \geq 0\}$
- Social planner's payoff function: $\mathcal{L}(z, u) = pz + u(q - Az)$
- Envelope Theorem $\Rightarrow \frac{\partial}{\partial q}\mathcal{L}(z, u) = u$
  $\Rightarrow$ dq extra constrained resource lifts planner's payoff by $u\ dq$.
- $u =$ shadow value of resource, as it indirectly shows true value



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Shadow Values in Shapley-Shubik Housing Model

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- buyer i and seller j trade $\Rightarrow$ gains from trade h_{ij}
- So ε more of *i and j* raises social payoff by εh_{ij}

$\Rightarrow$ All we can say is $v_i + w_j = h_{ij}$

- "It takes two to tango...but who matters more?"

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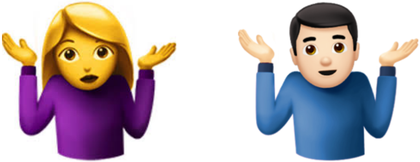

- We cannot separately identify buyers' & sellers' shadow values

1971 — Buyer-Seller Trade: Shapley and Shubik

- Assume three potential home buyers and three sellers

♂\♀	Seller Costs	Buyer Valuations		
		Buyer 1	Buyer 2	Buyer 3
House 1	18	23	26	20
House 2	15	22	24	21
House 3	19	21	22	17

- Match payoffs now are gains from trade, or zero, if negative:

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	$23 - 18 = 5$	$26 - 18 = 8$	$20 - 18 = 2$
Seller 2	$22 - 15 = 7$	$24 - 15 = 9$	$21 - 15 = 6$
Seller 3	$21 - 19 = 2$	$22 - 19 = 3$	$\max(17 - 19, 0) = 0$

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- Match payoffs now are gains from trade, or zero, if negative:

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	5	8	2
Seller 2	7	9	6
Seller 3	2	3	0

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- Match payoffs now are gains from trade, or zero, if negative:

♂\♀	Buyer 1	Buyer 2	Buyer 3
Seller 1	5	**8**	2
Seller 2	7	9	**6**
Seller 3	**2**	3	0

- gains from trade are the match payoffs

Solving the Housing Example

- Minimize the sum of shadow values $\sum_i v_i + \sum_j w_j$ subject to $v_i \geq 0$ and $w_j \geq 0$ as well as

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 \geq 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 \geq 6 \\ v_3 + w_1 \geq 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- Since the optimum occurs at the red matching, we just solve

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 = 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 = 6 \\ v_3 + w_1 = 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

- a solution: $(v_1, v_2, v_3) = (4, 5.5, 0)$ & $(w_1, w_2, w_3) = (2, 4, 0.5)$

$\Rightarrow$ home prices are $p_i = c_i + v_i$, or $p_1 = 22$, $p_2 = 20.5$, $p_3 = 19$

- Example: seller 1 sells his home (cost 18) to buyer 2 (who values it at 26) for a seller surplus $v_1 = 4$ and a buyer surplus $w_2 = 4$: from this, we deduce the price $p_1 = 22$

An Integer Price Solution of the Housing Example

♂\♀	y_1	y_2	y_3	Seller “wage” v_i
Seller 1	5	**8**	2	$v_1 = 4$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 0$
Buyer “wage”	$w_1 = 2$	$w_2 = 4$	$w_3 = 0$	

- We increase the price of home 2 to $p_2 = 21$, increasing the surplus of seller 2 to $v_2 = 6$ and reducing the surplus of buyer 3 to $w_3 = 0$.
- So house prices are now $p_1 = 22, p_2 = 21, p_3 = 19$

Worst Payoffs (“Wages”) for Sellers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 3$
Seller 2	7	9	**6**	$v_2 = 5$
Seller 3	**2**	3	0	$v_3 = 0$
Buyers	$w_1 = 2$	$w_2 = 5$	$w_3 = 1$	

- Buyer 1 does not buy house 1 $\Rightarrow v_1 \geq v_3 + 3$
  - Proof: $w_1 + v_1 \geq 5 = 3 + 2 = 3 + w_1 + v_3$ (Buyer 1 buys house 3)
- Buyer 1 does not buy house 2 $\Rightarrow v_2 \geq v_3 + 5$
  - Proof: $w_1 + v_2 \geq 7 = 5 + 2 = 5 + w_1 + v_3$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least seller payoffs $(\underline{v}_1, \underline{v}_2, \underline{v}_3) = (3, 5, 0)$
- Associated maximum buyer payoffs $(\bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 5, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\underline{v}_1 + \bar{w}_2 = 8$, $\underline{v}_2 + \bar{w}_3 = 6$, $\underline{v}_3 + \bar{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

Worst Payoffs ("Wages") for Buyers

♂\♀	y_1	y_2	y_3	Sellers v_i
Seller 1	5	**8**	2	$v_1 = 5$
Seller 2	7	9	**6**	$v_2 = 6$
Seller 3	**2**	3	0	$v_3 = 1$
Buyers	$w_1 = 1$	$w_2 = 3$	$w_3 = 0$	

- Buyer 1 does not buy house 2 $\Rightarrow w_1 \geq w_3 + 3$
  - Proof: $w_1 + v_2 \geq 7 = 1 + 6 = 1 + w_3 + v_2$ (Buyer 3 buys house 2)
- Buyer 2 does not buy house 2 $\Rightarrow w_2 \geq w_3 + 3$
  - Proof: $w_2 + v_2 \geq 9 = 3 + 6 = 3 + w_3 + v_2$
- *All other buying incentive constraints do not bind as tightly*
- Solution: Least buyer payoffs $(\underline{w}_1, \underline{w}_2, \underline{w}_3) = (1, 3, 0)$
- Associated maximum seller payoffs $(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (5, 6, 1)$
  - Proof: Equality constraints from matches that do occur imply: $\bar{v}_1 + \underline{w}_2 = 8$, $\bar{v}_2 + \underline{w}_3 = 6$, $\bar{v}_3 + \underline{w}_1 = 2$
- Then verify that payoffs $(\underline{v}, \bar{w})$ obey all incentive constraints!

The Welfare Theorems and Stigler's Proviso

Welfare Theorems *A competitive equilibrium yields an efficient matching. Conversely, an efficient matching is a competitive equilibrium, for a suitable set of wages.*

- Proof: We use linear programming duality.
- Consider the optimization of output, subject to the linear constraints of not overmatching any man or woman.
- The multipliers for these constraints are the wages.
- Duality: the maximum total output equals the minimum total wages, subject to all the competitive incentive constraints.
- This resolves the horrific complexity issue — we need only find n wages for men and n for women!

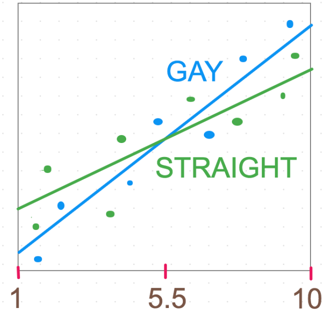
Who Matches with Whom: Becker's Marriage Model

- This paper argued to a large audience that matching was economically important.
- The paper then (re-) derived the welfare theorems for matching with transfers (unaware of Shapley and Shubik)
- The paper's highlight was a simple description of the efficient allocation of matching with transfers.

Positive Sorting is an Empirical Fact

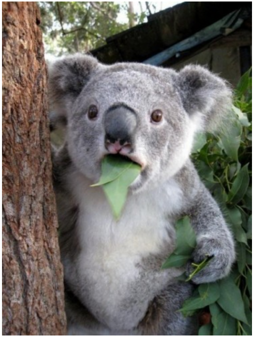
Fun Application (Yale 2006): The Dating Market

- Data Source 1: Facebook
- Data Source 2: Online beauty contest, such as www.rankmyphotos.com



General Type Distributions on Men and Women

- Allow now a continuum of men and women.



- Assume cdfs M for women and N for men
  - $M(x)$ gives the mass of women of type $x' \leq x$
  - $N(y)$ gives the mass of men of type $y' \leq y$
- let $\bar{M}, \bar{N}$ (respectively) be the total mass of men, women
- Let man $y(x)$ be the partner of woman x, if she is matched.

Assortative Matching: Basic Definitions

- Example: Easy finite terms, say with 50 men and 100 women.

$\Rightarrow$ men are on the **long side of the market** and women on the **short side of the market**
  - **positive assortative matching (PAM)**: woman k with man k, for $k = 1, 2, \ldots, 50$, and men $51, \ldots, 100$ unmatched
  - **negative assortative matching (PAM)**: woman k with man $51 - k$, for $k = 1, \ldots, 50$, & men $51, \ldots, 100$ unmatched
- Now consider the continuum analogues:
- PAM if $\bar{M} - M(x) = \bar{N} - N(y(x))$ for all matched women x.
- NAM if $\bar{M} - M(x) = N(y(x))$ for all matched women x.
- This definition allows for either women or men to be unmatched, since the mass of men and women might differ
- If $\bar{M} = \bar{N}$, normalize $\bar{M} = \bar{N} = 1$, and think of quantiles:
  - The q-th highest quantile man matches with the q-th highest quantile woman if there is PAM
  - The q-th highest quantile man is matched with the q-th lowest quantile woman if there is NAM

NTU Payoff Conditions for Assortative Matching

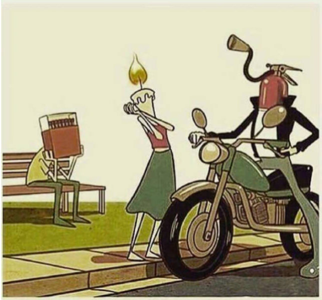
- Without transfers (NTU):
  - $f(y|x)$ = payoff of woman x matched with man y,
  - $g(x|y)$ = payoff of man y matched with woman x
- f, g are *comonotone* if $\forall y_2 > y_1$ and $x_2 > x_1$,

$$[f(y_2|x) - f(y_1|x)] \cdot [g(x_2|y) - g(x_1|y)] > 0 \;\; \forall x, y$$

- The opposite inequality is *reverse comonotone*
- If f and g are differentiable, then both partial derivatives (in first arguments) have the same sign if comonotone
- **Theorem:** The unique stable matching with NTU is PAM if f and g are comonotone, and NAM if reverse comonotone.

Proof of NTU Sorting Proposition

- Assume comonotonicity and but a stable match is not PAM
- Then $\exists x' > x$ and $y' > y$ with matches (x, y') and (x', y)
- Claim: either (x', y') or (x, y) is a blocking pair
- First case: $f(y'|x') > f(y|x') \Rightarrow g(x'|y') > g(x|y')$
- Second case: $f(y'|x) < f(y|x) \Rightarrow g(x'|y) < g(x|y)$



1973 — Becker's Marriage Model

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At left is **positive assortative matching (PAM)**
- Since men prefer higher women x and women prefer higher men y, *the stable matching without transfers is PAM*.

1973 — Becker's Marriage Model

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- At right is **negative assortative matching (NAM)**
- Why? Matches all profit from higher men, but the matches that profit most from higher men are those with lower women.
- This forces downward sorting.
- For instance, rematching the two sorted pairs $(1,1)$ and $(2,2)$ as $(1,2)$ and $(2,1)$ changes output by $(18+8)-(16+9) = 26-25 = 1$

Pairwise Efficiency and Efficiency

- A matching m is **pairwise efficient with TU** if for all matched pairs (x_1, y_1) and (x_2, y_2):

$$h(x_1, y_1) + h(x_2, y_2) - h(x_1, y_2) - h(x_2, y_1) \geq 0$$

- This is the analogue of the stability criterion with NTU, but it also measures the strength of the preferences
- If this fails, then rematching to (x_1, y_2) and (x_2, y_1) undoes the original matching with side payments.
- With NTU, losses of dumped partners do not matter
- An **efficient** matching maximizes the sum of all match outputs, and so rematching any set of couples cannot help.

Lemma

Any efficient matching $m \in \mathcal{M}$ is pairwise efficient.

- The converse of this lemma is false

Pairwise Efficiency $\not\Rightarrow$ Efficiency

	y_1	y_2	y_3
x_1	**3**	**3**	0
x_2	0	**3**	**3**
x_3	**2**	0	**3**

- The pairwise efficient green matching has a lower total payoff than the pairwise efficient cyan matching, and is inefficient.
- Q: What bribery scheme would unravel the green matching?

1973 — Strategic Substitutes Drives Negative Sorting

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

Cross Partial Payoff Differences (Synergies)

	12	23
23	$18 + 24 - 27 - 16 = -1$	$16 + 21 - 14 - 24 = -1$
12	$9 + 16 - 18 - 8 = -1$	$8 + 14 - 16 - 7 = -1$

- **Strategic substitutes**:
  - all cross partial differences of match payoffs are negative
  - pairwise efficiency ⇒ positive sorting is not locally efficiency
- **Strategic complements**:
  - all cross partial differences of match payoffs are positive
  - pairwise efficiency ⇒ negative sorting is not locally efficiency

1973 — Strategic Substitutes Drives Negative Sorting

NTU Matching

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$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

TU Matching

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- **Left**: payoffs are men get $2xy$ and women get $y(10 - 3x)$.
  - Men's payoffs $2xy$ increases in women's type x
  - Women's payoffs $y(10 - 3x)$ increases in men's type y
  - $\Rightarrow$ PAM is the stable allocation without transfers
- **Right**: match payoffs are $2xy + y(10 - 3x) = 10y - xy$.
  - Cross partial derivative is -1
  - $\Rightarrow$ strategic substitutes
  - $\Rightarrow$ NAM

Becker (1973): Assortative Matching with Transfers

- The match function $h(x, y)$ is *(strictly) supermodular* if

$$h(x', y') + h(x, y) \geq (>)\ h(x', y) + h(x, y') \qquad (1)$$

  for any pair of women $x' \geq x$ and men $y' \geq y$.
- $h(x, y)$ is *(strictly) submodular* if the reverse inequality holds
- For twice differentiable match payoffs, this says $h_{12}(x, y) \geq 0$

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Proposition (Becker's Marriage Model)

(a) If production is supermodular, then PAM is efficient.
If it is strictly supermodular, then PAM is uniquely efficient.
(b) If production is submodular, then NAM is efficient.
If it is strictly submodular, then NAM is uniquely efficient.
(c) If production is modular, then any matching is efficient.

Becker (1973): Assortative Matching with Transfers

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$$h(x', y') + h(x, y) \geq (>)\ h(x', y) + h(x, y') \qquad (1)$$

  for any pair of women $x' \geq x$ and men $y' \geq y$.
- $h(x, y)$ is *(strictly) submodular* if the reverse inequality holds
- For twice differentiable match payoffs, this says $h_{12}(x, y) \geq 0$

Proposition (Becker's Marriage Model)

(a) If production is supermodular, then PAM is efficient.
If it is strictly supermodular, then PAM is uniquely efficient.
(b) If production is submodular, then NAM is efficient.
If it is strictly submodular, then NAM is uniquely efficient.
(c) If production is modular, then any matching is efficient.

- Proof (by Buz Brock): Assume strictly supermodular
- Then not PAM is not pairwise efficient, and so not efficient.
- Corollary: If production is modular for a set of agents that match, then any re-matching among them is also efficient.

Example: Matching with and without Transfers

PAM

♂\♀	$x = 1$	$x = 2$	$x = 3$
$y = 3$	6,21	12,12	**18,3**
$y = 2$	4,14	**8,8**	12,2
$y = 1$	**2,7**	4,4	6,1

NAM

♂\♀	1	2	3
3	**27**	24	21
2	18	**16**	14
1	9	8	**7**

- Men earn $f(x|y) = 2xy$ and women earn $g(y|x) = y(10 - 3x)$

$\Rightarrow$ $\frac{\partial f(y|x)}{\partial x} = 2x > 0$ (men prefer higher women)
$\frac{\partial g(x|y)}{\partial y} = 10 - 3x > 0$ (women prefer higher men)

$\Rightarrow$ unique stable matching is PAM

$\Rightarrow$ Hence, the DAA delivered PAM

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$\Rightarrow$ unique stable matching is PAM

$\Rightarrow$ Hence, the DAA delivered PAM

- With transfers, match payoffs are strictly submodular:
  $h(x, y) = f(x|y) + g(y|x) = 2xy + y(10 - 3x) = 10y - xy$

$\Rightarrow$ unique efficient matching is NAM

How to Find Competitive Wages without Duality

- An Illustrative Example:
  - Match payoffs: $h(x, y) = x^2 y$
  - Types: women x and men y uniformly distributed on $[0, 1]$
- Since $h_{12} = 2x > 0$, PAM is the efficient outcome
  - Notice the cheat here: we are using Becker's PAM solution of the primal problem in the special case with PAM payoffs
- Let $w(x)$ and $v(y)$ be the competitive wage functions
- If the matchmaker pairs up x and y (paying them their wages but keeping the surplus), his profits are:

$$\pi(x, y) = x^2 y - w(x) - v(y)$$

- Competition among match makers forces a zero profit maximum at $y = x$ (by PAM):

$$\begin{aligned} \frac{\partial \pi}{\partial x} &= 0 \Rightarrow \left[2xy = w'(x)\right]_{y=x} \Rightarrow w'(x) = 2x^2 \\ \frac{\partial \pi}{\partial y} &= 0 \Rightarrow \left[x^2 = v'(y)\right]_{x=y} \Rightarrow v'(y) = y^2 \end{aligned}$$

Finding Competitive Wages, Continued

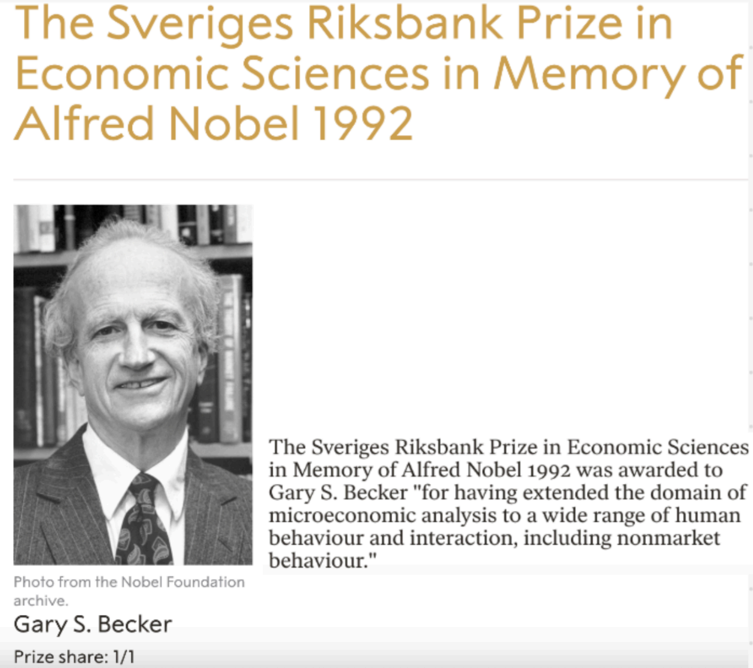
- Evaluating these at the efficient matches, (x, x) and (y, y),

$$\begin{aligned} w(x) &= \frac{2}{3}x^3 + \beta \\ v(y) &= \frac{1}{3}y^3 + \delta \end{aligned}$$

- By zero profits, $\pi(x, x) = 0 \ \forall x$, and so $\beta + \delta = 0$ because

$$0 = x^2 \cdot x - w(x) - v(x) = x^3 - \frac{2}{3}x^3 - \frac{1}{3}x^3 - (\beta + \delta)$$

- If unmatched pays everyone zero, then all wages must be nonnegative, and so $\beta = \delta = 0$
- A *dowry* $\delta > 0$ — a fixed transfer that women pay men — only arises if unmatched women earn a payoff at most $-\delta < 0$
- A *bride price* $\beta > 0$ — a fixed transfer that men pay women — only arises if unmatched men earn a payoff $\leq -\beta < 0$
- If both unmatched men and women earn negative payoffs, then a dowry or bride price will simply reflect a social norm (i.e. a Nash equilibrium)



The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992

Photo from the Nobel Foundation archive.

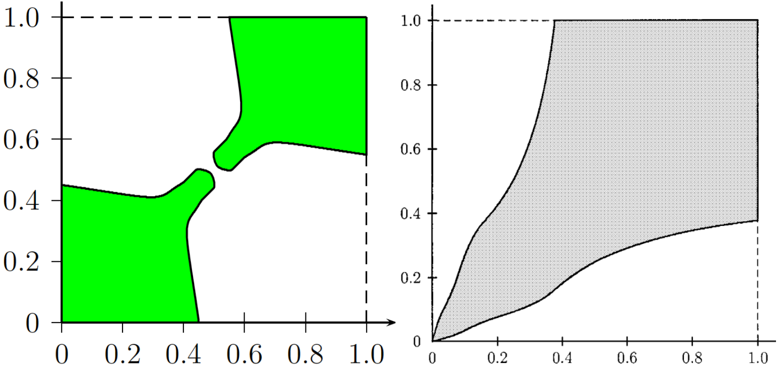
Gary S. Becker

Prize share: 1/1

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1992 was awarded to Gary S. Becker "for having extended the domain of microeconomic analysis to a wide range of human behaviour and interaction, including nonmarket behaviour."

Assortative Matching and Search (Shimer-Smith, 2000)

- There is no stock exchange for marriage partners, firm-worker pairs, etc. Search frictions matter $\Rightarrow$ Higher types might settle for lower parters because the cost of search is higher.
- With search frictions, PAM requires that $\log h_x(x, y)$ is SPM
- Eg: Matching with $h(x, y) = (x + y - 1)^2$ and $h(x, y) = e^{xy}$:



The Comparative Statics of Sorting (current)

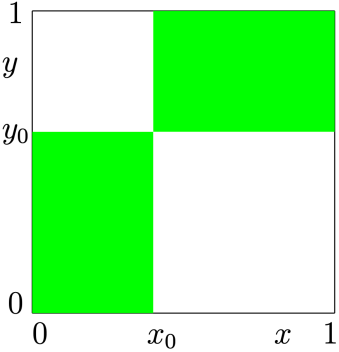
- We only fully understand two extreme cases: PAM or NAM.
- Since the transportation problem is not solved, we cannot characterize who matches with whom in any other case
- With my former PhD advisee Axel Anderson, we relate sorting to **synergies**: the cross partial differences of match output

  $$h(x_2, y_2) - h(x_2, y_1) + h(x_1, y_2) - h(x_1, y_2)$$

  when $x_2 \geq x_1, y_2 \geq y_1$
- If synergy is not always positive or always negative, then Becker's Theorem is silent on who matches with whom.

How to Mathematically Formaluate that Sorting Increases

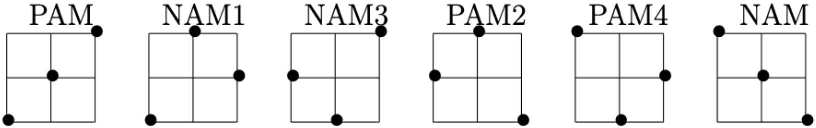
- Measure how sorting increases from NAM to PAM with the **positive quadrant dependence (PQD) order** on matches — namely, the mass of matches $(x, y) \geq (x_0, y_0)$ and $(x, y) \leq (x_0, y_0)$ increases, $\forall (x_0, y_0)$



All Pure Matchings with Three Types

- PQD is a partial order, $\succ_{PQD}$. For example,

$$\text{PQD} \succ_{PQD} \{\text{NAM1, NAM3}\} \succ_{PQD} \{\text{PAM2, PAM4}\} \succ_{PQD} \text{NAM}$$



Sorting Does not Rise in Cross Differences

- **Increasing Sorting Theorem**

  Sorting is higher with production function h^B than h^A if
  - synergy is higher with h^B than h^A, for every $x_2 \geq x_1, y_2 \geq y_1$
  - For every $x_2 \geq x_1, y_2 \geq y_1$, the synergy for each h^i obeys:

$$h^i(x_2, y_2) - h^i(x_2, y_1) + h^i(x_1, y_2) - h^i(x_1, y_2)$$

shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**14**	18
$y = 2$	5	2	**14**
$y = 1$	**1**	5	9

Matrix of Cross Differences

8	-8
-7	8

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  shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	16	**24**
$y = 2$	**5**	3	16
$y = 1$	1	**5**	9

Matrix of Cross Differences

9	-5
-6	9

Sorting Does not Rise in Cross Differences

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shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

NAM1 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	**20**	30
$y = 2$	5	6	**20**
$y = 1$	**1**	5	9

Matrix of Cross Differences

10	-4
-3	10

Sorting Does not Rise in Cross Differences

- **Increasing Sorting Theorem**

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shifts from negative to positive as x_1 or x_2 or y_1 or y_2 increases.

NAM3 is efficient

	$x = 1$	$x = 2$	$x = 3$
$y = 3$	9	22	**36**
$y = 2$	**5**	7	22
$y = 1$	1	**5**	9

Matrix of Cross Differences

11	−1
−2	11

Sorting in the Trade Paradigm

- Becker Marriage model applies to the Shapley-Shubik model
- The match payoff is $h_{ij} = \max\{0, \xi_{ij} - c_i\}$ when buyer j values seller i's house at ξ_{ij}
- If h_{ij} is supermodular, then there is sorting among traders.
- We now shift to a world with homogenous houses

Double Auctions

- We now relax Shapley-Shubik's double coincidence of wants
- The housing assignment model with homogeneous houses $\rightsquigarrow$ double auction model
- Buyer j's values all goods at ξ_j, so that $\xi_{ij} = \xi_j$ for all i
- $h(\xi, c) \equiv \max\{0, \xi - c\}$ are the gains from trade for a buyer with value ξ and a seller with cost c.
- Efficiency: maximize total trade surplus $\sum_i \sum_j m_{ij} h(\xi_j, c_i)$, where $m_{ij} = 1$ if seller i sells to buyer j, and $m_{ij} = 0$ otherwise.
- Shapley-Shubik: the sum of the shadow values of seller and buyer trading is the match output, $v_i + w_j = h_{ij}$ if $m_{ij} > 0$.
- The price p_i divides this surplus between matched traders
  - producer surplus: $v_i = p_i - c_i$
  - consumer surplus: $w_j = \xi_{ij} - p_i = \xi_j - p_i$

The Trade Surplus Function is Submodular

Lemma (Trade Surplus Function)

The trade surplus function h is submodular: If $\xi' \leq \xi''$ *and* $c' \leq c''$, *then* $h(\xi'', c'') + h(\xi', c') \leq h(\xi'', c') + h(\xi', c'')$, *with strict inequality iff* $\xi' < c' < c'' < \xi''$ *or* $c' < \xi' < \xi'' < c''$.

The Trade Surplus Function is Submodular

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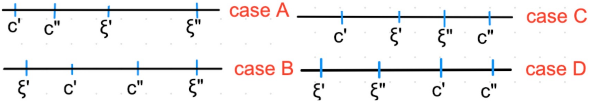
The trade surplus function h is submodular: If $\xi' \le \xi''$ and $c' \le c''$, then $h(\xi'', c'') + h(\xi', c') \le h(\xi'', c') + h(\xi', c'')$, with strict inequality iff $\xi' < c' < c'' < \xi''$ or $c' < \xi' < \xi'' < c''$.

Proof:



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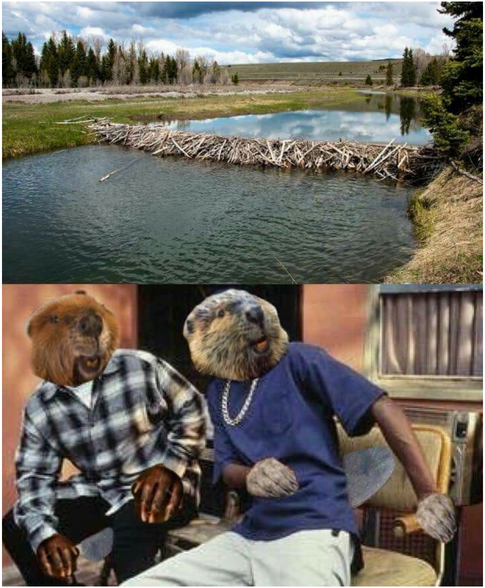
- Case A: Here, two trades should occur, and $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = \xi'' + \xi' - c' - c''$.
- Case B: Here, one trade should happen, and $h(\xi'', c'') + h(\xi', c') = \xi'' - c'' \le \xi'' - c' = h(\xi', c'') + h(\xi'', c')$. Case C: Here, one trade should happen, and $h(\xi'', c'') + h(\xi', c') = \xi' - c' \le \xi'' - c' = h(\xi', c'') + h(\xi'', c')$.
- Case D: Here, no trade should happen, and $h(\xi'', c'') + h(\xi', c') = h(\xi', c'') + h(\xi'', c') = 0$.
- Inequalities are strict if $c' < c''$ and $\xi' < \xi''$, since trade surplus falls when the wrong good is traded.

The Supply and Demand Paradigm

- The highest value buyers trade with the lowest cost sellers.
- Rank order buyers: $\xi_1 < \cdots < \xi_k < \xi_{k+1} < \cdots < \xi_N$
- Rank order sellers: $c_1 < \cdots < c_k < c_{k+1} < \cdots < c_N$
- Intuitively, the stronger/higher buyers have high values, *but the stronger/higher sellers have low costs*
- Since $h(\xi, c)$ is submodular, by Becker's Marriage Theorem, NAM arises: high value buyers trade with low cost sellers.
- Also, since $h(\xi, c)$ is modular matching among agents trading, and among those not trading:
  - *Matching among those trading sellers and buyers is irrelevant.*
  - *Matching among sellers and buyers not trading is irrelevant.*

Matching Model is a Foundation for Supply and Demand



Competitive Equilibrium in a Double Auction

Proposition (Double Auctions)

(*a*) *If* $\xi_N < c_1$*, there is no trade. Assume* $c_1 \leq \xi_N$ *henceforth.*
(*b*) *The* k^* *highest value buyers purchase from the* k^* *lowest cost sellers, where* k^* *is the largest* k *with* $c_k \leq \xi_{N+1-k}$*.*
(*c*) *The* ***law of one price*** *holds, with a common price*

$$p^* \in [\max(c_{k^*}, \xi_{N-k^*}), \min(c_{k^*+1}, \xi_{N+1-k^*})]$$

(*d*) *Any competitive equilibrium is efficient, and therefore maximizes the sum of gains from trade.*
(*e*) *The final allocation is immune to side bribes.*

- When supply balances demand, we say markets clear
- To understand deviations from the law of one price, which we see everywhere, one really needs to add search frictions to the model (as I teach in advanced theory).

Is There One Price? What is it?

- Proof of (*c*): The social planner equally values buyer *j*'s shadow value $w_j = \xi_j - p_i > 0$ in any optimal trade, namely from low cost sellers *i*, by the Becker Marriage Theorem

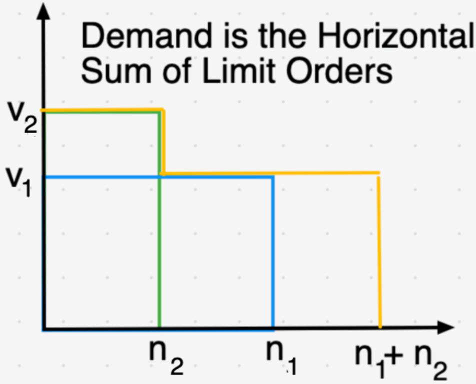
⇒ Seller prices p_i cannot vary with *i*, assuming they trade

- The price p^* encourages last transaction: $c_{k^*} \leq p^* \leq \xi_{N+1-k^*}$
- The price p^* deters another transaction: $\xi_{N-k^*} \leq p^* \leq c_{k^*+1}$
- Hence, crossing of supply and demand determines quantity:

$$\max(c_{k^*}, \xi_{N-k^*}) \leq p \leq \min(c_{k^*+1}, \xi_{N+1-k^*})$$

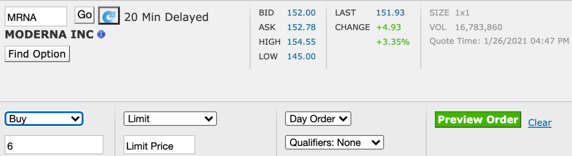
- The competitive price is not pinned down unless the last trade yields no surplus, whereupon the last unit needn't be traded
- "*Walrasian auctioneer*" secures a competitive equilibrium by raising the price with excess demand and reducing the price with excess supply
- Opening stock market prices are set to clear the market

Beyond Unit Supply and Demand: Limit Orders



- The same can be done to construct the supply curve.

Overnight Market in Stock Exchanges



MRNA Go 20 Min Delayed

MODERNA INC

Find Option

BID 152.00
ASK 152.78
HIGH 154.55
LOW 145.00

LAST 151.93
CHANGE +4.93 +3.35%

SIZE 1x1
VOL 16,783,860
Quote Time: 1/26/2021 04:47 PM

Buy
6
Limit
Limit Price
Day Order
Qualifiers: None

- To open/close, many stock exchanges use single price double auction
- The buyer must ask for a limit order (my choice) or a market order (limit order with unspecified price)

Double Auction Example

- Consider 20 agents, numbered from 1 to 20
- Even agents are buyers, and odd agents are sellers
- Buyer valuations are $\xi_i = 2i$ and sellers costs are $c_j = 3j$.
- Ordering the valuations from high to low:

$$40, 36, 32, 28, 24, 20, 16, 12, 8, 4$$

- Ordering costs from low to high:

$$3, 9, 15, 21, 27, 33, 39, 45, 51, 57$$

- An efficient matching clears the market: the high value buyers and low cost sellers $\Rightarrow k^* = 4$ (but actual pairing irrelevant)
- The price p^* encourages the value 28 buyer and cost 21 seller to trade:

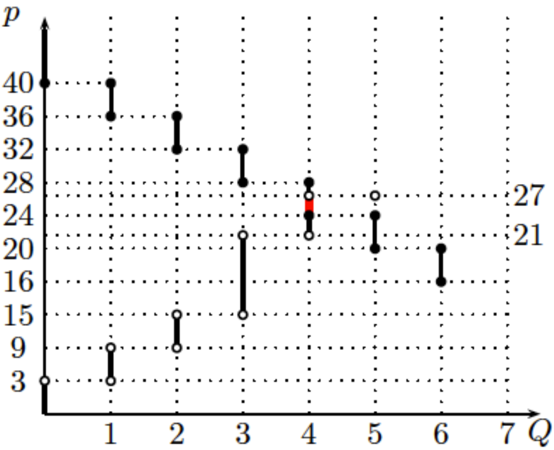
$$21 \leq p^* \leq 28$$

- The price p^* deters the value 24 buyer and cost 27 seller from trading:

$$24 \leq p^* \leq 27$$

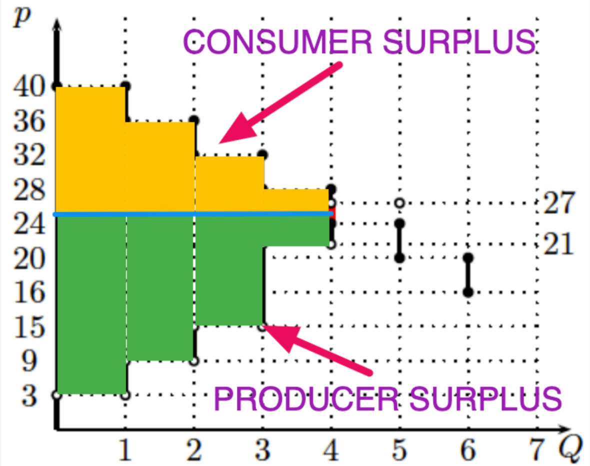
- any price in the interval $[24, 27]$ clears the market

Gains from Trade



- All traders earn positive surplus: e.g. at $p^* = 25$, the marginal buyer earns $28 - 25 = 3$ and the marginal seller $25 - 21 = 4$

Gains from Trade



- Heterogeneity is good and the source of all gains from trade.
    - If everyone had identical valuations, then no consumer secures consumer surplus at the market clearing price
    - the more heterogeneous are consumers or producers, the larger the total gains from trade.

How Paternalism Reduces Gains from Trade

- Example: Volunteer vs. Draft Army
    - A volunteer army maximizes gains from trade: it sets a wage so that the people who most want to serve willingly do so.
    - Milton Friedman's opposition the Draft helped end it in 1973.

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- Example: Regifting and scalping
    - U.S. Ticket Resale Laws vary hugely (my advisee Axel Anderson and football tix)
    - Jay Leno's freely gave away Tonight Show tickets to unemployed in Detroit in 2009. People tried resold tickets on eBay and Leno objected.

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- Example: Gift giving often means value $<$ cost
  - Waldfogel (1993), "The Deadweight Loss of Christmas'
  - Lost surplus was about ten billion dollars per holiday season!

Application to Market Power: Seller's or Buyer's Cartel

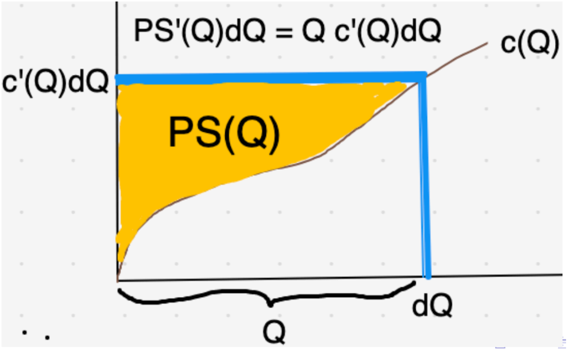
- Question: What if all sellers form a cartel?
- Buyer valuations are $v_i = 2i$ and sellers costs are $c_j = 3j$.
- Buyer valuations: $40, 36, 32, 28, 24, 20, 16, 12, 8, 4$
- Seller costs: $3, 9, 15, 21, 27, 33, 39, 45, 51, 57$
- Now: Cartel — not impartial auctioneer — sets the price p
- Sellers choose quantity n to maximize profits

$$\max_n \{ \sum_j^n (p - c_j) | v_{20-n} \geq p \}$$

- Example: Workers sell their unit supply of labor.
- If buyers act competitively, then intuitively sellers no longer take the price as given, but set the wage
- Market power ⇔ one side of the market influences the price

Market Power: A Union of Sellers

- Solution: use calculus, by assuming a continuous quantity
- The marginal buyer's value, or demand curve, is $v(Q)$
  - Demand curve falls if low Q buyers have high values
- The marginal seller's price, or the supply curve, is $c(Q)$
- The seller cartel's producer surplus $PS(Q)$ is the area over supply curve on $[0, Q]$ under $c(Q)$
  - Producer surplus slope $PS'(Q) = Qc'(Q)$

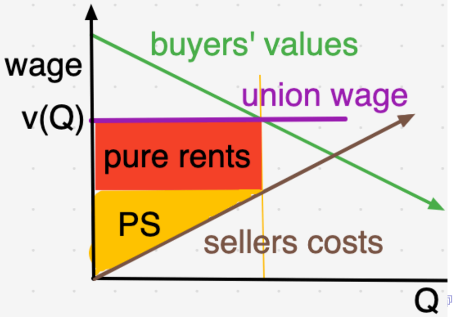


Market Power: Seller's Cartel (Union of Workers)

- The union's profits are $\Pi(Q) = PS(Q) + Q[v(Q) - c(Q)]$
- Second term is a pure rent, namely, a payment over and above that needed to keep all sellers trading
- Cartel demands a wage $v(Q)$ from the buyers for quantity Q
- Maximum union profits imply the FOC:

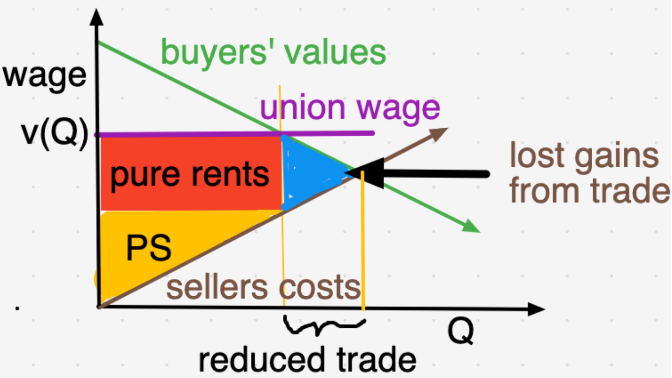
$$0 = \Pi'(Q) = PS'(Q) + [v(Q) - c(Q] + Q[v'(Q) - c'(Q)] = v(Q) - c(Q) + Qv'(Q)$$

- SOC holds if $c'(Q) > 0$ and $[Qv(Q)]'' = [v(Q) + Qv'(Q)]' \leq 0$
- E.g. $[Qv(Q)]'' = -2B < 0$ for linear functions $v(Q) = A - BQ$.



Market Power is Inefficient

- Market power prevents some positive surplus trades — albeit the lowest surplus ones — and thus reduces the gains from trade below competitive levels



Market Power: Seller's Cartel with an Integer # of Sellers

- Most profitable quantity for union: $0 = v(Q) - c(Q) + Qv'(Q)$
- Approximate integer demands by $v(Q) = 40 - 4Q$ and $c(Q) = 6Q + 3$
- Example: buyer $Q = 0$ has value 40 & seller $Q = 0$ has cost 3

$\Rightarrow$ $0 = 40 - 4Q - (6Q + 3) + Q(-4) \Rightarrow Q = 37/14 = 2.642$

- With integers, check that $k = 3$ is best for the cartel
    - $k = 1$: Price is 40, and maximum cartel profits are $40 - 3 = 37$
    - $k = 2$: Price is 36, and maximum cartel profits are $2 \times 36 - 3 - 9 = 60$
    - $k = 3$: Price is 32, and maximum cartel profits are $3 \times 32 - 3 - 9 - 15 = 69$
    - $k = 4$: Price is 28, and maximum cartel profits are $4 \times 28 - 3 - 9 - 15 - 21 = 64$
- Total gains from trade fall by the value of the lost fourth trade: $28 - 21 = 7$

Market Power: Buyer's Cartel (Union of Buyers)

(food for thought)

- Henceforth, we shift to a continuous quantity world