An Economic Theory Masterclass

Part VIII: General Equilibrium with Uncertainty

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How Markets Enable Risk Sharing

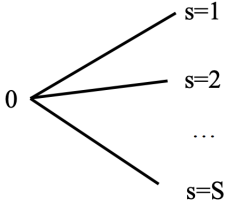
- Robinson Crusoe: shared ownership of firm exists to finance large firms that no one individual could own
- But shared ownership plays another key role: risk-sharing
- 1602, the Dutch East India Company officially was the world's first publicly traded company
  - issued shares of the company on Amsterdam Stock Exchange
  - Ships returning from the East Indies had a high chance of loss due to weather, war, or pirates.

  - Instead of investing in one voyage, investors could now purchase shares in multiple companies.

  - The company eventually went bankrupt in 1799.

Arrow-Debreu Securities and Risk Sharing

- Exchange economy with n traders and L goods
- Time-1: A state of the world $s \in S = \{1, \ldots, S\}$ is realized.
  - For simplicity, assume the state s is publicly known.
- Time-0: Only the probability π_s of each state s is known.
- Label the goods in the Arrow-Debreu model by the state.
- A state-contingent claim $x_{\ell s} \in \mathbb{R}^{LS}$ is a title to a unit of consumption of good ℓ in state s.
- p_s = price of the state s contingent claim, or *Arrow security*.
- So far, trade was contractually implemented, not using money. These are now LS forward contracts — binding agreements to buy/sell an underlying asset in the future, at a price set today
- The consumption vector of trader i is thus $\mathrm{x}^i \in \mathbb{R}^{LS}$.



Complete Markets

- Complete markets: if there is one Arrow security for every state (contingent claim), or if his securities span the states.
  - Sports: If two teams $i = 1, 2$ score X_1 and X_2 points,
    - the *spread* is $X_1 - X_2$
    - the *over/under line* is $X_1 + X_2$.
  - Together, these easily identify the scores X_1 and X_2.
  - *Moneyline*: bet \$100 on team i, and win Δ_i dollars if $X_i > X_{-i}$
- Example: Superbowl LV, in February 2021

  Moneylines: Buccaneers +145, Chiefs -165

  Total: O/U 56

  Spread: Chiefs -3

- Final score: Tampa Bay 31 beat Kansas City Chiefs 9

Insurance: The Value of Life in the Two State Model

- $L = 1$ good is denoted $x =$ "money"
- Bernoulli utility $u(x)$: increasing, concave, twice differentiable
- Willingness to accept for a cross town delivery trip, with a chance $p > 0$ of deadly accident (costing $L > 0$) is $\pi = \$200$.
- Case 1: linear function u (risk neutral) $\Rightarrow$ WLOG $u(x) = x$:

$$w = (1-p)(w+\pi) + p(w+\pi-L) \iff pL = \pi \iff L = \frac{\pi}{p}$$

- So if $p = 0.01\%$, then $L = \$200/0.0001 = \$2,000,000$
- Case 2: concave u (risk averse, in the sense of Arrow Pratt)

$$\begin{aligned} u(w) &= (1-p)u(w+\pi) + pu(w+\pi-L) \\ &\leq u\left((1-p)(w+\pi) + p(w+\pi-L)\right) \\ \Rightarrow w &\leq (1-p)(w+\pi) + p(w+\pi-L) \end{aligned}$$

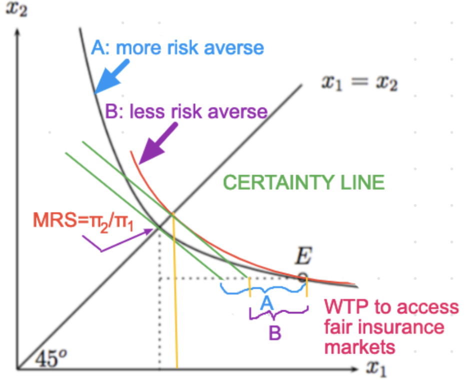
- Hence, $pL \leq \pi \iff L \leq \pi/p$
- Since individuals are willing to pay $\pi \geq pL$, insurance companies can make money if they are risk neutral

Yaari's Depiction of Risk Aversion in Two State World

- Consumption x_1 and x_2 in states 1 & 2 with chances π_1 & π_2
- Expected utility $U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2)$
- Risk aversion $\Rightarrow$ u concave $\Rightarrow$ U concave $\Rightarrow$ U quasiconcave
- A consumption vector x not on certainty line $(x_2 = x_1)$ is risky
- The MRS on full-insurance certainty line is π_1/π_2
- More risk averse $\Leftrightarrow$ willing to pay more for full insurance
- We now relate this economic notion to the concavity of $u(x)$

$$\left.\frac{dMRS_{1,2}}{dx_1}\right|_{x_1=x_2} = \frac{\pi_1}{\pi_2}\frac{u''(x)}{u'(x)}$$

|Slope| ↓ faster at certainty line
→ indifference curve more curved
→ ↑ Arrow-Pratt coefficient of risk aversion



Intensive Margin Choices: Optimal Insurance

- The value of life exercise explored an extensive 0-1 margin.
- The optimal insurance question turns on an intensive margin.
- Let disaster state wealth have price p in insurance premiums.

$$\max_{q \geq 0} \pi u(w - L + q - pq) + (1 - \pi) u(w - pq)$$

- At an interior solution, the FOC is:

$$\pi(1 - p) u'(w - L + q - pq) - p(1 - \pi) u'(w - pq) = 0$$

- *Actuarially fair insurance* when $p = \pi$, since the premiums paid pq equal expected value of compensation received πq

$$u'(w - L + q - pq) = u'(w - pq) \quad \Leftrightarrow \quad q^* = L \qquad \text{(full insurance)}$$

- Typical case is unfair insurance prices: $p > \pi$

$$\text{FOC:} \qquad \frac{u'(w - pq)}{u'(w - L + q - pq)} = \frac{\pi(1 - p)}{p(1 - \pi)} < 1$$

$$\Rightarrow u'(w - pq) < u'(w - L + q - pq)$$

- So $q < L$ if risk averse $\Rightarrow$ *not fully insured.*

The Fundamental Theorem of Risk Bearing

- Endowment of wealth across states $\bar{x} = (\bar{x}_s)$
- Expected utility $U(x_1, \ldots, x_S) = \sum_{s=1}^{S} \pi_s u(x_s)$
- Lagrangian $\mathcal{L} = \sum_{s=1}^{S} \pi_s u(x_s) + \lambda \sum_{s=1}^{S} p_s(\bar{x}_s - x_s)$.
- FOC: $\lambda = \pi_s u'(x_s)/p_s$ for all s (equal shadow value of money)

$\Rightarrow$ Equalize the bang per buck across states

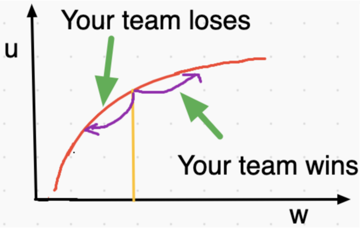
Proposition (Fundamental Theorem of Risk Bearing)

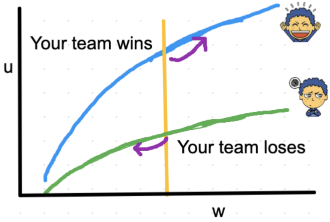
$$\frac{\pi_1 u'(x_1)}{p_1} = \cdots = \frac{\pi_S u'(x_S)}{p_S}$$

- Implications: the price of a state-contingent security rises in proportion to the likelihood of the state.
  - Eg. life insurance is really cheap for young buyers, and doubles in price when the death rates double.
  - This allows us to *infer event probabilities from insurance rates*

State Dependent Utility?

- Bad state $s = 1$ and good state $s = 2$ (your team loses / wins)
- Assume state-dependent utility functions $u_2(w) > u_1(w)$
- For this intensive margin question, we put an extra dollar where its *expected marginal utility* is highest
- An extra time-0 dollar, used to buy Arrow securities,
  - added to bad state raises expected utility by $\frac{\pi_1}{p_1} u_1'(w)$
  - added to good state raises expected utility by $\frac{\pi_2}{p_2} u_2'(w)$
- With fair prices $p_i = \pi_i$, transfer money to the higher u_i' state.
- Case 1: home team win is a wealth gain $\Rightarrow$ betting against them offers perfect insurance
- Case 2: home team win lifts marginal utility (namely, utility is state-dependent) $\Rightarrow$ bet for your team





Risk Sharing: Idiosyncratic Risk

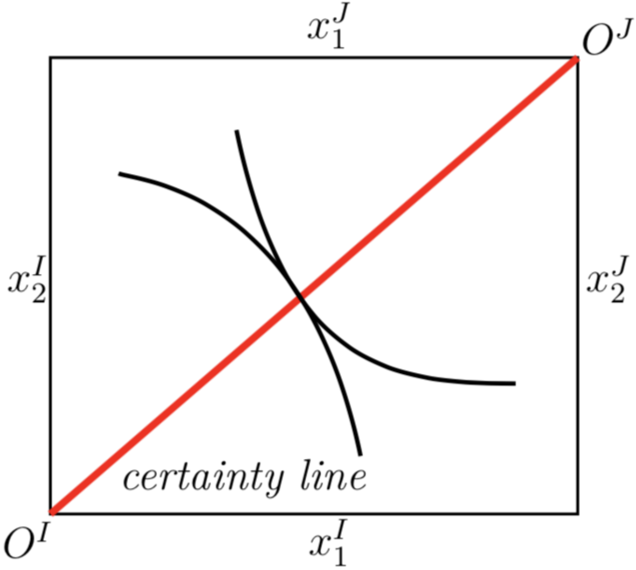
- Assume risk averse traders Iris and Joe, and $S = 2$ states.
- Iris and Joe obey the FOC $\pi_1 u'(x_1)/p_1 = \pi_2 u'(x_2)/p_2 = \lambda$.

$$x_1 \gtrless x_2 \Leftrightarrow \frac{p_1 \pi_2}{p_2 \pi_1} = \frac{u'(x_1)}{u'(x_2)} \lesseqgtr 1 \tag{1}$$

$\Rightarrow$ $x_1^I = x_2^I$ & $x_1^J = x_2^J$, or $x_1^I > x_2^I$ & $x_1^J > x_2^J$, or $x_1^I < x_2^I$ & $x_1^J < x_2^J$.

- Total endowment $\bar{x}_s = \bar{x}_s^I + \bar{x}_s^J$ in state s.
  - purely idiosyncratic risk: $\bar{x}_1 = \bar{x}_2$
  - aggregate risk: $\bar{x}_1 \neq \bar{x}_2$
- Case 1: **Idiosyncratic risk** $\Rightarrow x_1 = x_2$
  - $\Rightarrow$ fair prices: reflect probabilities of states: $p_1/p_2 = \pi_1/\pi_2$
  - $\Rightarrow$ traders fully insure
  - Life insurance premiums reflects death probabilities, and house insurance the chance of a home burning down.

Risk Sharing: Idiosyncratic Risk

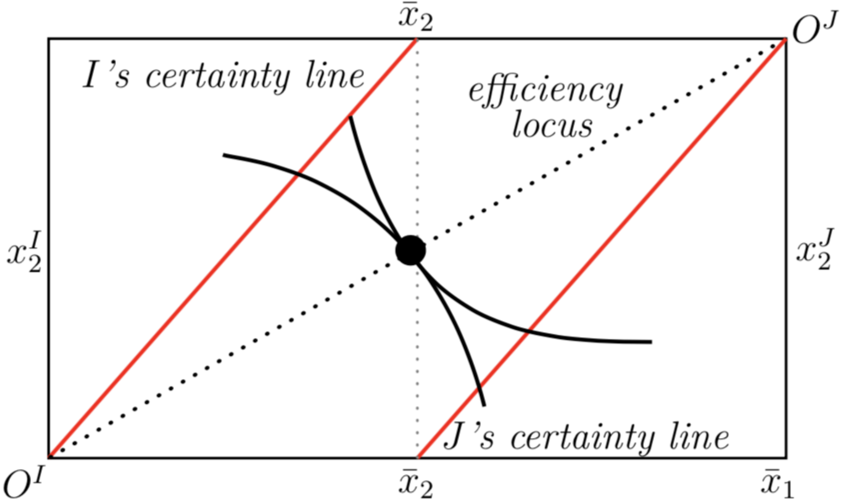


Risk Sharing: Aggregate Risk

- Case 2: **Aggregate risk**, with $\bar{x}_1 > \bar{x}_2$ (disaster state is $s = 2$)
  - Fundamental Theorem of Risk Bearing $\Rightarrow$ traders share risk.
  - $\bar{x}_1 > \bar{x}_2 \Rightarrow x_1^I > x_2^I$ and $x_1^J > x_2^J \Rightarrow p_2/p_1 > \pi_2/\pi_1$
  - Example: logarithmic Bernoulli utility $u^I(x) = u^J(x) = \log x$

    $\Rightarrow$ utility function over consumption bundles is Cobb Douglas
    - Ordinal utility $U(x_1, x_2) = \pi_1 \log x_1 + \pi_2 \log x_2$
    - We can now compute the earthquake insurance premium
    - The FOC (1) yields $p_2/p_1 = (\bar{x}_1/\bar{x}_2)(\pi_2/\pi_1) > \pi_2/\pi_1$.
- Example: earthquake insurance in California is extremely costly, since it only pays out in an overall disastrous state.
  - "force majeure" denies liability for catastrophes

Risk Sharing: Aggregate Risk

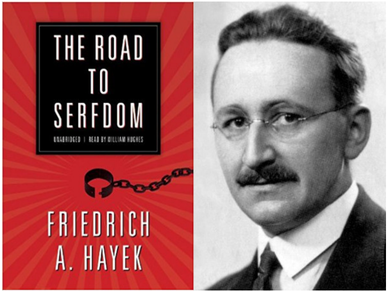


- In equilibrium, $\frac{p_2}{p_1} = MRS = \frac{\pi_2 u'(x_2)}{\pi_1 u'(x_1)} > \frac{\pi_2}{\pi_1}$ since $x_2 < x_1$
- What happens to risk sharing if Iris grows more risk averse?

Information Revelation and Rational Expectations

Information Revelation and Rational Expectations

- So far, prices serve as a mechanism to clear markets
- But prices also convey information about supply and demand, if traders are initially asymmetrically informed
  - Austrian economists, non Mises (1920) and Hayek (1935): social planners do not solve the *calculation problem*: aggregate idiosyncratic consumption / production information



The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1974

Photo from the Nobel Foundation archive.
Gunnar Myrdal
Prize share: 1/2

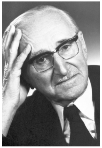


Photo from the Nobel Foundation archive.
Friedrich August von Hayek
Prize share: 1/2

- In a rational expectations equilibrium, agents fully extract information from prices (= Bayesian Nash equilibrium)
- 1970s *rational expectations* work (Radner, Lucas, Sargent,...)

Information Revelation and Rational Expectations

- Can prices "serve two masters": clear markets & convey info?
  - Information $\Rightarrow$ discontinuous demand as a function of price.
  - Resolution: Noisy prices $\Rightarrow$ small price changes reflect noise more than fundamentals.
  - Tatonnement process is now delicate:
    - Auctioneer calls out a price
    - Traders make demands
    - Before auctioneer revises his price, traders see demands, learn from them, and revise demands, etc.
    - Rinse and repeat

Prices Reveal Information in Prediction Markets

- These let people bet on sporting or presidential etc. events.
- Share price convey the expected probability of events.
- Example: Every individual i has log Bernoulli utility, wealth w_i, and can buy $x + i$ shares at price p ["Joe wins in 2020"]

$$\max_{x_i} \pi_i \log[w_i + x_i(1-p)] + (1-\pi_i)\log[w_i - x_i p]$$

- Individual $i = 1, \ldots, n$'s demand: $x_i^* = w_i \frac{\pi_i - p}{p(1-p)}$.
  - Traders buy if more optimistic than the price ($\pi_i > p$)
- Assume everyone is equally wealthy: $w_i = w$ for all i.
- Clear markets: Market excess demand is $\sum_{i=1}^n x_i^* = 0$, or

$$\sum_{\pi_i > p} (\pi_i - p) = \sum_{\pi_i \le p} (p - \pi_i) \Rightarrow p = \frac{1}{n}\sum_{i=1}^n \pi_i$$

Prediction Market Fail: March 3, 2020

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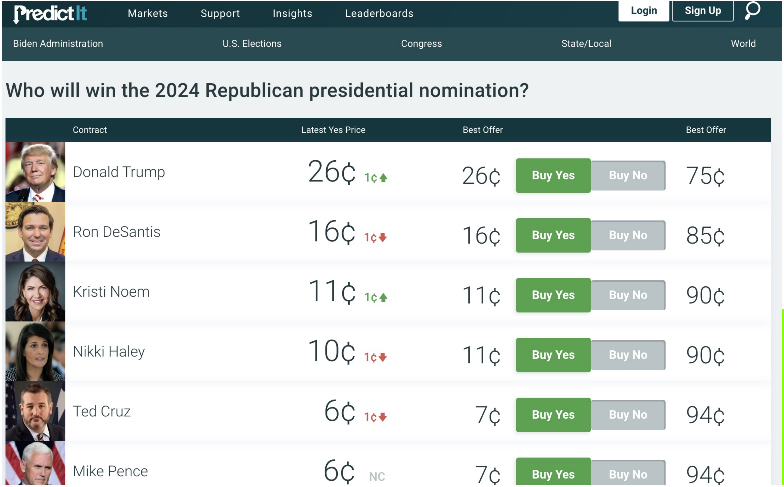
Who will win the 2020 U.S. presidential election?

Contract	Latest Yes Price		Best Offer			Best Offer
Donald Trump	55¢	1¢ ▲	56¢	Buy Yes	Buy No	45¢
Joe Biden	27¢	2¢ ▲	28¢	Buy Yes	Buy No	73¢
Bernie Sanders	25¢	1¢ ▲	25¢	Buy Yes	Buy No	78¢
Michael Bloomberg	3¢	NC	4¢	Buy Yes	Buy No	97¢
Hillary Clinton	2¢	NC	3¢	Buy Yes	Buy No	98¢
Elizabeth Warren	1¢	NC	2¢	Buy Yes	Buy No	99¢
Mike Pence	1¢	NC	2¢	Buy Yes	Buy No	99¢
Kamala Harris	1¢	NC	1¢	Buy Yes	Buy No	N/A
Cory Booker	1¢	NC	1¢	Buy Yes	Buy No	N/A

Prediction Market Forecast of Republican President 2024



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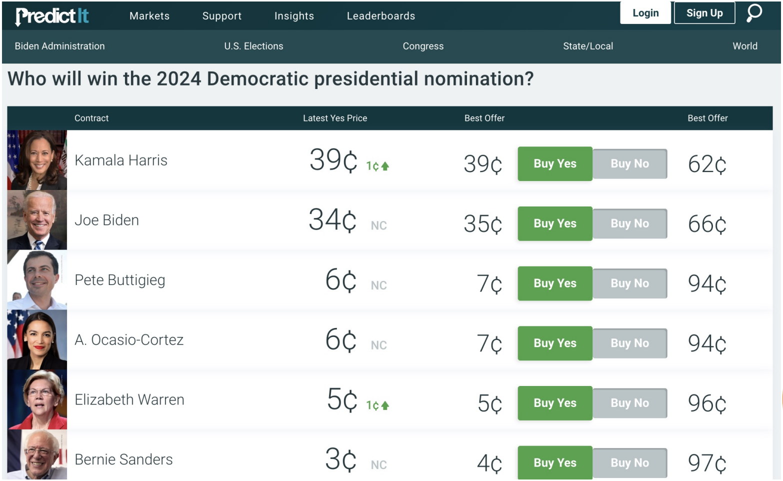
Who will win the 2024 Republican presidential nomination?

Contract	Latest Yes Price		Best Offer			Best Offer
Donald Trump	26¢	1¢ ▲	26¢	Buy Yes	Buy No	75¢
Ron DeSantis	16¢	1¢ ▼	16¢	Buy Yes	Buy No	85¢
Kristi Noem	11¢	1¢ ▲	11¢	Buy Yes	Buy No	90¢
Nikki Haley	10¢	1¢ ▼	11¢	Buy Yes	Buy No	90¢
Ted Cruz	6¢	1¢ ▼	7¢	Buy Yes	Buy No	94¢
Mike Pence	6¢	NC	7¢	Buy Yes	Buy No	94¢

Prediction Market Forecast of Democratic President 2024



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Who will win the 2024 Democratic presidential nomination?

Contract	Latest Yes Price		Best Offer			Best Offer
Kamala Harris	39¢	1¢ ▲	39¢	Buy Yes	Buy No	62¢
Joe Biden	34¢	NC	35¢	Buy Yes	Buy No	66¢
Pete Buttigieg	6¢	NC	7¢	Buy Yes	Buy No	94¢
A. Ocasio-Cortez	6¢	NC	7¢	Buy Yes	Buy No	94¢
Elizabeth Warren	5¢	1¢ ▲	5¢	Buy Yes	Buy No	96¢
Bernie Sanders	3¢	NC	4¢	Buy Yes	Buy No	97¢

Rational Expectations Equilibrium: Nonexistence (Kreps)

- Iris likes x more if $s = 2$: $u^I(x, y) = s \log x + y$ for $s = 1, 2$
- Joe likes x more if $s = 1$: $u^J(x, y) = (3 - s) \log x + y$
- Iris knows s, but Joe thinks $s = 1, 2$ each have 50% chance
- Endowments: $\bar{x} = 2$, and $\bar{y}$ is large. Naturally, $p = p_x/p_y$.
  - Iris's FOC is $x^I(p) = s/p$, provided $\bar{y}^I \geq 2p$.
  - Joe knows $s \Rightarrow x^J(p) = (3 - s)/p$, if $\bar{y}^J \geq 2p$.
- If Joe learns the state from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{3 - s}{p} = \frac{3}{p} \Rightarrow p(s) = 1.5$$

  - This price is the same in $s = 1, 2 \Rightarrow$ conceals Iris's information.
- If Joe learns nothing from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{1.5}{p} = 3 \Rightarrow p(s) = \frac{3}{s + 1.5}$$

  - This price is different in $s = 1, 2 \Rightarrow$ reveals Iris's information.
- $\nexists$ rational expectations equilibrium in this example.
- *Exercise: Find all REE if $u^I(x, y) = u^J(x, y) = s \log x + y$*