An Economic Theory Masterclass

Part VI: General Equilibrium in Exchange Economies

Lones Smith

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General Equilibrium and the Gold Rush

- Partial equilibrium: one-market world, often with quasi-linear utility — where "money" subsumes all other goods
- General equilibrium multi-market world: Markets interact!
- Sam Brannan
 - Richest man in California after Gold Rush of 1849
 - "Gold! Gold on the American River!"
 - Sutter's Mill in Coloma, California
 - Brennan owned only store between San Francisco & gold fields
 - paid 20 cents each for the pans, then sold them for \$15 each



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YREKA . NORTHERN Rich Bar CALIFORNIA SOLDEIEL DS SHASTA Bidwell's Ba SACRAMENTO RIVER BY B. R. BUCKELEW FEATHER RIVER SAN FRANCISCO, MARCH 15, 1848. SIFRR NEVADA SACRAMENTO Marmon *DELEDS* GOLD MINE FOUND.-In the newly RIVER SAN COLOMA FRANCISC made raceway of the Saw Mill recently Sutter's Mil erected by Captain Sutter, on the Ameri-California Gold Rush Map

General Equilibrium in the Movies

- Goldfinger: evil mastermind tried to irradiate Fort Knox gold ⇒ his own gold would ↑ in value
- ▶ Die Hard with a Vengeance: same plan for the gold in NY Fed.
- A View to a Kill: bad guy wants to trigger earthquake to destroy Silicon Valley, and then monopolize microchip market.



General Equilibrium in the Movies

- Casino Royale: bad guy shorts airline stocks, while planning to destroy a luxury jetliner on its maiden voyage.
- Quantum of Solace: bad guy wants to dam Bolivia's fresh water supply to create a Bolivian water monopoly (total joke).



General Equilibrium in the World: Corona Virus!

🕙 The Motley Fool

Latest Stock Picks

Picks St

Stocks - How to Invest - Ret

Retirement

Personal Finan

What are the biggest cruise line stocks?

The three biggest cruise line stocks are **Royal Caribbean** (<u>NYSE:RCL</u>), **Norwegian Cruise Line** (<u>NYSE:NCLH</u>), and market leader **Carnival** (<u>NYSE:CCL</u>). Let's take a closer look at each of these companies.



Conspiracy Theories with a GE Flavor

IF SIDEWALKS ARE SO SAFE

then why have billions of dollars been set aside to compensate victims of **sidewalk injuries?**

The truth? **Big Concrete** knows that sidewalks are unsafe, but that's the source of 99% of their income!



No more sidewalks for my family! From now on, we **walk in the street**!

Sure, my kids might get hit by cars every now and then, but car accidents are **very mild**. They'll spend two weeks out of school, during which time they'll be immune to other cars.

Do you live in a city that prohibits walking in the street? Apply for a personal or religious exemption today!

Know your rights!!!

General Equilibrium Notation

• Exchange economy $\mathcal{E} = (\{u^i\}, \bar{x}).$ ▶ L > 2 goods $\ell \in \{1, 2, ..., L\}$ ▶ n > 2 traders $i \in \{1, 2, ..., n\}$ • Consumer *i* has endowment $\bar{\mathbf{x}}^i = (\bar{\mathbf{x}}_1^i, \bar{\mathbf{x}}_2^i, \dots, \bar{\mathbf{x}}_l^i)' \in \mathbb{R}_+^L$ A goods allocation is a matrix $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^n) \in \mathbb{R}^{nL}_+$. ▶ Trader *i* has utility $u^i : \mathbb{R}^L_+ \to \mathbb{R}$. Trader i's income is the market value p · x̄ⁱ of his endowment So every trader solves a traditional consumer theory problem ▶ Prices $p = (p_1, p_2, ..., p_L) \in \mathbb{R}^L_+$ in some unit of account Jevons (1875): Money is a store of value, unit of account, and medium of exchange, standard of deferred payment Here, it is only a unit of account, and so ∃ degree of freedom. Each trader sells his endowment to the market, valued at the unit of account prices, and then buys his optimal bundle. We assume that all transactions realize by time-0 contracts Modern financial transactions, together with bankruptcy laws, violate this idyllic world (hence the 2008 Financial Crisis)

General Equilibrium

- A trader's wealth is the market value of his endowment
- Budget set $\mathcal{B}^{i}(\bar{x}^{i}, p) = \{x^{i} \in \mathbb{R}_{+}^{L} | p \cdot x^{i} \leq p \cdot \bar{x}^{i}\}$
- Traders optimize, given prices: Trader i = 1, 2..., n solves:

$$\max u^i(x^i)$$
 s.t. $x^i \in \mathcal{B}^i(\bar{x}^i, p)$

- Allocation $x \in \mathbb{R}^{nL}_+$ is feasible for \mathcal{E} if $\sum_{i=1}^n x_\ell^i \leq \sum_{i=1}^n \bar{x}_\ell^i \ \forall \ell$
 - ► free disposal of goods ⇒ weak inequality
 - We say that markets clear in this case
- A competitive equilibrium (x, p) of *E* is a feasible (market-clearing) allocation x s.t. all traders optimize, given p
- A feasible allocation x is socially optimal if *A* feasible allocation z with
 - no one worse off: $u^i(z^i) \ge u^i(x^i)$ for all i = 1, ..., n,
 - some trader *j* strictly better off: $u^j(z^j) > u^j(x^j)$ for some *j*
 - An allocation where one trader owns everything is efficient.

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Edgeworth Boxes for n = 2 Traders

- Francis Ysidro Edgeworth
 - Mathematical Psychics (1881)
 - introduced indifference curves
 - founding editor: Economic Journal
- Trader Iris and Trader Joe trade goods x, y
- They trade to an efficient allocation from endowment
- Assume an interior solution with smooth preferences.
- Equate marginal rate of substitution and price ratio: $\frac{u_x}{u_y} = \frac{p_x}{p_y}$







Competitive Equilibrium and Social Efficiency

- Contract curve: the locus of socially efficient allocations x
- ▶ Individually rational (IR) allocation x obeys $u^i(x^i) \ge u^i(\bar{x}^i) \forall i$
- The core x is the IR portion of the contract curve
- ► A competitive equilibrium for *E* is a pair (x, p) s.t. x is feasible, and optimal for traders, given prices p (via budget sets)
- A competitive equilibrium is in the core because $x^i \in \mathcal{B}^i(\bar{x}^i, p)$
- Divergent marginal rates of substitution \Rightarrow gains from trade
- In exchange economies, trade occurs due to differences in preferences and/or endowments



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Social Efficiency with Perfect Complements

- Utility functions $u'(x, y) = \min\{x, y\}$ & $u^J(x, y) = \min\{x, y\}$
- Endowments $\bar{x}^I = \bar{x}^J = 2$ and $\bar{y}^I = \bar{y}^J = 1$
- The contract curve is the shaded region, since preferences are not strictly monotone



Social Efficiency with Imperfect Complements

Increasing preferences that with at least one party strictly convex is needed to ensure a contract curve and not region



Social Efficiency with Smooth Strictly Convex Preferences

- Cobb-Douglas utility: $u'(x, y) = x^{\alpha}y$ and $u^{J}(x, y) = xy$
- Endowments $\bar{x}' = \bar{x}^J = \bar{y}^I = \bar{y}^J = 1$.
- Contract curve: $MRS_{x,y}^{I} = MRS_{x,y}^{J}$

$$\alpha y'/x' = y'/x \Rightarrow \alpha y'(2-x') = x'(2-y') \Rightarrow y_1 = \frac{2x'}{\alpha(2-x') + x'}$$

- Contract curve is above or below the diagonal as $\alpha \leq 1$.
- ▶ As $\alpha \uparrow$, Iris values good x more, and he efficiently gets more x





Competitive Equilibria are Socially Efficient

- Since trade is win-win, it makes sense that self-interest is good
- Adam Smith (1723–90)
 - 1759: "Theory of Moral Sentiments" explored empathy
 - 1776: "Inquiry into the Nature and Causes of the Wealth of Nations" explored the social benefits of self-interest
 - "It's not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard for their own interest"
- Smith attacked win-lose mercantilism: "We must always take heed that we buy no more from strangers than we sell them, for so should we impoverish ourselves and enrich them" (1549)



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 - "It's not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard for their own interest"
- "When a country is losing many billions of dollars on trade with virtually every country it does business with, trade wars are good, and easy to win" — Trump (2018)



The First Welfare Theorem

Proposition (Arrow (1951) & Debreu (1951), 1940s folk result) If (p, x) is a competitive equilibrium of \mathcal{E} , and preferences are locally non-satiated, then x is socially efficient.

- Intuition: If another allocation is better for all and strictly better for Joe, then it costs everyone at least as much (at the market price), and Joe strictly more. It thus costs more than the old allocation, and so more than the endowment.
- ▶ Proof: If x is socially inefficient, there is a feasible allocation z with $u^i(z^i) \ge u^i(x^i)$ for all *i*, and $u^j(z^j) > u^j(x^j)$ for some *j*.
- Claim 1: $p \cdot z^i \ge p \cdot x^i$ for all *i*
 - ▶ Proof: If not, $p \cdot z^i even though <math>u^i(z^i) \ge u^i(x^i)$
- By local nonsatiation, ∃yⁱ arbitrarily close to xⁱ (and so still affordable) but strictly preferred to xⁱ, contrary to x_i optimal
 Claim 2: p ⋅ z^j > p ⋅ x^j
- Proof: This follows since x^j is a utility maximizer for trader j
 Adding yields p · ∑_{i=1}ⁿ zⁱ > p · ∑_{i=1}ⁿ xⁱ.
 Since p ≥ 0, this contradicts ∑_{i=1}ⁿ zⁱ ≤ ∑_{i=1}ⁿ xⁱ; (z) + (z) = √(z)

The First Welfare Theorem

The proof logic used revealed preference theory.



The Second Welfare Theorem

Proposition (Second Welfare Theorem)

Assume that consumers have continuous, monotonic, and quasiconcave utility functions. If $x \in \mathbb{R}^{\ell n}_+$ is a socially efficient allocation, then there exists a price $p \in \mathbb{R}^{\ell}_+$ such that (x, p) is competitive equilibrium of $\mathcal{E} = (\{u^i\}, \bar{x})$.

Why do we need convex preferences?



The Second Welfare Theorem

- As in a double auction, equilibrium prices need not be unique.
- But here, nonuniqueness is harder to secure, given the intensive margin
- Question: When are competitive prices unique? Answer: At least one consumer has smooth convex preferences



The Second Welfare Theorem: Proof Idea

- The (Minkowski) Separating Hyperplane Theorem proof intuitively works for two traders
- Minkowski taught Einstein, and reformulated his 1905 special relativity as spacetime in 1908 (but then sadly died 1909 at age 44 of appendicitis)
- The Separating hyperplane Theorem easily works for n = 2:



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MY Second Welfare Theorem Proof: Step 1



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- Assume differentiable utility functions.
- Proof: At a socially efficient allocation x, any Trader Joe j maximizes his own utility, s.t. others' utility from x:

 $\max_{z} u^{j}(z^{j}) \quad \text{s.t.} \quad u^{i}(z^{i}) \ge u^{i}(x^{i}) \text{ for all } i \neq j$ $\sum_{z} z^{i} \le \sum_{z} x^{i} \text{ for } \ell = 1 \qquad l \neq j$

 $\sum_{i} \mathsf{z}_{\ell}^{i} \leq \sum_{i} \mathsf{x}_{\ell}^{i}$ for $\ell = 1, \dots, L$ (feasibility)

- As x is efficient, this maximum is realized at z = x.
 - objective function u^j is quasiconcave
 - constraint set is nonempty if no one is near a subsistence utility level (regularity condition on utility functions)
 - constraint set is convex if $u^{j}(z^{i})$ is quasiconcave
- \Rightarrow Lagrangian has saddle point property for some multipliers λ^i, p

$$\mathcal{L}^{j}(\mathbf{z}, \mathbf{p}^{j}, \lambda^{j}) = u^{j}(\mathbf{z}^{j}) + \sum_{i \neq j} \lambda^{j}_{i}[u^{i}(\mathbf{z}^{i}) - u^{i}(\mathbf{x}^{i})] + \sum_{\boldsymbol{x} \notin \ell} p^{j}_{\ell} \left[\sum_{\boldsymbol{x} \neq \ell} x^{j}_{\ell} - \sum_{\boldsymbol{x} \neq \ell} z^{j}_{\ell} \right]_{\substack{\boldsymbol{x} \neq 0 \\ 20/55}}$$

The Second Welfare Theorem: Simplifying the Lagrangian

• Optimality in z_{ℓ}^{i} and z_{ℓ}^{j} for all traders $i \neq j$ yield the FOC's:

$$\frac{\partial}{\partial z_{\ell}^{j}} \mathcal{L}^{j}(\mathbf{z}, \mathbf{p}^{j}, \lambda^{j}) = \frac{\partial}{\partial z_{\ell}^{j}} u^{j}(z^{j}) - p_{\ell}^{j} = 0$$
$$\frac{\partial}{\partial z_{\ell}^{i}} \mathcal{L}^{j}(\mathbf{z}, \mathbf{p}^{j}, \lambda^{j}) = \lambda_{i}^{j} \frac{\partial}{\partial z_{\ell}^{i}} u^{i}(z^{i}) - p_{\ell}^{j} = 0$$

• Equate p_{ℓ}^{j} for traders $i \neq j$ is:

$$\lambda_{i}^{j} = \frac{\partial}{\partial z_{\ell}^{j}} u^{j}(z^{j}) \Big/ \frac{\partial}{\partial z_{\ell}^{i}} u^{i}(z^{i})$$

Equate Planner's MRS between any traders i, j across goods l

$$\frac{\partial}{\partial z_{\ell_1}^i} u^i(z^i) \Big/ \frac{\partial}{\partial z_{\ell_1}^j} u^j(z^j) = \frac{\partial}{\partial z_{\ell_2}^i} u^i(z^i) \Big/ \frac{\partial}{\partial z_{\ell_2}^j} u^j(z^j)$$

 \Rightarrow The price ratio p_{ℓ_1}/p_{ℓ_2} is the same for any two traders i,j:

$$p_{\ell_1}^i/p_{\ell_2}^i=p_{\ell_1}^j/p_{\ell_2}^j$$

 \Rightarrow Multipliers are $p_{\ell}^{j} = c_{j}p_{\ell}$, some $c_{j} > 0$, in Lagrangian for all $\tilde{f}_{21/55}$

The Second Welfare Theorem: Saddle Point Property

 \Rightarrow Simplified Lagrangian

$$\mathcal{L}^{j}(\mathsf{z},\mathsf{p},\lambda^{j}) = u^{j}(\mathsf{z}^{j}) + \sum_{i \neq j} \lambda^{j}_{i}[u^{i}(\mathsf{z}^{i}) - u^{i}(\mathsf{x}^{i})] + c_{j}\mathsf{p}\cdot[\mathsf{x}^{j} - \mathsf{z}^{j}]$$

- By the saddle point property of L^j(x, p, λ^j), we know that (x, λ^j) is a maximum in z, and a minimum in (λ, p).
- ▶ By the maximum property, with the simplified Lagrangian, $\forall z^{j}$:

$$\Rightarrow \mathcal{L}^{j}(\mathsf{z},\mathsf{p},\lambda^{j}) \leq \mathcal{L}^{j}(\mathsf{x},\mathsf{p},\lambda^{j})$$

$$\Rightarrow \quad u^{j}(\mathbf{z}^{j}) + \sum_{i \neq j} \lambda^{j}_{i}[u^{i}(\mathbf{z}^{i}) - u^{i}(\mathbf{x}^{i})] + c_{j}\mathbf{p} \cdot [\mathbf{x}^{j} - \mathbf{z}^{j}] \leq u^{j}(\mathbf{x}^{j}). \quad (\bigstar)$$

- Claim: $u^{j}(z^{j}) > u^{j}(x^{j}) \Rightarrow p \cdot z^{j} > p \cdot x^{j}$ for all j
- If so, then (x, p) is competitive equilibrium, for no trader j can afford any bundle z^j with a higher utility than x^j at price p
- ▶ Proof of Claim: Since $u^j(z^j) > u^j(x^j)$, then (★) implies

$$u^{j}(\mathbf{z}^{j}) + c_{j}\mathbf{p}\cdot[\mathbf{x}^{j} - \mathbf{z}^{j}] \leq u^{j}(\mathbf{x}^{j})$$

(since saddle point of (\bigstar) holds in z^j , fixing $z^i = x^i \forall j \neq i$)

Prices as Shadow Values

Corollary The price of any good is its social shadow value.



Excess Demand Functions

- Strictly convex preferences \Rightarrow unique demands $x_{\ell}^{i}(p)$
- ▶ Trader *i*'s excess demand (net demand): $ED_{\ell}^{i}(p) = x_{\ell}^{i}(p) \bar{x}_{\ell}^{i}$
- The market excess demand for x_{ℓ} is $ED_{\ell}(p) = \sum_{i=1}^{n} ED_{\ell}^{i}(p)$
- Markets clear in a competitive eq (x(p), p): $ED_{\ell}(p) = 0 \ \forall \ell$

Lemma (Walras Law)

If traders consume their entire income at allocation x(p), then the market value of net excess demand vanishes: $\sum_{\ell=1}^{L} p_{\ell} ED_{\ell}(p) = 0$.

• Proof: Trader i's budget constraint $p \cdot x^{i}(p) \equiv p \cdot \bar{x}^{i}$:

$$\sum_{\ell=1}^{L} p_{\ell} E D_{\ell}^{i} \equiv \sum_{\ell=1}^{L} p_{\ell} [x_{\ell}^{i}(p) - \bar{x}_{\ell}^{i}] \equiv 0 \qquad \Box$$

- \Rightarrow By Walras, it suffices that L-1 of L markets clear
- Since demand is homogeneous of degree zero in (income, prices), there is a degree of freedom in the prices. So we can:

pick a good as numeraire, the de facto currency, with unit price

or, insist that all prices sum to one

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Existence Using Excess Demand Functions: L = 2 Goods

- So equilibrium amounts to L-1 equations in L-1 unknowns
- Assume L = 2 goods, x and y. Let x be the numeraire.
- \Rightarrow Measure the *price ratio* $p = p_y/p_x$ of y in units of x.
- \Rightarrow Equilibrium is one equation in one unknown: $ED_x(p) = 0$.

Theorem (Existence)

Assume every trader *i* has strictly monotone and strictly convex preferences over x and y, and owns a positive endowment (\bar{x}^i, \bar{y}^i) . There exists a Walrasian stable competitive equilibrium (x, y, p).

- Proof: Given strictly convex preferences, every trader i has a unique optimal consumption bundle xⁱ(p) at any price p > 0.
- ► The optimizer set is upper hemicontinuous in p, by (Berge's) Theorem of the Maximum ⇒ xⁱ(p) is thus continuous in p
- \Rightarrow Market excess demand $ED_x(p)$ is a continuous function
- Monotone preferences $\Rightarrow ED_x(0) < 0 < ED_x(\infty)$
- Intermediate Value Theorem $\Rightarrow ED_x(p) = 0$, for some p > 0.
- At least one zero of $ED_x(p) = 0$ is stable, crossing to +

Strictly Convex Preferences Leads to Unique Demands



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Excess demand curves can be almost anything, by suitably specifying consumers and endowments!

Local Uniqueness of Equilibria



- It is truly rare that the excess demand curve sits on the price axis for an interval of prices, with a continuum of equilibria
- Debreu used Sard's Theorem in differential topology to formalize the sense that this is rare: For generic endowments, it cannot happen.
- This mathematical minutia was studied in mathematical economics

Existence with Cobb Douglas Preferences & L = 2 Goods

- Utilities: Iris $u^{I}(x,y) = x^{\alpha}y^{1-\alpha}$ and Joe $u^{J}(x,y) = x^{\beta}y^{1-\beta}$
- Endowments: $(\bar{x}^{\prime}, \bar{y}^{\prime})$ and $(\bar{x}^{J}, \bar{y}^{J})$.
- Incomes: $w^I = \bar{x}^I + p\bar{y}^I$ and $w^J = \bar{x}^J + p\bar{y}^J$
- Demands: Iris $x'(p, w) = \alpha w'$ and Joe $x^J(p, w) = \beta w^J$
- Market excess demand:

$$ED_{x}(p) = \left(\alpha w^{I} - \bar{x}^{I}\right) + \left(\beta w^{J} - \bar{x}^{J}\right)$$

• It suffices to clear the x market: $ED_x(p) = 0$

$$p^* = \frac{\bar{x}^I (1 - \alpha) + \bar{x}^J (1 - \beta)}{\alpha \bar{y}^I + \beta \bar{y}^J}$$

- The unique competitive equilibrium price p*:
 - falls in α, β (greater love of x by either trader raises its price)
 - ▶ rises if \bar{x}^I or \bar{x}^J rises (gold discoveries led to inflation)

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Trade Offer Curves

- The trade offer curve (TOC) plots optimal consumption allocations as prices vary, fixing endowments.
- In consumer theory, it is called the price-consumption curve
- Note: Trade theory overlaps heavily with consumer theory
- ► TOCs are the best reply graphs of general equilibrium theory
- ▶ With L = 2 goods, TOC is tangent to the indifference curve through the endowment, and "more curved" than it



Trade Offer Curves and Substitutes and Complements



- The TOC can be nonmonotone, despite monotone preferences
- As the price p_i of y in terms of x rises p₁ < p₂ < p₃, the substitution effect pushes y ↓.
- As the price of y rises, "income" (value of endowment) rises, since Iris is a net supplier of y

If y is an normal good, then the TOC can fall or rise

If y is an inferior good, then the TOC is strictly falling

Backward Bending Trade Offer Curves Require Inferiority



- As the price of y in terms of x rises, the price of x in terms of y falls, and the substitution effect pushes x ↑
- ► As the price of *y* rises, and so the "income" rises
 - If x is an normal good, then the TOC pushes right
 - If x is an inferior good, the TOC can turn back (but need not)

Examples of Trade Offer Curves



- Perfect substitutes, perfect complements, and Cobb Douglas
- Assume Cobb-Douglas $u(x, y) = x^{\alpha}y^{1-\alpha}$
- TOC is the locus of indifference curve tangencies to the price line through the endowments (x̄, ȳ):

$$\frac{(1-\alpha)x}{\alpha y} = MRS = p = \frac{\bar{x} - x}{y - \bar{y}} \Rightarrow y(x) = \frac{(1-\alpha)\bar{y}x}{x - \alpha\bar{x}}$$

▶ The TOC starts at $y(\bar{x}) = \bar{y}$, for there is always a price for which it is efficient to consume the endowment.

Uniqueness: Trade Offer Curves

- As best reply graphs, their intersection yields an equilibrium
- At a crossing of TOC¹ and TOC³, each trader optimally chooses that bundle, and so markets clear
- The absolute slope of the price line is p_x/p_y



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Non-Uniqueness: Trade Offer Curves

- There are three equilibrium prices (of y): $p_1 > p_2 > p_3$
- Claim: p₁ and p₃ are Walrasian stable, and p₂ is not: If the price p ∈ (p₂, p₁) (flatter price line), then the excess demand for y pushes up p; so the price line swings away from p₂ more
- With multiple equilibria, alternating equilibria are stable



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Gross Substitutes and Uniqueness

- Recall that the TOC bends back with enough inferiority
- Demand has the gross substitutes property if an increase in price p_k raises the demand of every other good x_ℓ, for ℓ ≠ k.

Proposition (Uniqueness)

If the aggregate excess demand function satisfies gross substitutes, the economy has at most one Walrasian equilibrium

- Proof: We prove that z(p) has at most one (normalized) root.
- Assume z(p) = z(p') = 0 for p and p' not linearly dependent.
- By homogeneity of degree zero, normalize the price vectors so that p_ℓ ≥ p'_ℓ for all ℓ, and p_k = p'_k for some k

▶ p = (48, 12, 4) and $p' = (8, 4, 2) \Rightarrow$ scale p' to $\hat{p} = (16, 8, 4)$

- Change from p' to p in n-1 steps, raising p_{ℓ} for each $\ell \neq k$.
 - Raise \hat{p}_2 from 8 to $12 = p_2$, and then \hat{p}_1 from 16 to $48 = p_1$
- At each step, the aggregate demand for good x_k strictly increases, so that z_k(p) > z_k(p') = 0. Contradiction. QED
- ★ Complementarity ⇒ multiplicity of equilibria, while substitutes ⇒ uniqueness. Supermodular games may have₃> 1_equilibria_

Monopoly in the Edgeworth Box

Start with a competitive equilibrium with two goods, in which Joe sells y to the market and buys x



Monopoly Joe Replaces the Walrasian Auctioneer

- ▶ Joe seeks his highest indifference curve on Iris's TOC: $ar{u}_1^J > ar{u}_0^J$
- He sets a higher price ratio for y to x (now, the red price line)
- ▶ This monopoly is inefficient, or \exists (orange) gains from trade:
 - ▶ Proof: The indifference curve \bar{u}_1^J is tangent to TOC' at A
 - The (red) price line slices through TOC', and thus through \bar{u}_1^J
 - But indifference curve \bar{u}^{l} is tangent to the (red) price line at \bar{A}



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Monopoly Kingpin Joe Sets a Two Part Tariff

- ▶ Joe now secures an even higher utility $\bar{u}_2^J > \bar{u}_1^J$
- He sets a new price ratio for y to x, but now sets a trading fee
- Omnipotent monopoly is efficient: B is on the contract curve!



• EXAMPLE: $u^{J}(x, y) = x + y$ and $\bar{x}^{J} = 20$ and $\bar{y}^{J} = 0$. • $u^{I}(x, y) = x(9 - x) + y$ and $\bar{x}^{I} = 0$ and $\bar{y}^{I} = 20$.

Zoom in on the last Edgeworth Box



- Iris' utility u^l(x, y) = x(9 x) + y with x
 _l = 0 and y
 _l = 20.
 Her indifference curves have the form: y = u
 + x(x 9)
 Joe's utility u^J(x, y) = x + y with x
 _J = 20 and y
 _J = 0.
 Joe has constant MRS = 1 ⇒ equilibrium price of y is p = 1
 an advantage of one consumer having perfect substitutes utility
 Iris has quasi-linear preferences and is linear in y
 ⇒ pick y as numeraire, with relative price π = 1/p of x
 Iris equates 9 2x_I ≡ u^I_x/u^I_y = π = 1 ⇒ x_I = 4 and y_I = 16
 So Joe demands the residual x_J = 16 and y_J = 4.
- First Welfare Theorem: The outcome is efficient (tangency)





Joe is a monopolist ⇒ sets the price π of x
Iris is a price taker ⇒ demands x_l(π) = (9 − π)/2 (★) ⇒ y_l(π)=20 − π(9 − π)/2, from her budget constraint (y
_l = 20)
We now solve for the (quadratic) trade offer curve of Iris. ⇒ TOC_l: y=20 − πx=20 − (9 − 2x)x by budget constraint, (★)
Joe maximizes indirect utility

$$V_J(\pi) = x_J + y_J = [20 - x_I(\pi)] + [20 - y_I(\pi)] = 20 - (9 - \pi)/2 + \frac{1}{2}\pi(9 - \pi)$$

 \Rightarrow FOC: $1 + 9 - 2\pi = 0 \Rightarrow \pi = 5$.

▶ Joe sets higher than competitive price π for his endowed good $\frac{2}{42/55}$



Iris' demands

► $x_l(\pi) = (9 - \pi)/2 = (9 - 5)/2 = 2$ ► $y_l(\pi) = 20 - \pi(9 - \pi)/2 = 5(9 - 5)/2 = 10$

• Joe's demands are the residual: $x_J = 18$ and $y_J = 10$.

- Joe's utility rises from 4 + 16 = 20 to 18 + 10 = 28
- ► Iris's utility falls from 4(9-4) + 16=36 to 2(9-2) + 10=24.

This still beats Iris's endowment utility of 20.

▶ Inefficiency of monopoly: Joe's *MRS* is constant at one, whereas Iris ends up with $MRS = u_x^I/u_y^I = 9 - 2x_J = 9 - 4 = 5$.



▶ Iris' reservation utility is $u'_0(0, 20) = 20$ at endowment (0, 20). ⇒ Iris needs $u'(x, y) = x(9 - x) + y \ge 20$

► Joe maximizes welfare given Iris's demands (x, 20 - x(9 - x))

$$[20-x] + [20-y] = (20-x) + x(9-x) \quad \Rightarrow -1 + 9 - 2\hat{x} = 0$$

- So Iris consumes $(\hat{x}_I, \hat{y}_I) = (4, 0)$ and Joe $(\hat{x}_J, \hat{y}_J) = (16, 20)$
- Two part tariff: Iris pays a fee $y = 16 \varepsilon$, then a price $\pi = 1$
- Inefficiency of monopoly: vanishes

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