# An Economic Theory Masterclass

# Part VIII: General Equilibrium with Uncertainty

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# How Markets Enable Risk Sharing

- Robinson Crusoe: shared ownership of firm exists to finance large firms that no one individual could own
- But shared ownership plays another key role: risk-sharing
- 1602, the Dutch East India Company officially was the world's first publicly traded company
  - issued shares of the company on Amsterdam Stock Exchange
  - Ships returning from the East Indies had a high chance of loss due to weather, war, or pirates.

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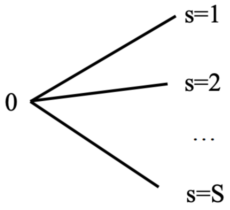
  - Instead of investing in one voyage, investors could now purchase shares in multiple companies.
  - The company eventually went bankrupt in 1799

# Arrow-Debreu Securities and Risk Sharing

- Exchange economy with $n$ traders and $L$ goods
- Time-1: A state of the world $s \in S = \{1, \ldots, S\}$ is realized.
- Time-0: Only the probability $\pi_s$ of each state $s$ is known.
- Label the goods in the Arrow-Debreu model by the state.
- A state-contingent claim $x_{\ell s} \in \mathbb{R}^{LS}$ is a title to a unit of consumption of good $\ell$ in state $s$.
- $p_s$ = price of the state $s$ contingent claim, or *Arrow security*.
- So far, trade was contractually implemented, not using money. These are now $LS$ forward contracts — binding agreements to buy/sell an underlying asset in the future, at a price set today
- The consumption vector of trader $i$ is thus $\mathbf{x}^i \in \mathbb{R}^{LS}$.



# Complete Markets

- Complete markets: if there is one Arrow security for every state (contingent claim), or if his securities span the states.
  - Sports Example: If two teams score $X$ and $Y$ points, individuals can often bet on the *spread* $X - Y$ and the *over/under line* $X + Y$. Together, these easily allow complete identification of the scores $X$ and $Y$.

**Super Bowl Odds 2020: 49ers vs. Chiefs Spread, Over/Under Betting Guide**

- **Spread**: Kansas City (-1.5)
- Over/Under: 54.
- Money Line: Kansas City (-127; bet $127 to win $100); San Francisco (+107; bet $100 to win $107)

# The Value of Life in the Two State Model

- $L = 1$ good, denoted $x =$ "money"
- *twice differentiable and concave* Bernoulli utility function $u(x)$.
- Willingness to accept for a cross town delivery trip, with a chance $p > 0$ of deadly accident (costing $L > 0$) is $\pi = \$200$.
- Case 1: linear function $u$ (risk neutral) $\Rightarrow$ WLOG $u(x) = x$:

$$w = (1-p)(w+\pi) + p(w+\pi-L) \iff pL = \pi \iff L = \frac{\pi}{p}$$

- So if $p = 0.01\%$, then $L = \$200/0.0001 = \$2,000,000$
- Case 2: concave $u$ (risk averse, in the sense of Arrow Pratt)

$$\begin{aligned} u(w) &= (1-p)u(w+\pi) + pu(w+\pi-L) \\ &\leq u((1-p)(w+\pi) + p(w+\pi-L)) \\ \Rightarrow w &\leq (1-p)(w+\pi) + p(w+\pi-L) \end{aligned}$$

- Hence, $pL \leq \pi \iff L \leq \pi/p$

# Risk Bearing and Optimal Insurance

- The WTP exercise explored an extensive margin. The optimal insurance question turns on the intensive margin.
- Let disaster state wealth have price $p$ in insurance premiums.

$$\max_{q \geq 0} \pi u(w - L + q - pq) + (1 - \pi) u(w - pq)$$

- At an interior solution, the FOC is:

$$\pi(1 - p) u'(w - L + q - pq) - p(1 - \pi) u'(w - pq) = 0$$

- *Actuarially fair insurance* when $p = \pi$, since the premiums paid $pq$ equal expected value of compensation received $\pi q$

$$u'(w - L + q - pq) = u'(w - pq) \quad \Leftrightarrow \quad q^* = L \qquad \text{(full insurance)}$$

- Typical case is unfair insurance prices: $p > \pi$

$$\text{FOC:} \qquad \frac{u'(w - pq)}{u'(w - L + q - pq)} = \frac{\pi(1 - p)}{p(1 - \pi)} < 1$$

$$\Rightarrow u'(w - pq) < u'(w - L + q - pq)$$

- So $q < L$ if risk averse $\Rightarrow$ *not fully insured*.

# The Fundamental Theorem of Risk Bearing

- Endowment of wealth across states $\bar{\mathbf{x}} = (\bar{x}_s)$
- Expected utility $U(x_1, \ldots, x_S) = \sum_{s=1}^{S} \pi_s u(x_s)$
- Lagrangian $\mathcal{L} = \sum_{s=1}^{S} \pi_s u(x_s) + \lambda \sum_{s=1}^{S} p_s(\bar{x}_s - x_s)$.
- FOC: $\lambda = \pi_s u'(x_s)/p_s$ for all $s$.

## Proposition (Fundamental Theorem of Risk Bearing)

$$\frac{\pi_1 u'(x_1)}{p_1} = \cdots = \frac{\pi_S u'(x_S)}{p_S}$$

- Implications: the price of a state-contingent security rises in proportion to the likelihood of the state.
  - Eg. life insurance is really cheap for young buyers, and doubles in price when the death rates double.
  - This allows us to infer event probabilities from insurance rates.

# RIsk Bearing Encore: Must You Bet Against Your Team?

- Bad state $s = 1$ and good state $s = 2$ (your team loses / wins)
- Assume state-dependent utility functions $u_2(w) > u_1(w)$
- An extra time-0 dollar, used to buy Arrow securities,
  - added to bad state raises expected utility by $\frac{\pi_1}{p_1} u_1'(w)$
  - added to good state raises expected utility by $\frac{\pi_2}{p_2} u_2'(w)$
- With fair prices, $p_i = \pi_i$, one transfers a dollar to the higher marginal utility state.
- Is Joy best captured by the marginal utility of spending?

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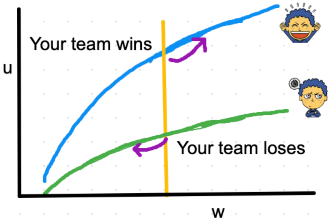
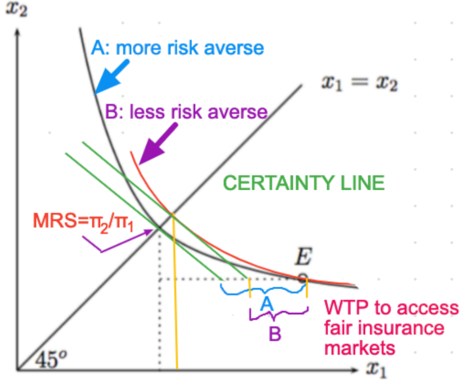


# Yaari's Depiction of Risk Aversion in Two State World

- Consumption $x_1$ and $x_2$ in states 1 & 2 with chances $\pi_1$ & $\pi_2$
- Expected utility $U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2)$
- Risk aversion $\Rightarrow$ $u$ concave $\Rightarrow$ $U$ concave $\Rightarrow$ $U$ quasiconcave
- A consumption vector $x$ not on certainty line $(x_2 = x_1)$ is risky
- The MRS on certainty line is $\pi_1/\pi_2$
- Define more risk averse $\Leftrightarrow$ willing to pay more to avoid risk
- We now relate this economic notion to the concavity of $u(x)$

$$\left.\frac{dMRS_{1,2}}{dx_1}\right|_{x_1=x_2} = \frac{\pi_1}{\pi_2}\frac{u''(x)}{u'(x)}$$

|Slope| ↓ faster at certainty line
→ indifference curve more curved
→ ↑ Arrow-Pratt coefficient of risk aversion



# Risk Sharing: Idiosyncratic Risk

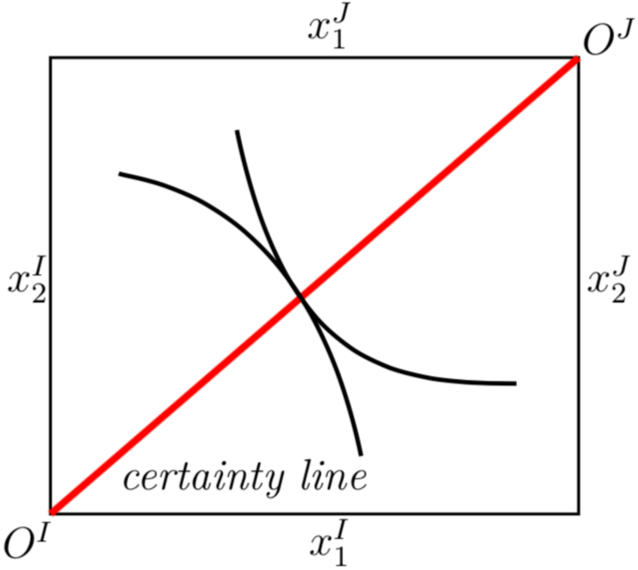
- Assume risk averse traders Iris and Joe, and $S = 2$ states.
- Iris and Joe obey $\pi_1 u'(x_1)/p_1 = \pi_2 u'(x_2)/p_2 = \lambda$.

$$x_1 \gtreqless x_2 \Leftrightarrow \frac{p_1 \pi_2}{p_2 \pi_1} = \frac{u'(x_1)}{u'(x_2)} \lesseqgtr 1 \tag{1}$$

$\Rightarrow$ $x_1^I = x_2^I$ & $x_1^J = x_2^J$, or $x_1^I > x_2^I$ & $x_1^J > x_2^J$, or $x_1^I < x_2^I$ & $x_1^J < x_2^J$.

- Total endowment $\bar{x}_s = \bar{x}_s^I + \bar{x}_s^J$ in state $s$.
  - purely idiosyncratic risk: $\bar{x}_1 = \bar{x}_2$
  - aggregate risk: $\bar{x}_1 \neq \bar{x}_2$
- Case 1: **Idiosyncratic risk** $\Rightarrow x_1 = x_2$
  - $\Rightarrow$ fair prices: reflect probabilities of states: $p_1/p_2 = \pi_1/\pi_2$
  - $\Rightarrow$ traders fully insure
  - Life insurance premiums reflects death probabilities, and house insurance the chance of a home burning down.

# Risk Sharing: Idiosyncratic Risk



# Risk Sharing: Aggregate Risk

- Case 2: **Aggregate risk**, with $\bar{x}_1 > \bar{x}_2$ (disaster state is $s = 2$)
  - Fundamental Theorem of Risk Bearing $\Rightarrow$ traders share risk.
  - $\bar{x}_1 > \bar{x}_2 \Rightarrow x_1^I > x_2^I$ and $x_1^J > x_2^J \Rightarrow p_1/p_2 < \pi_1/\pi_2$
  - logarithmic Bernoulli utility $u^I(x) = u^J(x) = \log x$
  - $\Rightarrow$ utility function over consumption bundles is Cobb Douglas
  - Ordinal utility $U(x_1, x_2) = \pi_1 \log x_1 + \pi_2 \log x_2$
  - $p_1/p_2 = (\bar{x}_2/\bar{x}_1)(\pi_1/\pi_2) > \pi_1/\pi_2$.
- Example: earthquake insurance in California is extremely costly, since it only pays out in an overall disastrous state.
  - "force majeure" denies liability for catastrophes

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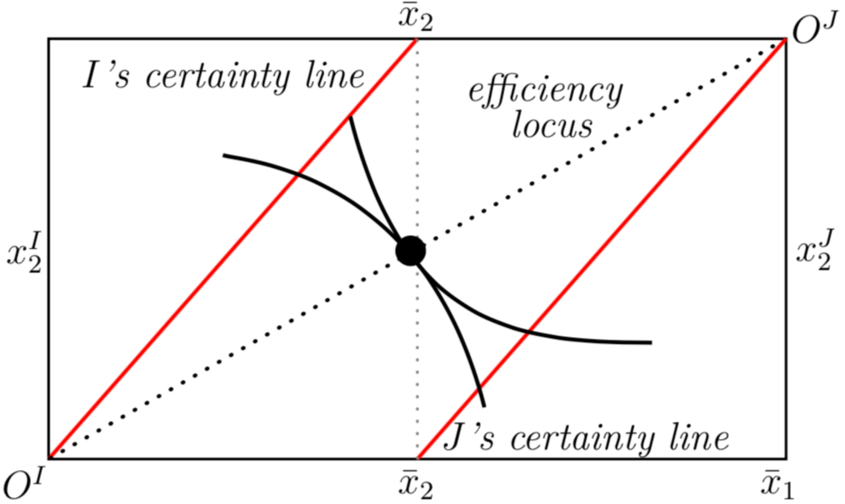

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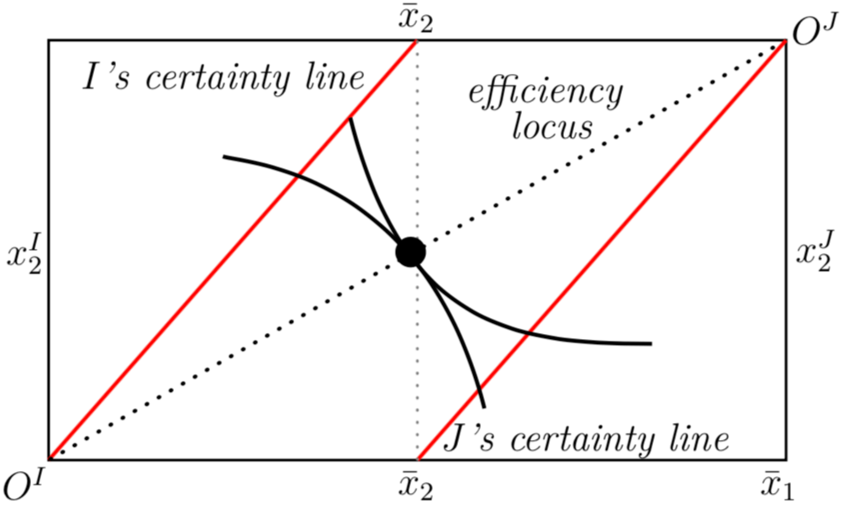
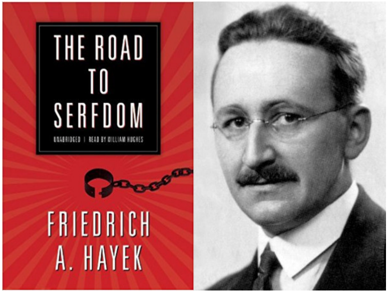



- What happens to risk sharing if Iris grows more risk averse?

# Information Revelation and Rational Expectations

# Information Revelation and Rational Expectations

- So far, prices serve as a mechanism to clear markets
- But prices also convey information about supply and demand
  - Austrian economists, non Mises (1920) and Hayek (1935): social planners do not solve the *calculation problem*: aggregate idiosyncratic consumption / production information





- In a rational expectations equilibrium, agents fully extract information from prices (= Bayesian Nash equilibrium)
- 1970s *rational expectations* work (Radner, Lucas, Sargent,...)

# Information Revelation and Rational Expectations

- Can prices "serve two masters": clear markets & convey info?
  - Information $\Rightarrow$ discontinuous demand as a function of price.
  - Resolution: Noisy prices $\Rightarrow$ small price changes reflect noise more than fundamentals.
  - Tatonnement process is now delicate:
    - Auctioneer calls out a price
    - Traders make demands
    - Traders see demands, and revise demands
    - Rinse and repeat

# Prices Reveal Information in Prediction Markets

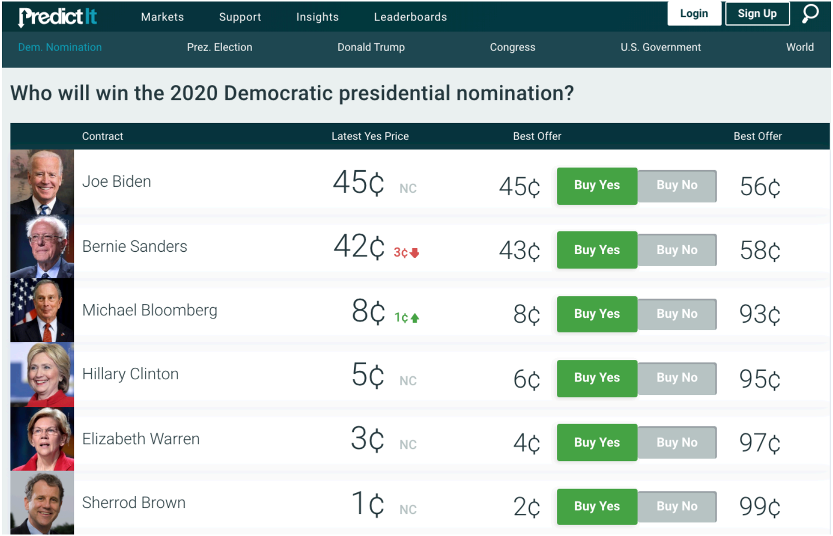
- These let people bet on sporting or presidential etc. events.
- Share price convey the expected probability of events.
- Example: Every individual $i$ has log Bernoulli utility, wealth $w_i$, and can buy $x + i$ shares at price $p$ ["Joe wins in 2020"]

$$\max_{x_i} \pi_i \log[w_i + x_i(1-p)] + (1-\pi_i) \log[w_i - x_i p]$$

- Individual $i = 1, \ldots, n$'s demand: $x_i^* = w_i \frac{\pi_i - p}{p(1-p)}$.
  - Traders buy if more optimistic than the price ($\pi_i > p$)
- Assume everyone is equally wealthy: $w_i = w$ for all $i$.
- Clear markets: Market excess demand is $\sum_{i=1}^n x_i^* = 0$, or

$$\sum_{\pi_i > p} (\pi_i - p) = \sum_{\pi_i \leq p} (p - \pi_i) \Rightarrow p = \frac{1}{n} \sum_{i=1}^n \pi_i$$

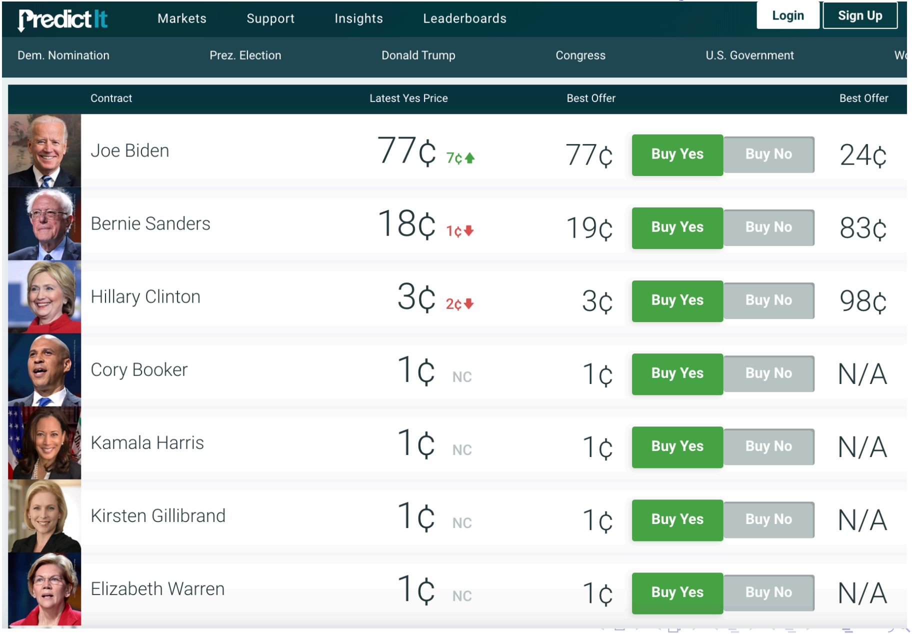
# Political Prediction Market, as of Monday




Who will win the 2020 Democratic presidential nomination?

| Contract | Latest Yes Price | | Best Offer | | | Best Offer |
|---|---|---|---|---|---|---|
| Joe Biden | 45¢ | NC | 45¢ | Buy Yes | Buy No | 56¢ |
| Bernie Sanders | 42¢ | 3¢↓ | 43¢ | Buy Yes | Buy No | 58¢ |
| Michael Bloomberg | 8¢ | 1¢↑ | 8¢ | Buy Yes | Buy No | 93¢ |
| Hillary Clinton | 5¢ | NC | 6¢ | Buy Yes | Buy No | 95¢ |
| Elizabeth Warren | 3¢ | NC | 4¢ | Buy Yes | Buy No | 97¢ |
| Sherrod Brown | 1¢ | NC | 2¢ | Buy Yes | Buy No | 99¢ |

# Political Prediction Market, as of Wednesday




| Contract | Latest Yes Price | | Best Offer | | | Best Offer |
|---|---|---|---|---|---|---|
| Joe Biden | 77¢ | 7¢ ↑ | 77¢ | Buy Yes | Buy No | 24¢ |
| Bernie Sanders | 18¢ | 1¢ ↓ | 19¢ | Buy Yes | Buy No | 83¢ |
| Hillary Clinton | 3¢ | 2¢ ↓ | 3¢ | Buy Yes | Buy No | 98¢ |
| Cory Booker | 1¢ | NC | 1¢ | Buy Yes | Buy No | N/A |
| Kamala Harris | 1¢ | NC | 1¢ | Buy Yes | Buy No | N/A |
| Kirsten Gillibrand | 1¢ | NC | 1¢ | Buy Yes | Buy No | N/A |
| Elizabeth Warren | 1¢ | NC | 1¢ | Buy Yes | Buy No | N/A |

# Prediction Market Fail, as of Monday

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Dem. Nomination — Prez. Election — Donald Trump — Congress — U.S. Government — World

## Who will win the 2020 U.S. presidential election?

| Contract | Latest Yes Price | Best Offer | | | Best Offer |
|---|---|---|---|---|---|
| Donald Trump | 55¢ 1¢▲ | 56¢ | Buy Yes | Buy No | 45¢ |
| Joe Biden | 27¢ 2¢▲ | 28¢ | Buy Yes | Buy No | 73¢ |
| Bernie Sanders | 25¢ 1¢▲ | 25¢ | Buy Yes | Buy No | 78¢ |
| Michael Bloomberg | 3¢ NC | 4¢ | Buy Yes | Buy No | 97¢ |
| Hillary Clinton | 2¢ NC | 3¢ | Buy Yes | Buy No | 98¢ |
| Elizabeth Warren | 1¢ NC | 2¢ | Buy Yes | Buy No | 99¢ |
| Mike Pence | 1¢ NC | 2¢ | Buy Yes | Buy No | 99¢ |
| Kamala Harris | 1¢ NC | 1¢ | Buy Yes | Buy No | N/A |
| Cory Booker | 1¢ NC | 1¢ | Buy Yes | Buy No | N/A |

# Prediction Market Fail, as of Wednesday

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Dem. Nomination  Prez. Election  Donald Trump  Congress  U.S. Government  W

| Contract | Latest Yes Price | | Best Offer | | | Best Offer |
|---|---|---|---|---|---|---|
| Donald Trump | 55¢ | 2¢ ▲ | 56¢ | Buy Yes | Buy No | 45¢ |
| Joe Biden | 40¢ | 1¢ ▲ | 40¢ | Buy Yes | Buy No | 61¢ |
| Bernie Sanders | 8¢ | 3¢ ▼ | 8¢ | Buy Yes | Buy No | 93¢ |
| Hillary Clinton | 2¢ | NC | 2¢ | Buy Yes | Buy No | 99¢ |
| Mike Pence | 1¢ | NC | 2¢ | Buy Yes | Buy No | 99¢ |
| Kamala Harris | 1¢ | NC | 1¢ | Buy Yes | Buy No | N/A |
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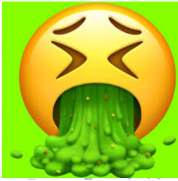

# Rational Expectations Equilibrium: Nonexistence (Kreps)

- Iris likes $x$ more if $s = 2$: $u^I(x, y) = s \log x + y$ for $s = 1, 2$
- Joe likes $x$ more if $s = 1$: $u^J(x, y) = (3 - s) \log x + y$
- Iris knows $s$, but Joe thinks $s = 1, 2$ each have 50% chance
- Endowments: $\bar{x} = 2$, and $\bar{y}$ is large.
  - Iris's FOC is $x^I(p) = s/p$, provided $\bar{y}^I \geq 2p$.
  - Joe's FOC is $x^J(p) = (3 - s)/p$, provided $\bar{y}^J \geq 2(1 - p)$.

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  - Joe's FOC is $x^J(p) = (3 - s)/p$, provided $\bar{y}^J \geq 2(1 - p)$.
- If Joe learns the state from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{3 - s}{p} = \frac{3}{p} \Rightarrow p(s) = 1.5$$

  - This price is the same in $s = 1, 2 \Rightarrow$ conceals Iris's information.
- If Joe learns nothing from the price, then market demand is

$$x^I(p) + x^J(p) = \frac{s}{p} + \frac{1.5}{p} = 3 \Rightarrow p(s) = \frac{3}{s + 1.5}$$

  - This price is different in $s = 1, 2 \Rightarrow$ reveals Iris's information.
- $\nexists$ rational expectations equilibrium in this example.
- *Exercise: Find all REE if $u^I(x, y) = u^J(x, y) = s \log x + y$*