An Economic Theory Masterclass

Part VII: General Equilibrium in Production Economies

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The Walrasian Existence Problem

- Leon Walras (1874), *Éléments d'économie politique pure*
- Formulated the marginal theory of value
- Father of the general equilibrium theory

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- He deduced four sets of equations solving for
  - (a) the price of each good and
  - (b) factor of production,
  - (c) the quantity of each good and
  - (d) factor bought by businesses.
- Fifth equation: Walras Law: all money received is spent

Nash (1951) Inspires Arrow and Debreu (1954)
Existence of an Equilibrium for a Competitive Economy

- *"Walras first formulated the state of the economic system at any point of time as the solution of a system of simultaneous equations* . . . *Walras did not, however, give any conclusive arguments to show that the equations have a solution"*

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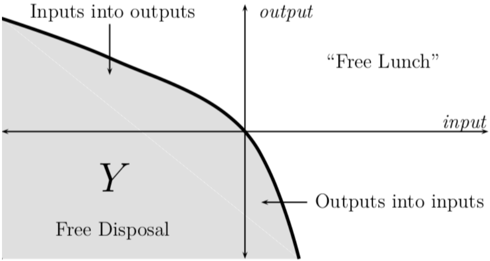
- Idea: *Professor Nash has formally introduced the notion of an equilibrium point for a game...*
- Goal: introduce an $(m + n + 1)$-player game with
  - m consumers maximize utility, and n firms maximize profits
  - One fictitious Walrasian chooses prices to maximize the value of net excess demand $\Rightarrow$ reduce prices of goods in excess supply and raise the prices of goods in excess demand

Arrow and Debreu's Damn Clever Formulation of Firms

- A firm is a subset $Y \subset \mathbb{R}^L$, given $L \geq 2$ goods.
  - Assume m firms $\Rightarrow$ exchange economy is $m = 0$ special case
    - A firm transforms inputs into outputs
    - $y_k \in Y$ is outputs if $y_k > 0$ and input if $y_k < 0$

    $\Rightarrow$ firm profits are $p \cdot y$, the dot product of prices and quantities.

  Closed convex technology
    - *no free lunch* $\Rightarrow Y \cap \mathbb{R}^L_+ = \{0\} \Rightarrow 0 \in Y$ *(free exit)*
    - *free disposal* $\Rightarrow Y \supset \mathbb{R}^L_-$
    - $Y \subseteq \mathbb{R}^\ell$ is closed and convex



Formal Model of Capitalism

- Let consumer $i = 1, \dots n$ own a share $\theta_{ij} \geq 0$ of the profits of each firm $j = 1, \dots, m$
- A **competitive equilibrium** of a private ownership economy

$$\left(\{Y^j\}_{j=1}^m; \{X^i, u^i, \overline{\mathbf{x}}^i, \theta_{i1}, \theta_{i2}, ..., \theta_{im}\}_{i=1}^n\right)$$

is an allocation $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n\ell} \times \mathbb{R}^{m\ell}$ and a price $\mathbf{p} \in \mathbb{R}^\ell$ so that

  - $\forall j$: $\mathbf{y}^j \in Y^j$ maximizes profits, namely $\mathbf{p} \cdot \hat{\mathbf{y}}^j \leq \mathbf{p} \cdot \mathbf{y}^j$ $\forall \hat{\mathbf{y}}^j \in Y^j$
  - $\forall i$: $\mathbf{x}^i \in X^i$ maximizes utility u^i in the budget set:

$$B^i(\mathbf{p}) = \{\mathbf{x}^i \in X^i : \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \overline{\mathbf{x}}^i + \sum_{j=1}^m \theta_{ij} \mathbf{p} \cdot \mathbf{y}^j\}$$

  - Markets clear, namely the excess demand vector is nonpositive:

$$\mathbf{z} = D(p) - \overline{x} - S(p) \equiv \sum_{i=1}^n \mathbf{x}^i - \sum_{i=1}^n \overline{\mathbf{x}}^i - \sum_{j=1}^m \mathbf{y}^j \leq 0$$

and if $z_k < 0$, then $p_k = 0$

Existence Theorem

Theorem (Arrow and Debreu, 1954)

Assume every consumer $i = 1, \ldots, n$ has a continuous, nonsatiated and strictly quasiconcave utility u_i, endowment $\bar{\mathbf{x}}^i \in \mathbb{R}^\ell_+$, and dividend shares (θ_{ij}). Assume firms $j = 1, \ldots, m$ have closed and convex production technologies. A competitive equilibrium exists.

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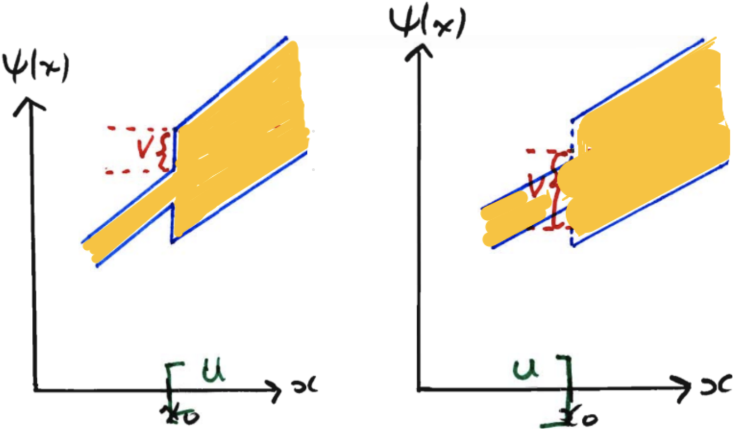
Corollary (Existence in General Exchange Economies)

Assume every consumer $i = 1, \ldots, n$ has a continuous, nonsatiated and strictly quasiconcave utility u_i, and an endowment $\bar{\mathbf{x}}^i \in \mathbb{R}^\ell_+$. A competitive equilibrium exists.

- Nash only proved existence of mixed strategy equilibrium for games with finitely many actions
- Arrow and Debreu generalized Nash's existence for games with quasiconcave and continuous payoff functions of action in a compact convex space.
- They parallel Nash's proof.

Upper Hemi Continuity

- Upper hemicontinuity precludes sudden implosions of a correspondence function $\psi(x)$: left but not right graph
- Lower hemicontinuity precludes sudden implosions of a correspondence function $\psi(x)$: right but not left graph
  - Cool aside: game theory refinements, like the Intuitive Criterion, essentially argue LHC ought not fail



Kakutani Fixed Point Theorem

Theorem (Kakutani Fixed Point Theorem, 1944)

Let $S \subset \mathbb{R}^n$ be non-empty, compact and convex. Let ϕ be a set-valued function on S with a closed graph and $\phi(x) \neq \varnothing$ and convex-valued for all $x \in S$. Then ϕ has a "fixed point" $x \in \phi(x)$

- "Sir, tell us about the Kakutani FPT." Him: "What's that?"

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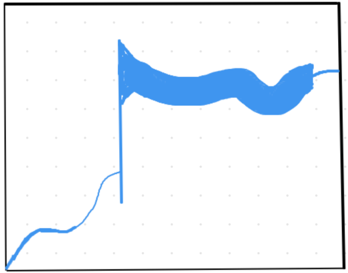
- Kakutani used the von Neumann Approximation lemma to draw a continuous function very close to any closed graph.



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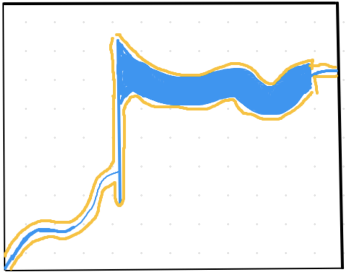

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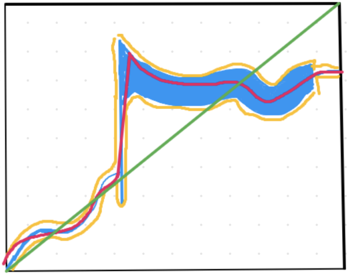
- Each such function has a fixed point, by Brouwer. Take limits.



Is the domain compact and convex?

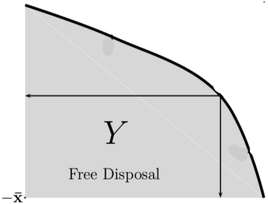
- A competitive equilibrium is a triple (x, y, p) such that:
  - Given p, consumers and firms choose x, y.
  - Given x, y, auctioneer chooses price p (& it clears the market).
- Price domain compact: Use $P = \{p \in \mathbb{R}^L_+ | p_1 + \cdots + p_L = 1\}$.
- Markets clear $\Rightarrow$ excess demand $\mathbf{z} = D(p) - S(p) - \bar{\mathbf{x}} \leq 0$.

$\Rightarrow$ $S(p) = D(p) - \mathbf{z} - \bar{\mathbf{x}} \geq -\bar{\mathbf{x}}$, since $D(p) \geq 0 \geq S(p)$.

$\Rightarrow$ Since Y_j is convex, and $Y_j \cap \mathbb{R}^L_+ = \{0\}$, it is bounded above

$\Rightarrow$ So every firm's optimization is one a compact domain Y_j.

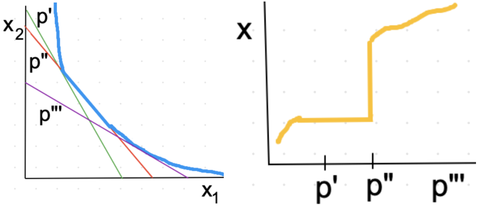
- Likewise, $D(p) \leq S(p) + \bar{\mathbf{x}}$ is then uniformly bounded above



Arrow-Debreu (1954) Proof Sketch Theorem

$$\begin{aligned} \phi : (x, y, p) &\rightarrow \text{all utility maximizers } x \\ &\rightarrow \text{all profit maximizers } y \\ &\rightarrow \text{all maximizers } p \text{ of net excess demand value} \end{aligned}$$

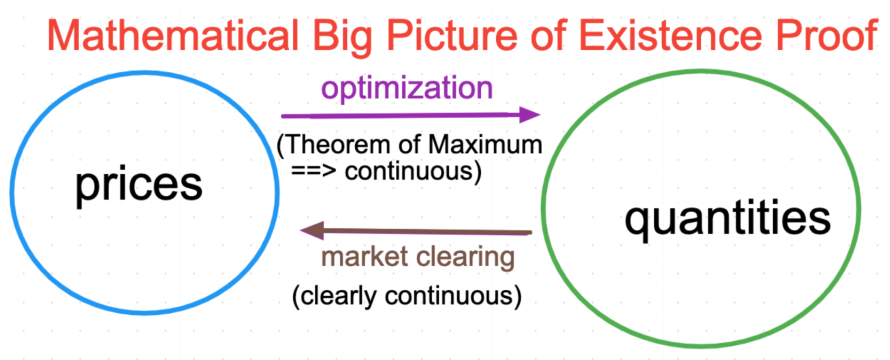
- Theorem of the Maximum $\Rightarrow \phi(x, y, p)$ has a closed graph
- Continuous u & compact domain $\Rightarrow \phi(x, y, p) \neq \varnothing$
- Convex preferences and technologies $\Rightarrow \phi(x, y, p)$ convex



$\Rightarrow$ correspondence $\phi(x, y, p)$ is uhc and nonempty/convex-valued

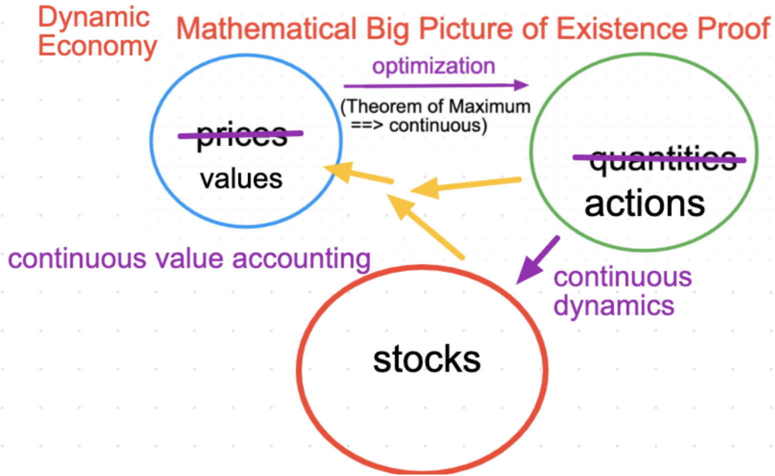
- By Kakutani's Fixed Point Theorem, $\exists(x, y, p) \in \phi(x, y, p)$

Insights for Iterative Computer Equilibrium Computation

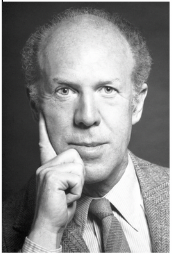


Equilibrium Computation for Dynamical Economic Systems

(Foretaste of my Advanced Theory Search MiniCourse)



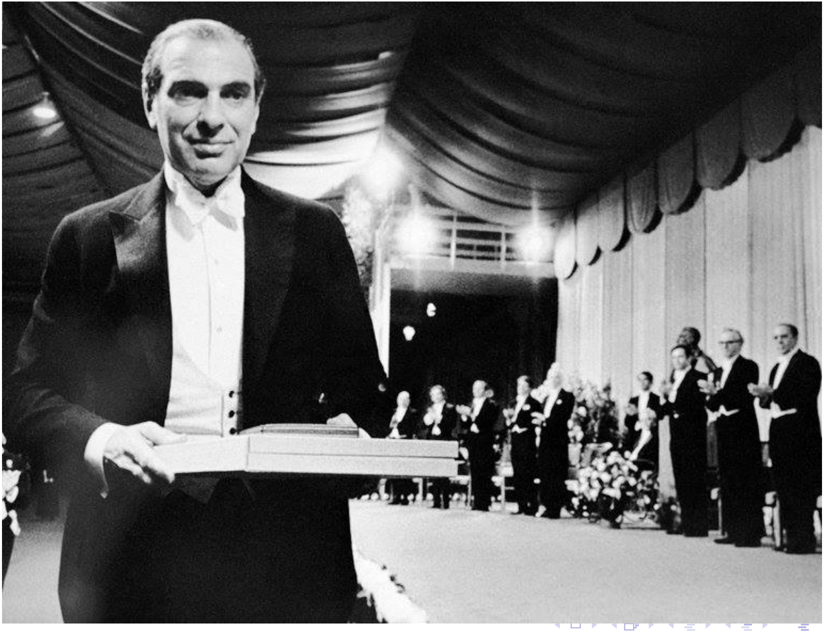
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1983



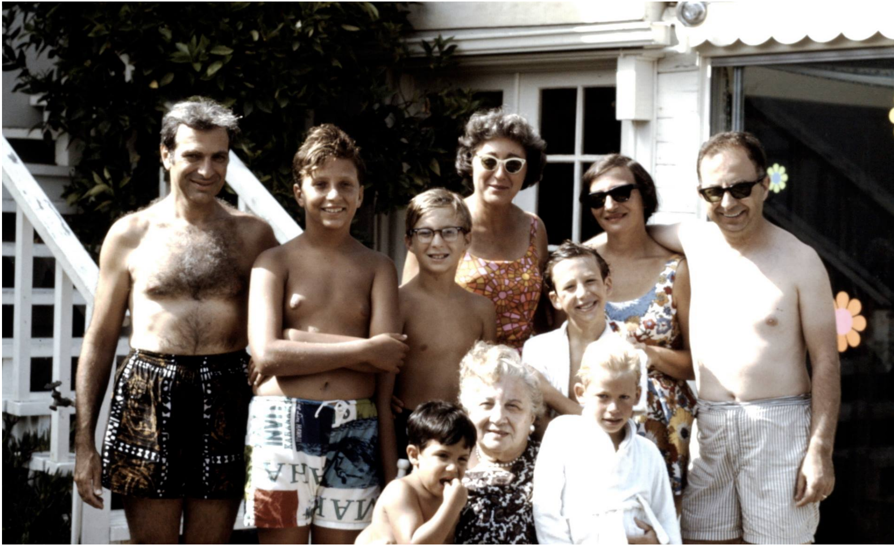
Gerard Debreu

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1983 was awarded to Gerard Debreu "for having incorporated new analytical methods into economic theory and for his rigorous reformulation of the theory of general equilibrium."

Ken Arrow



Ken Arrow, young Larry Summers, et al



Mackenzie's Parallel Related Paper

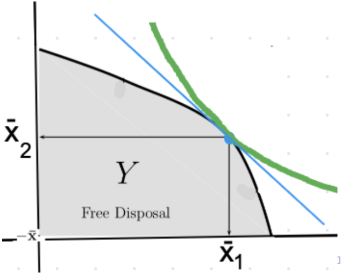
- Mackenzie (1954) also had the idea to use Kakutani.
  - But he did not model consumers
  - He did not have the parsimonious description of firms.
- He did not cite Arrow-Debreu (1954), not did they cite him!!
- The editor Strotz wrote him in 1953: *"I have given up. Letters have gone to both referees requesting the return of your manuscript to this office right away. I hope to God I can have better luck with the next people. I don't know whether this is a matter of concern to you, but let me assure you that it is my intention not to publish the paper by Arrow and Debreu (which has also been submitted) before the publication of your paper (if both are found acceptable). I think this would only be fair to you.'*
- Mackenzie founded the Rochester economics department

Welfare Theorem with Production

Theorem (Efficiency $\Leftrightarrow$ Competition)

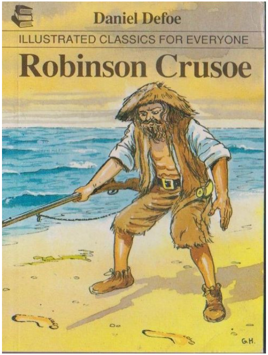
(a) If $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ *is a competitive equilibrium and preferences are not locally satiated, then* $(\mathbf{x}, \mathbf{y})$ *is an efficient allocation.*
(b) Conversely, assume monotonic and convex preferences, and closed convex technologies. If $(\mathbf{x}, \mathbf{y})$ *is socially efficient, then* $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ *is a competitive equilibrium, for some prices* $\mathbf{p}$*, endowments* $\bar{\mathbf{x}}$*, and ownership shares* θ*.*

- Idea: Choose the origin of Y to be the endowment vector $\bar{\mathbf{x}}$, as that corresponds to the zero production exchange economy



Robinson Crusoe Economies

- $M = 1$ firms, $N = 1$ consumers
- Karl Marx made this metaphor famous in *Das Kapital*



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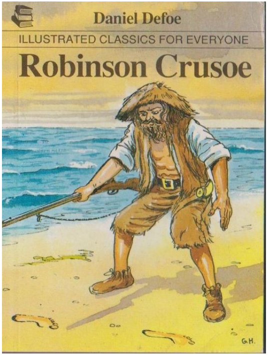
Karl Marx's London memorial vandalised for second time

The words 'doctrine of hate' and 'architect of genocide' were painted on Highgate cemetery memorial

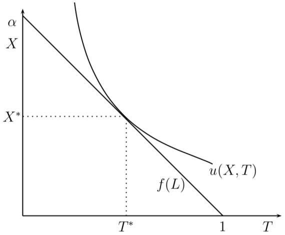


Example: Constant Return to Scale Technology

- $L = 2$ goods, produced by $M = 1$ firm, for $N = 1$ consumer
  - Technology: fish or fowl $f(L) = \alpha L$, where L is labor.
  - Preferences: $u(X, T) = X^{\alpha} T^{1-\alpha}$, where T is leisure.
  - Endowment: one unit of time $1 = \bar{T} = L + T$
  - Need not specify firm ownership shares: it earns no profits
- Solution: Let T be numéraire, and p the relative price of X
- Crusoe Inc. maximizes $pf(L) - L$ iff $p\alpha = 1$.
- As endowment income is $\bar{T} = 1$, Cobb Douglas demands are:

$$X = \alpha/p = \alpha^2 \quad \text{and} \quad T = 1 - \alpha$$

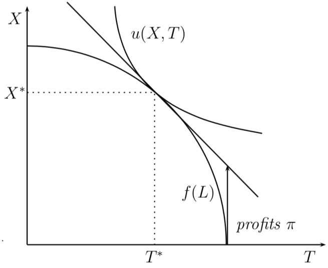
- Robinson works $L = \alpha$ hours (producing $X = \alpha^2$)
- Crusoe, Inc. produces $X = \alpha^2$ from labor $L = \alpha$



Example: Technology with Diminishing Returns

- New technology: fish $f(L) = \sqrt{L}$
  - Crusoe Inc. maximizes $p\sqrt{L} - L$.
  - The FOC is $L = p^2/4$
  - $\Rightarrow$ Production is $X = p/2$
  - Profits $pX - L$ are as depicted:

$$\pi = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4} > 0$$



  - Robinson's income is his endowment value and profits: $1 + \pi$.

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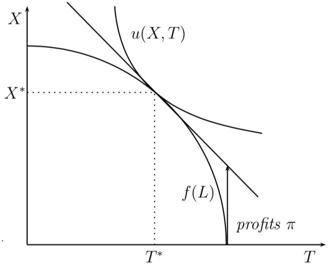

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  - Robinson's income is his endowment value and profits: $1 + \pi$.
- Let's choose to clear the labor-leisure market:
  - Leisure demand (using $L = p^2/4$) is

$$T = (1-\alpha)(1 + p^2/4) = (1-\alpha)(1+L)$$

  - $T + L = 1 \Rightarrow 1 - L = (1-\alpha)(1+L) \Rightarrow L^* = \alpha/(2-\alpha)$
  - $T^* = 1 - L^* = 2(1-\alpha)/(2-\alpha)$.
  - Supply $X^* = \sqrt{\alpha/(2-\alpha)} \Rightarrow p = 2X^* = 2\sqrt{\alpha/(2-\alpha)}$
- Quicker to find competitive equilibrium than social optimum