An Economic Theory Masterclass

Part VI: General Equilibrium in Exchange Economies

Lones Smith

April 28, 2020

General Equilibrium and the Gold Rush

- *Partial equilibrium*: one-market world, often with quasi-linear utility — where "money" subsumes all other goods
- *General equilibrium* multi-market world: Markets interact!
- Sam Brannan
  - Richest man in California after Gold Rush of 1849
  - "Gold! Gold on the American River!"
  - Brennan owned only store between San Francisco & gold fields
  - paid 20 cents each for the pans, then sold them for $15 each

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CALIFORNIAN

BY B. R. BUCKELEW.

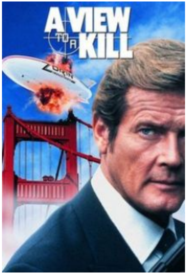
SAN FRANCISCO, MARCH 15, 1848.

GOLD MINE FOUND.—In the newly made raceway of the Saw Mill recently erected by Captain Sutter, on the Ameri-

California Gold Rush Map

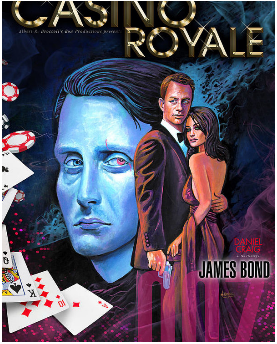
General Equilibrium in the Movies

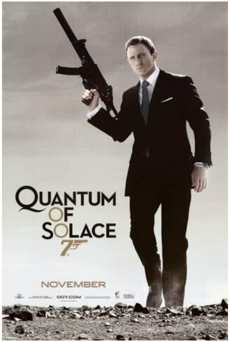
- Goldfinger: evil mastermind tried to irradiate Fort Knox gold $\Rightarrow$ his own gold would $\uparrow$ in value
- Die Hard with a Vengeance: same plan for the gold in NY Fed.
- A View to a Kill: bad guy wants to trigger earthquake to destroy Silicon Valley, and then monopolize microchip market.



General Equilibrium in the Movies

- Casino Royale: bad guy shorts airline stocks, while planning to destroy a luxury jetliner on its maiden voyage.
- Quantum of Solace: bad guy wants to dam Bolivia's fresh water supply to create a Bolivian water monopoly (total joke).





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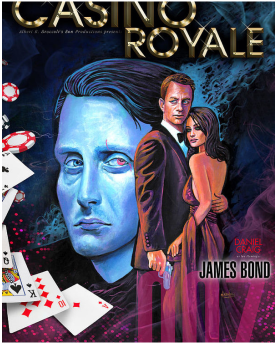

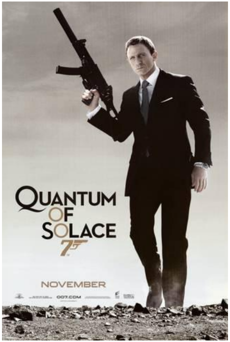

General Equilibrium in the World: Corona Virus!

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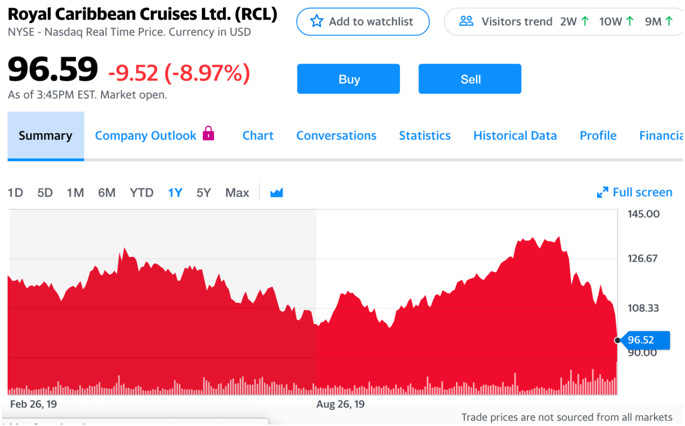
What are the biggest cruise line stocks?

The three biggest cruise line stocks are **Royal Caribbean** (NYSE:RCL), **Norwegian Cruise Line** (NYSE:NCLH), and market leader **Carnival** (NYSE:CCL). Let's take a closer look at each of these companies.



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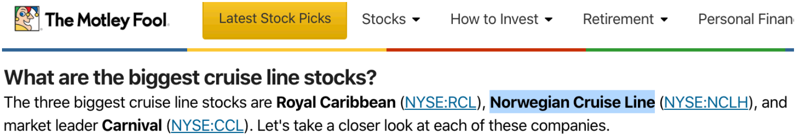
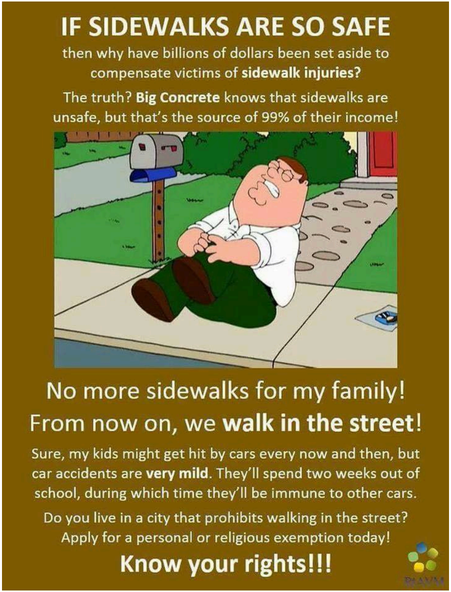


Conspiracy Theories with a GE Flavor



General Equilibrium Notation

- *Exchange economy* $\mathcal{E} = (\{u^i\}, \bar{\mathbf{x}})$.
  - $L \geq 2$ goods $\ell \in \{1, 2, \ldots, L\}$
  - $n \geq 2$ traders $i \in \{1, 2, \ldots, n\}$
  - Consumer i has *endowment* $\bar{\mathbf{x}}^i = (\bar{x}_1^i, \bar{x}_2^i, \ldots, \bar{x}_L^i)' \in \mathbb{R}_+^L$
  - A goods *allocation* is a matrix $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \in \mathbb{R}_+^{nL}$.
  - Trader i has utility $u^i : \mathbb{R}_+^L \to \mathbb{R}$.
  - Trader i's income is the market value $\mathbf{p} \cdot \bar{\mathbf{x}}^i$ of his endowment
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- Prices $\mathbf{p} = (p_1, p_2, \ldots, p_L) \in \mathbb{R}^L_+$ in some unit of account
  - Jevons (1875): Money is a *store of value*, *unit of account*, and *medium of exchange*, *standard of deferred payment*
  - Here, it is only a unit of account, and so $\exists$ degree of freedom.
  - Each trader sells his endowment to the market, valued at the unit of account prices, and then buys his optimal bundle.
  - We assume that all transactions realize by time-0 contracts
  - Modern financial transactions, together with bankruptcy laws, violate this idyllic world (hence the 2008 Financial Crisis)

General Equilibrium

- A trader's wealth is the market value of his endowment
- *Budget set* $\mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p}) = \{\mathbf{x}^i \in \mathbb{R}^L_+ | \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \bar{\mathbf{x}}^i\}$
- Traders optimize, given prices: Trader $i = 1, 2 \ldots, n$ solves:

$$\max u^i(\mathbf{x}^i) \text{ s.t. } \mathbf{x}^i \in \mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p})$$

- Allocation $\mathbf{x} \in \mathbb{R}^{nL}_+$ is feasible for $\mathcal{E}$ if $\sum_{i=1}^n x^i_\ell \leq \sum_{i=1}^n \bar{x}^i_\ell \; \forall \ell$
- free disposal of goods $\Rightarrow$ weak inequality
- We say that markets clear in this case

- A feasible allocation $\mathbf{x}$ is (Pareto) efficient

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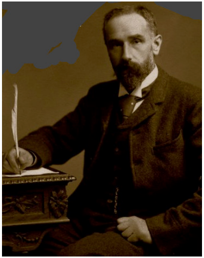
- We say that markets clear in this case
- A competitive equilibrium $(\mathbf{x}, \mathbf{p})$ of $\mathcal{E}$ is a feasible allocation $\mathbf{x}$ such that all traders optimize, given prices
- A feasible allocation $\mathbf{x}$ is socially optimal if $\nexists$ feasible allocation $\widehat{\mathbf{x}}$ with
  - no one worse off: $u^i(\widehat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for all $i = 1, \ldots, n$,
  - some trader j strictly better off: $u^j(\widehat{\mathbf{x}}^j) > u^j(\mathbf{x}^j)$ for some j
  - analogous to a blocking pair for stability

Edgeworth Boxes for $n = 2$ Traders

- Francis Ysidro Edgeworth
  - *Mathematical Psychics* (1881)
  - introduced indifference curves
  - founding editor: *Economic Journal*



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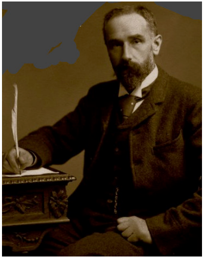

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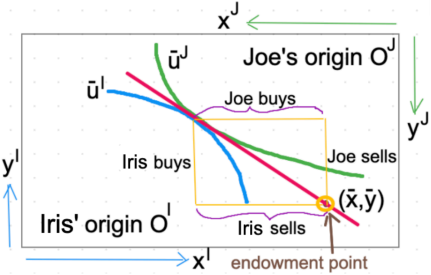
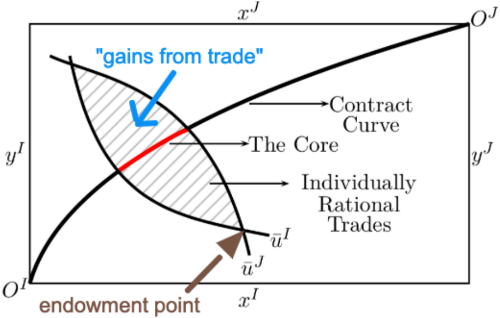
- Assume Trader Iris and Trader Joe trade goods x and y
- Here, they trade to an efficient allocation from endowment
- Assume an interior solution with smooth preferences.
- Equate marginal rate of substitution and price ratio: $\frac{u_x}{u_y} = \frac{p_x}{p_y}$



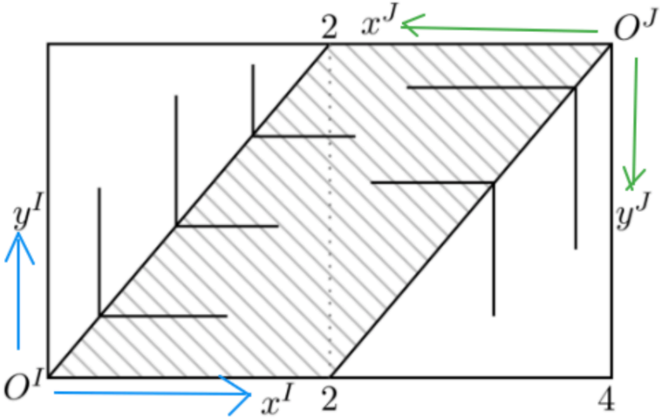
Competitive Equilibrium and Social Efficiency

- Contract curve: the locus of socially efficient allocations $\mathbf{x}$
- Individually rational (IR) allocation $\mathbf{x}$ obeys $u^i(\mathbf{x}^i) \geq u^i(\bar{\mathbf{x}}^i)\ \forall i$
- The core $\mathbf{x}$ is the IR portion of the contract curve
- A competitive equilibrium for $\mathcal{E}$ is a pair $(\mathbf{x}, \mathbf{p})$ s.t. $\mathbf{x}$ is feasible, and optimal for traders, given prices $\mathbf{p}$ (via budget sets)
- A competitive equilibrium is in the core because $\mathbf{x}^i \in \mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p})$
- Divergent marginal rates of substitution $\Rightarrow$ gains from trade
- In exchange economies, trade occurs due to differences in preferences and/or endowments



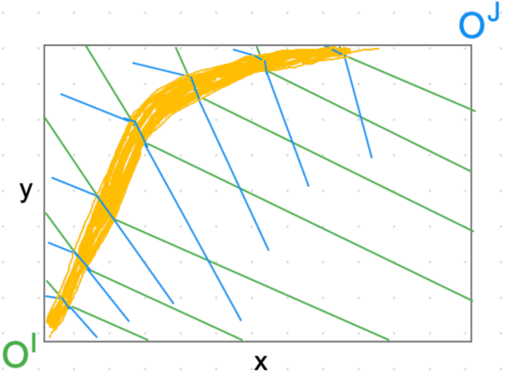
Social Efficiency with Perfect Complements

- Utility functions $u^I(x, y) = \min\{x, y\}$ & $u^J(x, y) = \min\{x, y\}$
- Endowments $\bar{x}^I = \bar{x}^J = 2$ and $\bar{y}^I = \bar{y}^J = 1$
- The contract curve is the shaded region, since preferences are not strictly monotone



Social Efficiency with Imperfect Complements

- Increasing preferences that with at least one party strictly convex is needed to ensure a contract curve and not region

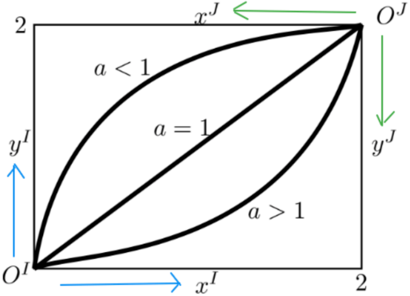


Social Efficiency with Smooth Strictly Convex Preferences

- Cobb-Douglas utility functions $u^I(x,y) = x^\alpha y$ and $u^J(x,y) = xy$
- Endowments $\bar{x}^I = \bar{x}^J = \bar{y}^I = \bar{y}^J = 1$.
- Contract curve: $MRS^I_{x,y} = MRS^J_{x,y}$

$$\alpha y^I/x^I = y^J/x^J \Rightarrow \alpha y^I(2-x^I) = x^I(2-y^I) \Rightarrow y_1 = \frac{2x^I}{\alpha(2-x^I)+x^I}$$

- Contract curve is above or below the diagonal as $\alpha \lesseqgtr 1$.
- As $\alpha \uparrow$, Iris values good x more, and he efficiently gets more x



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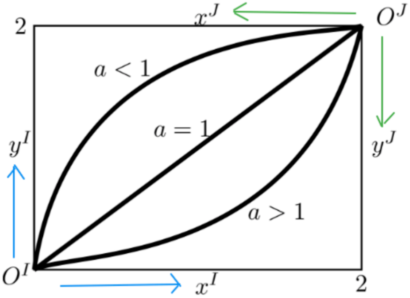

Competitive Equilibria are Socially Efficient

- Since trade is win-win, it makes sense that self-interest is good
- Adam Smith (1723–90)
  - 1759: "Theory of Moral Sentiments" explored empathy
  - 1776: "Inquiry into the Nature and Causes of the Wealth of Nations" explored the social benefits of self-interest
  - "It's not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard for their own interest"
- Smith attacked win-lose mercantilism: "We must always take heed that we buy no more from strangers than we sell them, for so should we impoverish ourselves and enrich them" (1549)

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- "When a country is losing many billions of dollars on trade with virtually every country it does business with, trade wars are good, and easy to win" — Trump (2018)

The First Welfare Theorem

Proposition (Arrow (1951) & Debreu (1951), 1940s folk result)

If $(\mathbf{p}, \mathbf{x})$ *is a competitive equilibrium of* $\mathcal{E}$*, and preferences are locally non-satiated, then* $\mathbf{x}$ *is socially efficient.*

- Intuition: If another allocation is better for all and strictly better for Joe, then it costs everyone at least as much (at the market price), and Joe strictly more. It thus costs more than the old allocation, and so more than the endowment.

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- Claim 1: $\mathbf{p} \cdot \widehat{\mathbf{x}}^i \geq \mathbf{p} \cdot \mathbf{x}^i$ for all i
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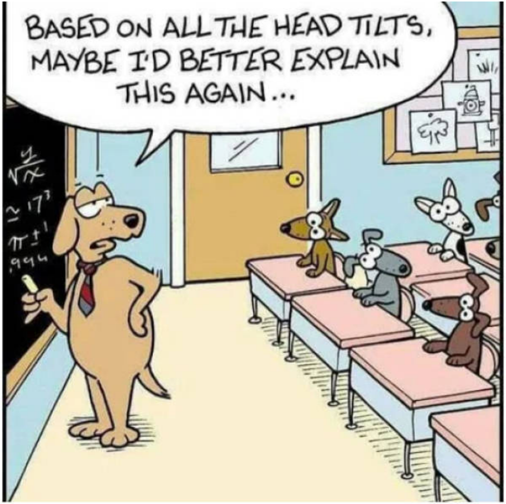
- Claim 2: $\mathbf{p} \cdot \widehat{\mathbf{x}}^j > \mathbf{p} \cdot \mathbf{x}^j$
  - Proof: This follows since $\mathbf{x}^j$ is a utility maximizer for trader j
- Adding yields $\mathbf{p} \cdot \sum_{i=1}^n \widehat{\mathbf{x}}^i > \mathbf{p} \cdot \sum_{i=1}^n \mathbf{x}^i$.
- Since $\mathbf{p} \geq 0$, this contradicts $\sum_{i=1}^n \widehat{\mathbf{x}}^i \leq \sum_{i=1}^n \mathbf{x}^i$. □

The First Welfare Theorem

- The proof logic used revealed preference theory.

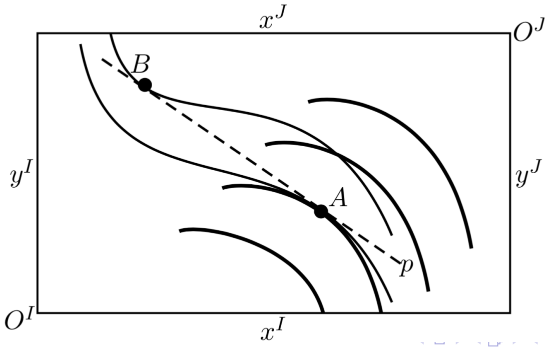


The Second Welfare Theorem

Proposition (Second Welfare Theorem)

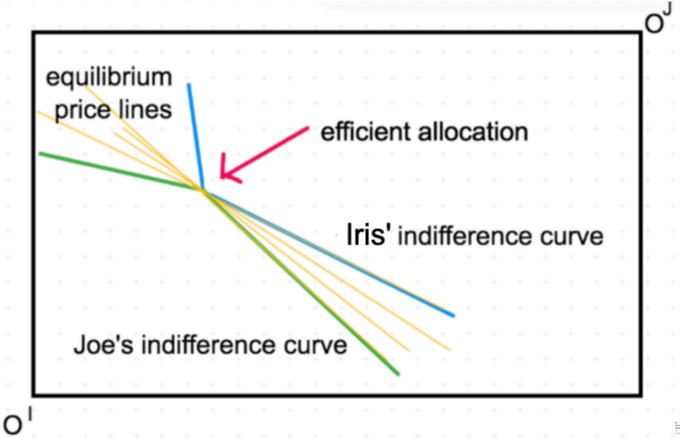
Assume that consumers have continuous, monotonic, and quasiconcave utility functions. If $\mathbf{x} \in \mathbb{R}_+^{\ell n}$ *is an efficient allocation, then there exists a price* $\mathbf{p} \in \mathbb{R}_+^{\ell}$ *such that* $(\mathbf{x}, \mathbf{p})$ *is competitive equilibrium of* $\mathcal{E} = (\{u^i\}, \bar{\mathbf{x}})$.

- Why do we need convex preferences?



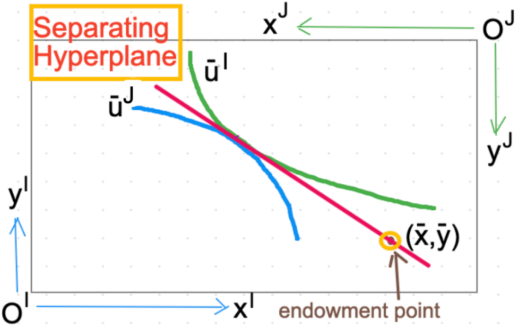
The Second Welfare Theorem

- As in a double auction, equilibrium prices need not be unique.
- But here, nonuniqueness is harder to secure, given the intensive margin
- Question: When are competitive prices unique?
  Answer: At least one consumer has smooth convex preferences



The Second Welfare Theorem: Proof Idea

- The (Minkowski) Separating Hyperplane Theorem proof intuitively works for two traders
- Minkowski taught Einstein, and reformulated his 1905 special relativity as spacetime in 1908 (but then sadly died 1909 at age 44 of appendicitis)
- The Separating hyperplane Theorem easily works for $n = 2$:



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$$\max_{\mathbf{z}} u^j(\mathbf{z}^j) \quad \text{s.t.} \quad u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i) \text{ for all } i \neq j$$

$$\textstyle\sum_i \mathbf{z}^i_\ell \leq \sum_i \mathbf{x}^i_\ell \text{ for } \ell = 1, \ldots, L \text{ (feasibility)}$$

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- As $\mathbf{x}$ is efficient, this maximum is realized at $\mathbf{z} = \mathbf{x}$.
  - objective function u^j is quasiconcave
  - constraint set is nonempty if no one is near a subsistence utility level (regularity condition on utility functions)
  - constraint set is convex if $u^j(\mathbf{z}^i)$ is quasiconcave

$\Rightarrow$ Lagrangian has saddle point property for some multipliers $\lambda^i, \mathbf{p}$

$$\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) = u^j(\mathbf{z}^j) + \sum_{i \neq j} \lambda^j_i [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + \sum_\ell p^j_\ell \left[\sum_i x^i_\ell - \sum_i z^i_\ell\right]$$

The Second Welfare Theorem: Proof by Lones Smith

- Optimality in z^i_ℓ and z^j_ℓ for all traders $i \neq j$ yield the FOC's:

$$\begin{aligned}
\frac{\partial}{\partial z^j_\ell}\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) &= \frac{\partial}{\partial z^j_\ell}u^j(z^j) - p^j_\ell = 0 \\
\frac{\partial}{\partial z^i_\ell}\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) &= \lambda^j_i \frac{\partial}{\partial z^i_\ell}u^i(z^i) - p^j_\ell = 0
\end{aligned}$$

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$$\frac{\partial}{\partial z^i_\ell}\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) = \lambda^j_i \frac{\partial}{\partial z^i_\ell} u^i(z^i) - p^j_\ell = 0$$

- Equate p^j_ℓ for traders $i \neq j$ is:

$$\lambda^j_i = \frac{\partial}{\partial z^j_\ell} u^j(z^j) \Big/ \frac{\partial}{\partial z^i_\ell} u^i(z^i)$$

- Equate Planner's MRS between any traders i, j across goods ℓ

$$\frac{\partial}{\partial z^i_{\ell_1}} u^i(z^i) \Big/ \frac{\partial}{\partial z^j_{\ell_1}} u^j(z^j) = \frac{\partial}{\partial z^i_{\ell_2}} u^i(z^i) \Big/ \frac{\partial}{\partial z^j_{\ell_2}} u^j(z^j)$$

$\Rightarrow$ The price ratio p_{ℓ_1}/p_{ℓ_2} is the same for any two traders i, j:

$$p^i_{\ell_1}/p^i_{\ell_2} = p^j_{\ell_1}/p^j_{\ell_2}$$

$\Rightarrow$ Multipliers are $\mathbf{p}^j = c_j \mathbf{p}$, some $c_j > 0$, in Lagrangian for all j

The Second Welfare Theorem: Proof by Lones Smith

$$\Rightarrow \mathcal{L}^j(\mathbf{z}, \mathbf{p}, \lambda^j) = u^j(\mathbf{z}^j) + \sum_{i \neq j} \lambda_i^j [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + c_j \mathbf{p} \cdot [\mathbf{x}^j - \mathbf{z}^j]$$

- Fixing the optima $\mathbf{z}^i = \mathbf{x}^i$ for all $i \neq j$, the saddle point property of $\mathcal{L}^j$ gives $\mathcal{L}^j(\mathbf{z}, \mathbf{p}, \lambda^j) \leq \mathcal{L}^j(\mathbf{x}, \mathbf{p}, \lambda^j)$ for all $\mathbf{z}^j$:

$$u^j(\mathbf{z}^j) + \sum_{i \neq j} \lambda_i^j [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + c_j \mathbf{p} \cdot [\mathbf{x}^j - \mathbf{z}^j] \leq u^j(\mathbf{x}^j)$$

  Since $u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i)$, we must have

$$u^j(\mathbf{z}^j) + c_j \mathbf{p} \cdot [\mathbf{x}^j - \mathbf{z}^j] \leq u^j(\mathbf{x}^j)$$

$\Rightarrow$ If $u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j)$ then $\mathbf{p} \cdot \mathbf{z}^j > \mathbf{p} \cdot \mathbf{x}^j$. At price $\mathbf{p}$, no trader j can afford any bundle $\mathbf{z}^j$ with a higher utility than $\mathbf{x}^j$

$\Rightarrow$ $(\mathbf{x}, \mathbf{p})$ is competitive equilibrium $\square$

Prices as Shadow Values

Corollary

The price of any good is its social shadow value.

Prices as Shadow Values

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Excess Demand Functions

- Strictly convex preferences $\Rightarrow$ unique demands $x^i_\ell(p)$
- Trader i's excess demand (net demand): $ED^i_\ell(\mathbf{p}) = x^i_\ell(p) - \bar{x}^i_\ell$
- The market excess demand for x_ℓ is $ED_\ell(\mathbf{p}) = \sum_{i=1}^n ED^i_\ell(\mathbf{p})$
- Markets clear in a competitive eq $(\mathbf{x}(\mathbf{p}), \mathbf{p})$: $ED_\ell(\mathbf{p}) = 0 \ \forall \ell$

Lemma (Walras Law)

If traders consume their entire income at allocation $\mathbf{x}(p)$, *then the market value of net excess demand vanishes:* $\sum_{\ell=1}^L p_\ell ED_\ell(\mathbf{p}) = 0$.

- *Proof*: Trader i's budget constraint $\mathbf{p} \cdot \mathbf{x^i}(\mathbf{p}) \equiv \mathbf{p} \cdot \bar{\mathbf{x}}^i$:

$$\sum_{\ell=1}^L p_\ell ED^i_\ell \equiv \sum_{\ell=1}^L p_\ell [x^i_\ell(p) - \bar{x}^i_\ell] \equiv 0 \qquad \square$$

$\Rightarrow$ By Walras, it suffices that $L-1$ of L markets clear

- Since demand is homogeneous of degree zero in (income, prices), there is a degree of freedom in the prices. So we can:
  - pick a good as numeraire, the de facto currency, with unit price
  - or, insist that all prices sum to one

Existence Using Excess Demand Functions: $L = 2$ Goods

- So equilibrium amounts to $L - 1$ equations in $L - 1$ unknowns
- Assume $L = 2$ goods, x and y. Let x be the numeraire.

$\Rightarrow$ Measure the *price ratio* $p = p_y / p_x$ of y in units of x.

$\Rightarrow$ Equilibrium is one equation in one unknown: $ED_x(p) = 0$.

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Theorem (Existence)

Assume every trader i has strictly monotone and strictly convex preferences over x and y, and owns a positive endowment $(\bar{x}^i, \bar{y}^i)$. There exists a Walrasian stable competitive equilibrium $(\mathbf{x}, \mathbf{y}, p)$.

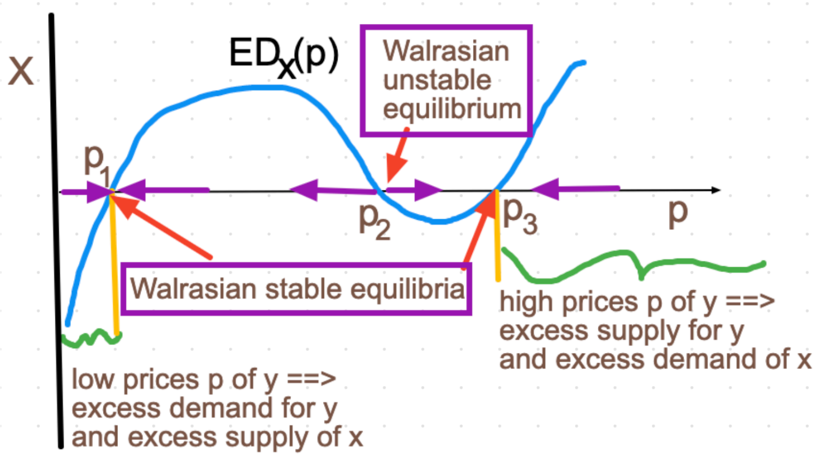
- *Proof:* Given strictly convex preferences, every trader i has a unique optimal consumption bundle $x^i(p)$ at any price $p > 0$.
- The optimizer set is upper hemicontinuous in p, by (Berge's) Theorem of the Maximum $\Rightarrow x^i(p)$ is thus continuous in p

$\Rightarrow$ Market excess demand $ED_x(p)$ is a continuous function

- Monotone preferences $\Rightarrow ED_x(0) < 0 < ED_x(\infty)$
- Intermediate Value Theorem $\Rightarrow ED_x(p) = 0$, for some $p > 0$.
- At least one zero of $ED_x(p) = 0$ is stable, crossing - to +

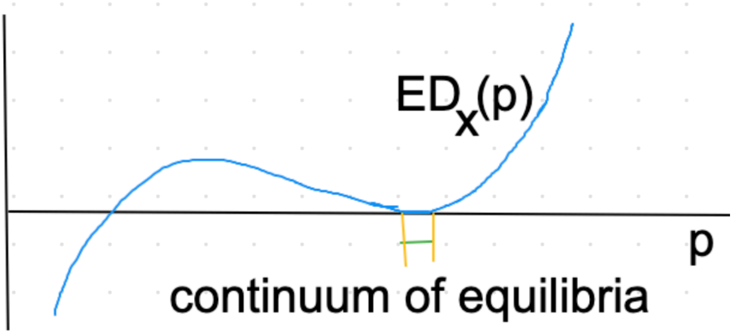
Existence and Stability of Competitive Equilibrium

- Monotone preferences $\Rightarrow ED_x(0) < 0 < ED_x(\infty)$



- Debreu-Mantel-Sonnenschein Theorem (1973/1974):
  Excess demand curves can be almost anything, by suitably specifying consumers and endowments!

Local Uniqueness of Equilibria



- It is truly rare that the excess demand curve sits on the price axis for an interval of prices, with a continuum of equilibria
- Debreu used Sard's Theorem in differential topology to formalize the sense that this is rare: For generic endowments, it cannot happen.
- This mathematical minutia was studied in mathematical economics

Existence with Cobb Douglas Preferences & $L = 2$ Goods

- Utilities: Iris $u^I(x, y) = x^\alpha y^{1-\alpha}$ and Joe $u^J(x, y) = x^\beta y^{1-\beta}$
- Endowments: $(\bar{x}^I, \bar{y}^I)$ and $(\bar{x}^J, \bar{y}^J)$.
- Incomes: $w^I = \bar{x}^I + p\bar{y}^I$ and $w^J = \bar{x}^J + p\bar{y}^J$
- Demands: Iris $x^I(p, w) = \alpha w^I$ and Joe $x^J(p, w) = \beta w^J$
- Market excess demand:

$$ED_x(p) = \left(\alpha w^I - \bar{x}^I\right) + \left(\beta w^J - \bar{x}^J\right)$$

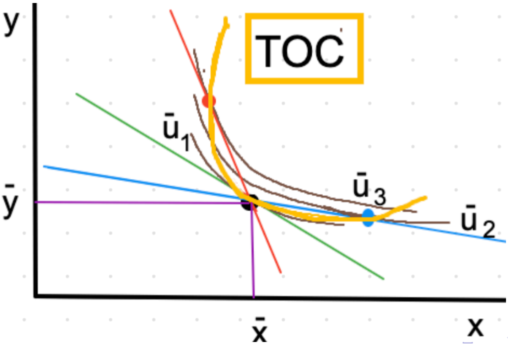
- It suffices to clear the x market: $ED_x(p) = 0$

$$p^* = \frac{\bar{x}^I(1-\alpha) + \bar{x}^J(1-\beta)}{\alpha\bar{y}^I + \beta\bar{y}^J}$$

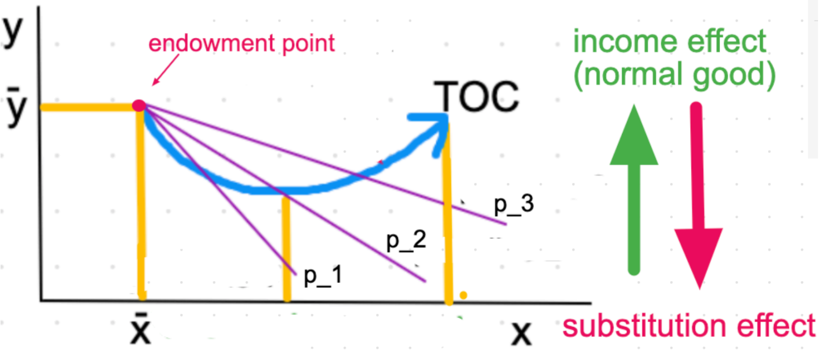
- The unique competitive equilibrium price p^*:
  - falls in α, β (greater love of x by either trader raises its price)
  - rises if $\bar{x}^I$ or $\bar{x}^J$ rises (gold discoveries led to inflation)

Trade Offer Curves

- The trade offer curve (TOC) plots optimal consumption allocations as prices vary, fixing endowments.
- In consumer theory, it is called the price-consumption curve
- Note: Trade theory overlaps heavily with consumer theory
- TOCs are the best reply graphs of general equilibrium theory
- With $L = 2$ goods, TOC is tangent to the indifference curve through the endowment, and "more curved" than it

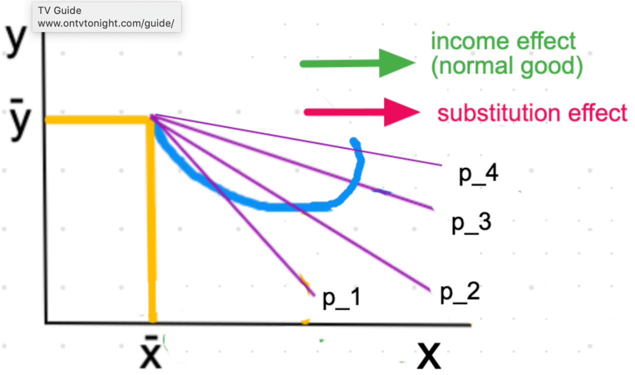


Trade Offer Curves and Substitutes and Complements



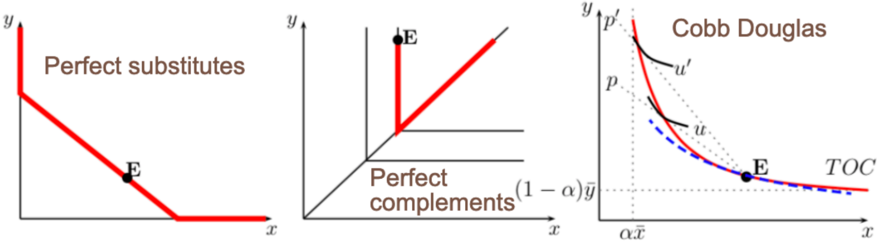
- The TOC can be nonmonotone, despite monotone preferences
- As the price p_i of y in terms of x rises $p_1 < p_2 < p_3$, the substitution effect pushes $y \downarrow$.
- As the price of y rises, "income" (value of endowment) rises, since Iris is a net supplier of y
  - If y is an normal good, then the TOC can fall or rise
  - If y is an inferior good, then the TOC is strictly falling

Backward Bending Trade Offer Curves Require Inferiority



- As the price of y in terms of x rises, the price of x in terms of y falls, and the substitution effect pushes $x\uparrow$
- As the price of y rises, and so the "income" rises
  - If x is an normal good, then the TOC pushes right
  - If x is an inferior good, the TOC can turn back (but need not)

Examples of Trade Offer Curves



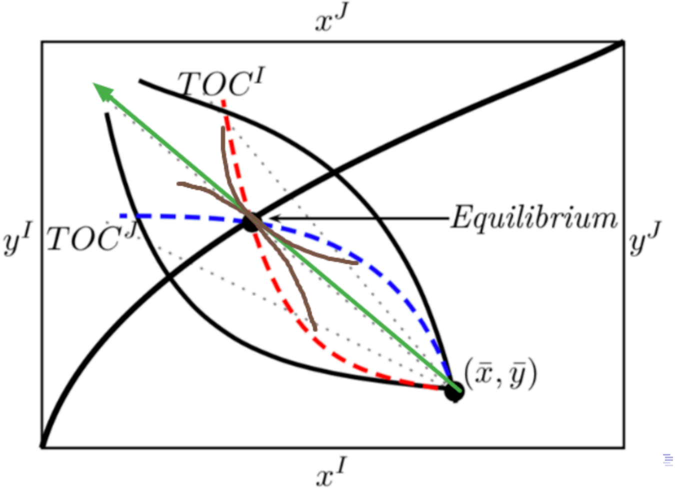
- Perfect substitutes, perfect complements, and Cobb Douglas
- Assume Cobb-Douglas $u(x, y) = x^{\alpha} y^{1-\alpha}$
- TOC is the locus of indifference curve tangencies to the price line through the endowments $(\bar{x}, \bar{y})$:

$$\frac{(1-\alpha)x}{\alpha y} = MRS = p = \frac{\bar{x} - x}{y - \bar{y}} \Rightarrow y(x) = \frac{(1-\alpha)\bar{y}x}{x - \alpha\bar{x}}$$

- The TOC starts at $y(\bar{x}) = \bar{y}$, for there is always a price for which it is efficient to consume the endowment.

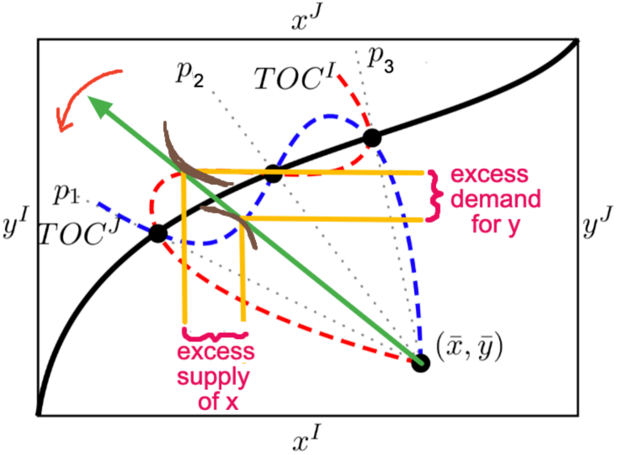
Uniqueness: Trade Offer Curves

- As best reply graphs, their intersection yields an equilibrium
- At a crossing of TOC^I and TOC^J, each trader optimally chooses that bundle, and so markets clear
- The absolute slope of the price line is p_x/p_y



Non-Uniqueness: Trade Offer Curves

- There are three equilibrium prices (of y): $p_1 > p_2 > p_3$
- Claim: p_1 and p_3 are Walrasian stable, and p_2 is not: If the price $p \in (p_2, p_1)$ (flatter price line), then the excess demand for y pushes up p; so the price line swings away from p_2 more
- With multiple equilibria, alternating equilibria are stable



Gross Substitutes and Uniqueness

- Recall that the TOC bends back with enough inferiority
- Demand has the gross substitutes property if an increase in price p_k raises the demand of every other good x_ℓ, for $\ell \neq k$.

Proposition (Uniqueness)

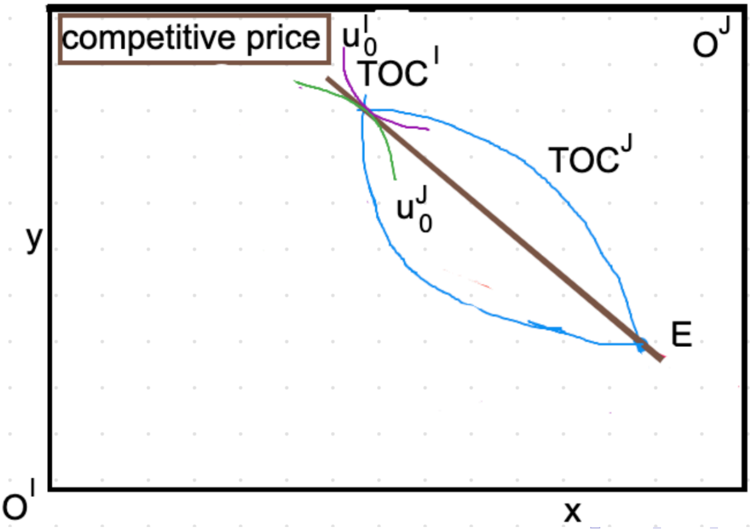
If the aggregate excess demand function satisfies gross substitutes, the economy has at most one Walrasian equilibrium

- Proof: We prove that $z(p)$ has at most one (normalized) root.
- Assume $z(p) = z(p') = 0$ for p and p' not linearly dependent.
- By homogeneity of degree zero, normalize the price vectors so that $p_\ell \geq p'_\ell$ for all ℓ, and $p_k = p'_k$ for some k
- Move from p' to p in $n-1$ steps, increasing p_ℓ for each $\ell \neq k$.
- At each step, the aggregate demand for good x_k strictly increases, so that $z_k(p) > z_k(p') = 0$. Contradiction. QED

★ Economics pro-tip: complementarity is the source of all multiplicity of equilibria, while substitutes leads to uniqueness. Supermodular games, eg., struggles with multiplicity

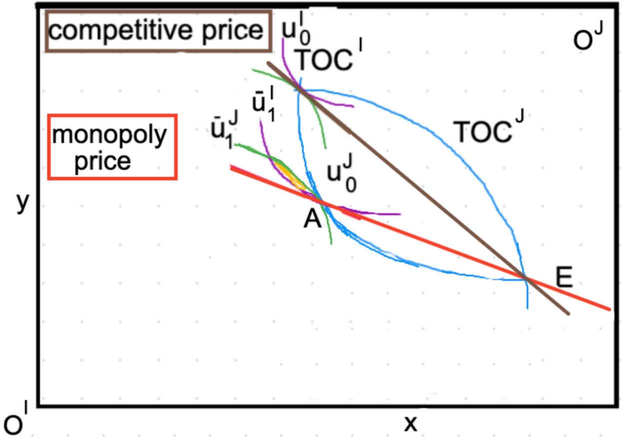
Monopoly in the Edgeworth Box

- Start with a competitive equilibrium with two goods, in which Joe sells y to the market and buys x



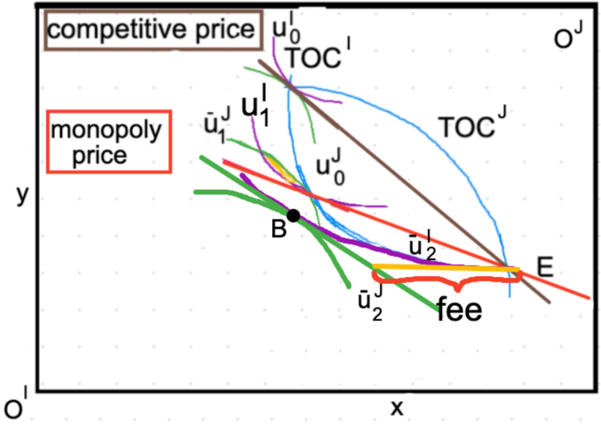
Monopoly Joe Replaces the Walrasian Auctioneer

- Joe seeks his highest indifference curve on Iris's TOC: $\bar{u}_1^J > \bar{u}_0^J$
- He sets a higher price ratio for y to x (now, the red price line)
- This monopoly is inefficient, or $\exists$ (orange) gains from trade:
  - Proof: The indifference curve $\bar{u}_1^J$ is tangent to TOC^I at A
  - The (red) price line slices through TOC^I, and thus through $\bar{u}_1^J$
  - But indifference curve $\bar{u}^I$ is tangent to the (red) price line at A



Monopoly Kingpin Joe Sets a Two Part Tariff

- Joe now secures an even higher utility $\bar{u}_2^J > \bar{u}_1^J$
- He sets a new price ratio for y to x, but now sets a trading fee
- Omnipotent monopoly is efficient: B is on the contract curve!



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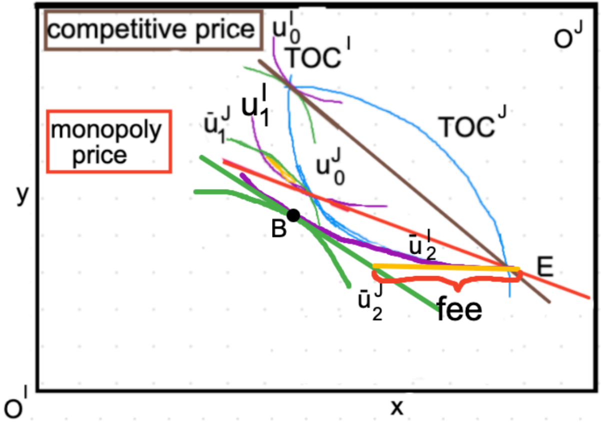
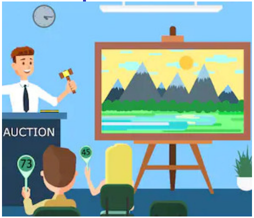


- EXAMPLE: $u^J(x, y) = x + y$ and $\bar{x}^J = 20$ and $\bar{y}^J = 0$.
- $u^I(x, y) = x(9 - x) + y$ and $\bar{x}^I = 0$ and $\bar{y}^I = 20$.

Beyond Markets: Cooperative Games and Core Theory

- We now return to a world with just an extensive margin, and develop a framework that subsumes markets with and without market power, as well as public economics
- Allow arbitrary coalitions of individuals to form, like unions, or god forbid, gangs. Is this the "law of the jungle" (Kipling)?
- These naturally arise in the networked world we have entered
- $N =$ set of all players (cardinality N too)
- A coalition is a group of players $S \subseteq N$ (grand coalition)
- A game with transferable payoffs associates to any coalition $S \subseteq N$ a value $v(S)$, where $v(\varnothing) = 0$
- A coalition S blocks a payoff vector $x \in \mathbb{R}^N$ if $\sum_{i \in S} x_i < v(S)$
- Pairwise matching model: $v(S) =$ highest sum of match values from pairing off members of S
- The core is all unblocked feasible payoffs x: $\sum_{i \in N} x^i = v(N)$
  - computer scientist Donald Gillies coined this in 1959

Core Theory: House Example



- Seller S values painting at 100, buyers B_1 & B_2 at 120 & 150

$$\Rightarrow V(B_1) = V(B_2) = V(B_1, B_2) = 0$$

$$V(S) = 100,\ V(B_1, S) = 120,\ V(B_2, S) = V(B_1, B_2, S) = 150$$

- SOLUTION:
- Buyers B_1 and B_2 earn payoffs (consumer surpluses) π_1 and π_2
- Payoffs for seller S: price p
- IR constraints: $\pi_1 \geq 0, \pi_2 \geq 0, p \geq 100$.
- Pairwise constraints: $p + \pi_1 \geq 120, p + \pi_2 \geq 150, \pi_1 + \pi_2 \geq 0$
- Grand coalition earns $V(B_1, B_2, S) = p + \pi_1 + \pi_2 = 150$.

$\Rightarrow$ $\pi_1 = 0$ and $120 \leq p \leq 150$ and $\pi_2 = 150 - p$.

Core Theory: Carrying a Table

- A table must be carried by ≥ 2 people to secure a gain of 50.
- IR constraints: $x_i \geq V(i) = 0$.
- Pairwise constraints:

$$\begin{aligned} x_1 + x_2 &\geq V(1,2) = 50 \\ x_2 + x_3 &\geq V(2,3) = 50 \\ x_1 + x_3 &\geq V(1,3) = 50 \end{aligned}$$

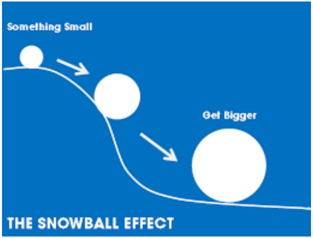
- Summing: $x_1 + x_2 + x_3 \geq 75 > 50 = V(1,2,3) \Rightarrow$ *empty core!*

Properties of Transferable Utility Cooperative Games

- *Monotone:* $S \subseteq T \Rightarrow v(S) \leq v(T)$
- *Supermodular:* $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\ \forall S, T$
- *Superadditive*: $v(S \cup T) \geq v(S) + v(T)$ when $S \cap T = \emptyset$.
- Supermodular $\Rightarrow$ superadditive, if $v(0) = 0$
- A supermodular game is also called a convex game, since:
- Shapley's Claim: v is supermodular if and only if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad \forall S \subseteq T \subseteq N \backslash \{i\}, \forall i \in N$$

  - A supermodular valuation implies increasing returns to size.
  - "Snowballing effect" emerges: *incentives for joining a coalition increase in its size* $\Rightarrow$ precludes table carrying example!



Convex Games

Lemma (Bondareva-Shapley Theorem)

A convex game has a non-empty core.

- Idea: convexity $\Rightarrow$ core constraints do not preclude feasibility.
- Player i is paid his marginal addition to the coalition $\{1, \ldots, i-1\}$, namely, $x_i = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$
- Claim: The payoff $x = (x_1, \ldots, x_N)$ is in the core, i.e. no coalition $A_k = \{i_1, \ldots, i_{k-1}\}$ blocks it, where $i_1 < i_2 < \cdots < i_k$

$$
\begin{aligned}
\sum_{j=1}^{k} x_{i_j} &= \sum_{j=1}^{k} [v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_j - 1\})] \\
&\geq \sum_{j=1}^{k} [v(\{i_1, \ldots, i_j\}) - v(\{i_1, \ldots, i_{j-1}\})] \\
&= v(\{i_1, \ldots, i_k\})
\end{aligned}
$$

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$$\begin{aligned}
\sum_{j=1}^{k} x_{i_j} &= \sum_{j=1}^{k} [v(\underbrace{\{1, \ldots, i_j\}}_{B_j \cup i_j = S \cup T}) - v(\underbrace{\{1, \ldots, i_j - 1\}}_{B_j = S})] \\
&\geq \sum_{j=1}^{k} [v(\underbrace{\{i_1, \ldots, i_j\}}_{A_j = T}) - v(\underbrace{\{i_1, \ldots, i_{j-1}\}}_{A_{j-1} = S \cap T})] \\
&= v(\{i_1, \ldots, i_k\}) \quad \uparrow \text{telescoping sum, eg } 1 + 3 + \cdots + (2i-1) = i^2
\end{aligned}$$

- Why? Supermodularity $\Rightarrow v(B_j \cup i_j) - v(B_j) \geq v(A_j) - v(A_{j-1})$, given $A_{j-1} = \{i_1, \ldots, i_{j-1}\} \subset \{1, \ldots, i_j - 1\} = B_j$

Core Generalization of First Welfare Theorem

Proposition (Welfare Theorem)

If $(\mathbf{x}, \mathbf{p})$ *is a competitive equilibrium, then* $\mathbf{x}$ *is in the core.*

- *Proof*: Let $(\mathbf{x}, \mathbf{p})$ be a competitive equilibrium, but $\mathbf{x} \notin$ core.
- Then some coalition S has a feasible allocation $\widehat{\mathbf{x}}$ with $u^i(\widehat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for all $i \in S$, strictly so for some $j \in S$.
- Revealed preference $\Rightarrow \mathbf{p} \cdot \widehat{\mathbf{x}}^i \geq \mathbf{p} \cdot \mathbf{x}^i \ \forall i \in S$, and $\mathbf{p} \cdot \widehat{\mathbf{x}}^j > \mathbf{p} \cdot \mathbf{x}^j$.
- Adding over $i \in S$ yields $\mathbf{p} \cdot \left(\sum_{i \in S} \widehat{\mathbf{x}}^i\right) > \mathbf{p} \cdot \left(\sum_{i \in S} \mathbf{x}^i\right) = \mathbf{p} \cdot \left(\sum_{i \in S} \mathbf{x}^i\right)$.
- Then $\widehat{\mathbf{x}}$ is infeasible for the coalition S: $\sum_{i \in S} \widehat{\mathbf{x}}^i \leq \sum_{i \in S} \mathbf{x}^i$. □

Core Generalization of First Welfare Theorem

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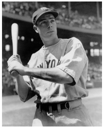

Core Generalization of First Welfare Theorem

Proposition (Welfare Theorem)

If $(\mathbf{x}, \mathbf{p})$ *is a competitive equilibrium, then* $\mathbf{x}$ *is in the core.*

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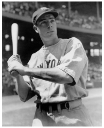
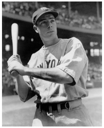

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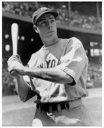 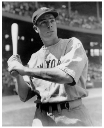 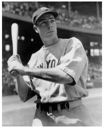 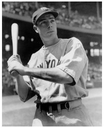 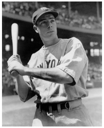 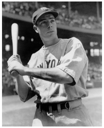

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The Shrinking Core of a Market Economy

- Agent $k = \{I, J\}$ with utility function $u^k(x, y) = xy$.
- Endowments diverge: $(\bar{x}^I, \bar{y}^I) = (2, 0)$ and $(\bar{x}^J, \bar{y}^J) = (0, 2)$.
- The core is the diagonal $y^I = x^I$ of the Edgeworth box
- We now clone each trader: two Irises and two Joes.
- Any allocation with $y^k = x^k$ for $k = I, J$ is still efficient.
  - E.g. $(x^I, y^I) = (0.4, 0.4)$ for Irises and $(x^J, y^J) = (1.6, 1.6)$ for Joes is efficient and IR
  - This allocation yields $u^I = 0.16$ and $u^J = 2.56$.
- $\{I_1, I_2, J_1\}$ blocks with $(x^I, y^I) = (1.2, 0.2), (x^J, y^J) = (1.6, 1.6)$
  - This is feasible: two Irises and one Joe are endowed with $(4, 2)$
  - Irises strictly better off: $u^I(1.2, 0.2) = 0.24 > 0.16 = u^I(0.4, 0.4)$
  - Joe is indifferent. (The excluded Joe is worse off.)

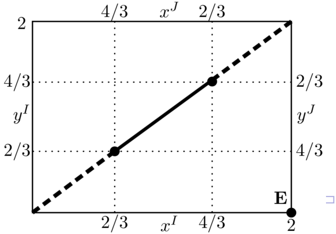
$\Rightarrow$ $(x^I, y^I) = (0.4, 0.4)$ and $(x^J, y^J) = (1.6, 1.6)$ not in the core.

The Shrinking Core of a Market Economy

- The coalition $\{I_1, I_2, J\}$ blocks more allocations.
- Start at the symmetric efficient allocation $(x^I, y^I) = (a, a)$ and $(x^J, y^J) = (2-a, 2-a)$, with $u^I = a^2$ and $u^J = (2-a)^2$.
- Reallocate the coalition's $(4, 2)$ endowment so that $(\hat{x}^I, \hat{y}^I) = (1 + a/2, a/2)$ and $(\hat{x}^J, \hat{y}^J) = (2-a, 2-a)$.
- This blocks the symmetric allocation iff $a < 2/3$:

$$u^I(\hat{x}^I, \hat{y}^I) = \left(\frac{a}{2} + 1\right)\left(\frac{a}{2}\right) > a^2 = u^I(x^I, y^I)$$

- The core weakly shrinks with each replication, since each adds more coalition constraints.
- EXERCISE. Show that 3 Irises and 3 Joes block any $a < 4/5$



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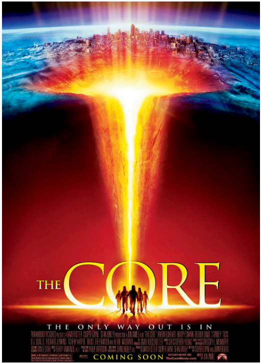
- Debreu and Scarf (1963) proved the reverse of the Core Welfare Theorem holds in large economies
- This is an amazing endorsement of the competitive model
- Let C_M be the core of the M-clone model.

Proposition (Core Convergence Theorem)

If $\mathbf{x}^* \in C_M$ for all M, then $\mathbf{x}^*$ is a competitive outcome. So the limit of the M-replica cores $\cap_{M=1}^{\infty} C_M$ is a competitive outcome.



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