Contagious Matching Games

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(preliminary, and yet fun)

Abstract

This paper explores a simple class of matching games in which individuals meet pairwise, unwittingly passing along a bad in a contagion fashion. It may be a private "bad", like a counterfeit money or stolen art. Or it may be a collective "bad", like a disease or a computer virus. Either way, individuals expend effort to avoid acquiring the "bad". With a private "bad", these efforts are complements, and the game is submodular. With a collective "bad", they are substitutes, and the game is supermodular.

The symmetric equilibria of these games share a common feature, that the marketplace often produces fewer "infections" as the bad grows more prevalent. One cannot, for instance, infer that counterfeiting is less severe when there is less passed counterfeit money.

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1 Introduction

What do forged art and AIDS have in common? Answer: Both exchange hands when two individuals meet, and at least the recipient unwittingly acquires it through inattention or carelessness. This note introduces and explores a new class of matching games, characterized by individuals pairwise meeting, possibly unwittingly passing along a "bad" in a contagion-like fashion. It may be a private "bad", like counterfeit money, stolen goods, or forged art. Or it may be a collective "bad", like a disease, email virus, or an undesirable accent. In either case, we assume that individuals may expend effort to acquire or avoid the contagious good. In this synthesis, a common theme emerges: the market place confounds the signals that emerge about the disease prevalence. Individuals strive harder not to acquire the disease the more common it becomes. When the Secret Service finds more passed counterfeit money, does that signal more circulating counterfeit money? When we observe more AIDS cases reported, is AIDS a worse problem? At low levels of either, it turns out the answer is yes. But at higher levels, this switches.

This paper relates to the growing literature on network games. We are aware of only one paper that relates to our collective good contagion game: Kremer (1996) introduces choice of parter into an epidemiological context. The role of effort choice here is completely different.

2 The Strategic Infection Games

Individuals meet pairwise. Each individual i exerts costly effort to acquire a filter of quality $q_i \in (0,1)$. We consider a "bad" χ like counterfeit money or a disease. If A has χ and B does not, then χ must pass through both filters to change hands.

For instance, if χ is possibly counterfeit money, then one wishes to avoid acquiring it. Since knowingly passing it on is illegal, it is worthless if B discovers that it is counterfeit; this has chance q_B if the note A hands him is counterfeit, as A may briefly study the money for authenticity. If a fraction γ of all notes are counterfeit, then the chance that A hands him counterfeit money is γ times the chance $1-q_A$ that A misses it. Of course, B may already possess counterfeit money, but each new acquisition brings its own loss, and this is a true loss. Hence, the chance that B acquires counterfeit money when meeting A equals $\gamma q_A(1-q_B)$.

If χ is an infectious disease, then B catches the disease with chance $1-q_B$ if A passes it on to him. For instance, A may sneeze and not cover his mouth, and B may directly inhale. The protective actions for STDs are more obvious. If a fraction γ of all individuals are infected, then the chance that A passes it to him is γ times the chance $1 - q_A$ that A misses it. The chance that B acquires the disease when meeting A thus equals $\gamma(1 - q_A)(1 - q_B)$. We now assume that B is unaware of his disease status. While this is a simplifying assumption, it is clearly realistic for many diseases, that are latent and only revealed by blood tests until their symptoms emerge. He thus has chance $1 - \gamma$ of not having the disease. Altogether, the chance that B *newly acquires* the disease when meeting A equals $\gamma(1 - \gamma)(1 - q_A)(1 - q_B)$.

In our world, producing a greater screening action q is naturally a costly endeavor. We wish to assume an increasing and convex cost function, and shall focus on $C(e) = ce^r/r$ with $r > 1$. Here, c is the marginal cost of the maximum screening effort $e = 1$, measured as a fraction of the unit magnitude of the gain or loss from matching.

	collective consumption	private consumption
"bad"	$-\gamma(1-\gamma)(1-\bar{e})(1-e) - C(e)$	$-\gamma(1-e)\bar{e} - C(e)$
"good"'	$\gamma(1-\gamma)\bar{e}e - C(e)$	$\gamma e(1-\bar{e}) - C(e)$

Figure 1: **Payoff Functions for the Infection Models.** Here we imagine that one individual chooses screening level e, and he faces someone choosing level $\bar{e}$.

	collective bad	private bad
"bad"	submodular	supermodular
"good"	supermodular	submodular

These examples fall into two separate economic camps that we explore. First, counterfeit money is a "bad" but is private: Neither desires it, and by passing it on, one loses it. A disease is a collective[1] "bad", being retained as it is passed on.

3 Private Bads

3.1 Equilibrium

We first consider private bads χ, like counterfeit money. We seek the symmetric Nash equilibria, where everyone opts for the common verification level $\bar{e}$. Given a meeting, each individual maximizes $\pi(e, \bar{e}) = -\gamma\bar{e}(1 - e) - C(e)$ in e. Using the first order

[1]A disease is not non-rivalrous as it can only be possessed by the "club" of individuals infected.

condition, $\gamma \bar{e} = C'(e) = ce^{r-1}$ solved by $e = \bar{e}$. The *equilibrium first order condition* becomes

$$\gamma \bar{e} = c\bar{e}^{r-1} \Leftrightarrow \gamma = c\bar{e}(\gamma)^{r-2} \tag{1}$$

The second order condition clearly holds in our paper, when $r > 1$, for then costs are convex and benefits are linear in e. There is one final possibility, which is a corner solution at $e = 0$ or $e = 1$. No such corner solution exists when $r > 2$ and $\gamma \leq c$ for then $\gamma \bar{e} = ce^{r-1}$ is always solvable with $\bar{e}^{r-2} = \gamma/c \leq 1$. Observe that $\lim_{\gamma \to 0} \bar{e}(\gamma) = 0 = \bar{e}(0)$ only if $r \leq 2$. *We assume that $r > 2$ throughout this section.*

The equation $\gamma = c\bar{e}^{r-2}$ defines a function $\bar{e}(\gamma) = \left(\frac{\gamma}{c}\right)^{\frac{1}{r-2}}$. When the prevalence of χ attains $\gamma = c$, the equilibrium calls for $\bar{e} = 1$ and new transmissions are choked off. Observe that $\bar{e}(\gamma) < 1$ exactly when $\gamma < c$, which we have assumed.

Lemma 1 *The screening effort rises in γ for a private bad with cost convexity $r > 2$.*

Observe that with payoff $\pi(e, \bar{e}) = -\gamma \bar{e}(1 - e) - C(e)$, *screening efforts for the private bad are inefficiently high in equilibrium*, as they confer a negative externality on others. Since someone is eventually stuck with the private bad, individuals play a zero sum game, and any effort is inefficient. For example, scrutinizing possibly counterfeit currency that one is handed obviously hurts the individual passing it to you. One can check that the marginal value (to oneself) of others' efforts equals $\pi_{\bar{e}}(e, \bar{e}) = -\gamma(1 - e) = -\gamma[1 - \left(\frac{\gamma}{c}\right)^{\frac{1}{r-2}}] < 0$ when $e = \bar{e}$. So this externality grows in the prevalence γ. To be sure, there is another equilibrium here: an efficient equilibrium involves zero screening by everyone.

One can check that the cross-partial of efforts equals the prevalence $\pi_{\bar{e}p}(e, \bar{e}) = \gamma > 0$. Actions are therefore strategic complements: The more carefully others examines the currency you pass to them, the better one wishes to scrutinize the money one is handed. In other words, own efforts raise the marginal product of others' efforts, so that game is *supermodular*. The equation $\pi_{\bar{e}}(e, \bar{e}) = 0$ defines the best response function $e(\bar{e})$. The slope of the best response function is then found by solving $\pi_{pp}dp + \pi_{x\bar{e}}d\bar{e} = 0$. Since $\pi_{x\bar{e}} > 0$, the best response graphs is increasing functions the others' actions. Just as in Diamond (1982), this encourages multiple equilibria. Further, as γ grows, the supermodularity works to ameliorate the negative externality, since the greater others' screening efforts, the steeper my marginal product of screening, and the faster I raise it.

3.2 The Market Response to Infections

Our focus is on the observable consequences of the contagion transmission. The *new "infections"* in equilibrium are $I(\gamma) = \bar{e}(\gamma)[1 - \bar{e}(\gamma)]\gamma$. This vanishes with zero screening effort, for no one ever discovers the bad. Equally well, it vanishes with perfect screening, as the good is never passed on. Infections require some inattention by the giver ($\bar{e} < 1$), and diligence by the receiver ($\bar{e} > 0$).

Easily, if $\bar{e}(\gamma) < 1/2$ and $r > 2$, then $\bar{e}(\gamma)[1 - \bar{e}(\gamma)]$ rises in $\bar{e}(\gamma)$ and the screening effort satisfies $\bar{e}'(\gamma) > 0$. On balance, infections rise too, or $I'(\gamma) > 0$. Our focus is on the surprising case of a decreasing infection rate, where the marketplace response entirely counteracts the adverse change in nature.

Proposition 1 *Assume a private bad with $r > 2$. Infections move in opposition to the prevalence $\gamma \leq c$ when the bad is sufficiently common.*

Proof: Observe that

$$\bar{e}'(\gamma) = \frac{\bar{e}(\gamma)}{(r-2)\gamma} \tag{2}$$

Substituting (1) into (2), we find that (where $a \propto b$ if a, b have the same sign):

$$\begin{aligned} I'(\gamma) &= \bar{e}(1-\bar{e}) + \gamma(1-2\bar{e})\bar{e}'(\gamma) = \bar{e}(1-\bar{e}) + \frac{(1-2\bar{e})\bar{e}}{r-2} \\ &\propto (r-2)(1-\bar{e}) + (1-2\bar{e}) = (r-1) - r\bar{e} \end{aligned}$$

Then $I'(\gamma) < 0$ if and only if $\bar{e}(\gamma) > 1 - 1/r$, namely, if and only if $\gamma > c(1-1/r)^{r-2}$, because $\bar{e}(\gamma) = \left(\frac{\gamma}{c}\right)^{\frac{1}{r-2}}$. As r explodes, this says that $\gamma > c/e$, where $e \approx 2.718$. $\square$

Large levels of counterfeit money are not without precedent. For instance, during the American Revolution, the British so successfully counterfeited American money that the Continental currency soon became worthless — hence the saying "Not worth a Continental". The Secret Service reports that later on, during the Civil War, one-third to one-half of the circulating currency was counterfeit.

Clearly, infections cannot always move in opposition to the prevalence γ, since $I(\gamma)$ vanishes when $\gamma = 0$, and is otherwise positive. But Proposition 1 says that infections thus move when the screening effort exceeds $1/2$ — which eventually holds for large enough γ, by Lemma 1.

The market for a private bad plays a zero sum game. As such, the game per se has obvious welfare properties. On the other hand, we have ignored an ex ante stage where the private bad was introduced into the market place. Suppose that the government wishes to discourage the admission of new private bads, by somehow sub-

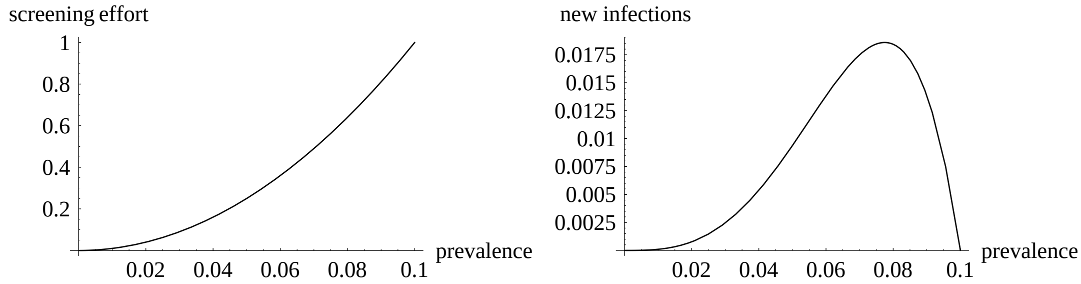

Figure 2: **Private Bads.** This illustrates the screening effort and resulting infections given the disease rate γ, assuming cost convexity $r = 2.5$ and cost parameter $c = 0.1$. With a constant screening, new infections are forever rising in γ. But here effort levels constantly rise in response to greater γ, and thus the peak infection level is at $\gamma \approx 0.08$.

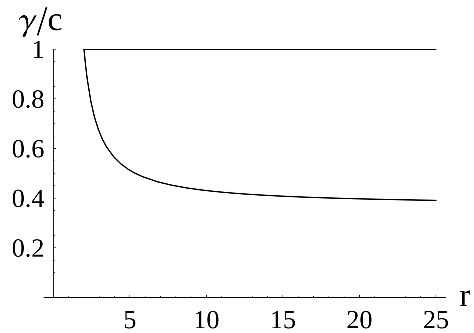

Figure 3: **Feasible Parameters** γ, r**.** All levels of the private bad γ/c below 2 and above the lower frontier $(1-1/r)^{r-2}$ yield falling "infections" in the private bad level. Provided $c < 1$ is small enough, infections may be falling at a low prevalence γ.

sidizing or otherwise encouraging the screening effort e. Such efforts must surely lessen the parameter c, and thereby raise the screening effort $\bar{e}(\gamma) = (\gamma/c)^{\frac{1}{r-2}}$. Since $I(\gamma) = \bar{e}(\gamma)[1 - \bar{e}(\gamma)]\gamma$, we have $dI(\gamma) \propto -[1 - 2\bar{e}(\gamma)]\gamma de(\gamma) < 0$, for $\bar{e} > 1/2$.

3.3 The Prevalence Dynamics

Assume now an exogenous entry flow of δ of the characteristic χ into the matching pool. For instance, this may be new counterfeit money or forged art into the economy by criminal elements. The infections now constitutes exits from the prevalence pool. Modifying the standard disease dynamics, we find that

$$\dot{\gamma} = \delta - I(\gamma) = \delta - \bar{e}(\gamma)[1 - \bar{e}(\gamma)]\gamma = \delta - (\gamma/c)^{\frac{1}{r-2}}[1 - (\gamma/c)^{\frac{1}{r-2}}]\gamma$$

The steady-state level of χ clearly satisfies $\dot{\gamma} = 0$, or

$$(\gamma/c)^{\frac{1}{r-2}}[1 - (\gamma/c)^{\frac{1}{r-2}}]\gamma = \delta \tag{3}$$

Proposition 2 *Assume a private bad with $r > 2$. The steady-state prevalence of χ is rising in c if $\gamma < 4c/2^r$, and falling if $\gamma > 4c/2^r$ (for a sufficiently prevalent bad).*

Proof: Just as $x(1-x)$ falls in x when $x < 1/2$, when the steady-state level of γ obeys $(\gamma/c) < 2^{2-r}$, the left side of (3) is falling in c. This inequality condition reduces to $\gamma < 4c/2^r$. Also, in order to maintain equality, γ is increasing in c over this range. But above that, the steady-state level of γ is falling in the cost c of effort. □

The intuitive result is γ rising in c. But when χ is prevalent enough, then paradoxically, cost-lowering efforts counter-productively the prevalence.

4 Collective Bads

4.1 Equilibrium

The analysis now entails maximizing $-\gamma(1-\bar{e})(1-e)-C(e)$ with respect to e, where the symmetric Nash equilibrium with $e = \bar{e}$ requires:

$$(1 - \bar{e})\gamma = C'(e) = c\bar{e}^{r-1} \tag{4}$$

Direct substitution reveals that $e(0) = 0$. As this cannot be solved in closed form, we proceed indirectly.

Lemma 2 *The screening effort rises in γ for a collective bad for any convexity $r \geq 1$.*

Proof: Immediately, (4) yields $(1 - \bar{e}) - \bar{e}'(\gamma)\gamma = c(r - 1)\bar{e}^{r-2}\bar{e}'(\gamma)$, and so

$$\bar{e}'(\gamma) = \frac{1 - \bar{e}}{\gamma + c(r-1)\bar{e}^{r-2}} = \frac{\bar{e}(1 - \bar{e})}{\gamma(\bar{e} + (r - 1)(1 - \bar{e}))} > 0 \tag{5}$$

where we have simplified the expression using (4). □

Observe that with the payoff function $-\gamma(1 - \bar{e})(1 - e) - C(e)$, screening efforts will be inefficiently low in equilibrium, since they confer an unaccounted positive externality on others. For instance, protecting oneself from disease transmission clearly helps any individuals one encounters. The same effect is true in the matching setting of Mortensen (1982), where individuals exert effort in advance that affects others' probabilistic futures. One can check that the marginal value of others' efforts to oneself in equilibrium equals $\gamma(1-\bar{e}) = \gamma[1-\left(\frac{\gamma}{c}\right)^{\frac{1}{r-2}}]$. In other words, this is a dynamic externality, at least initially growing in the prevalence γ.

On the other hand, actions are strategic substitutes: The more carefully one protects against disease transmission, the more one lowers the marginal product of others efforts. In other words, each individual's actions lower the marginal product of others' actions, and the resulting game is *submodular*. This means that the best response graphs are decreasing functions of others actions, and precludes the possibility of multiple equilibria. One can check that the cross-partial of efforts equals the prevalence $-\gamma$. Consequently, as γ grows, the supermodularity aggravates the negative externality, since the greater others' screening efforts, the smaller is the slope of my marginal product of screening, and the slower I raise it.

4.2 The Market Response to Infections

Equilibrium infections are now $I(\gamma) = \gamma(1-\gamma)(1-\bar{e})^2$. In other words, an infected meets an uninfected individual, and the disease passes through both effort screens.

Proposition 3 *Assume a collective bad and $r > 1$. New infections move in opposition to the disease prevalence for $\gamma > 1/2$. In particular, when $r \leq 2$ it suffices that $\gamma > 1/2$. When $r > 2$, it suffices that $\gamma > \frac{(r-1)c}{2c(r-1)+1}$, and so for $\gamma > 1/2$.*

Proof: The slope of new infections equals

$$\begin{aligned} I'(\gamma) &= (1-2\gamma)(1-\bar{e})^2 - 2\gamma(1-\gamma)(1-\bar{e})\bar{e}'(\gamma) \\ &= (1-\bar{e})^2\left[1-2\gamma-\frac{2(1-\gamma)\bar{e}}{\bar{e}+(r-1)(1-\bar{e})}\right] \end{aligned}$$

When $1 < r \leq 2$, this is negative at least when $[1-2\gamma-2(1-\gamma)\bar{e}] < 0$, and thus in particular when $\gamma > 1/2$. On the other hand, when $r > 2$, we have $I'(\gamma) < 0$ iff $(r-1)(1-2\gamma) < \bar{e}/(1-\bar{e})$. Finally, first rewriting the premise inequality of this proposition, and then applying $\bar{e}^{2-r} \geq 1$ and (4), we find that

$$(r-1)(1-2\gamma) < \gamma/c \leq (\gamma/c)\bar{e}^{2-r} = \bar{e}/(1-\bar{e})$$

In other words, $I'(\gamma) < 0$ given our condition on γ. $\square$

How realistic is the condition $\gamma > \frac{(r-1)c}{2c(r-1)+1}$? Observe that we have normalized the cost of the disease to 1, to secure a common model for collective and private goods. We may however capture a more deadly or less disease within our model by simply

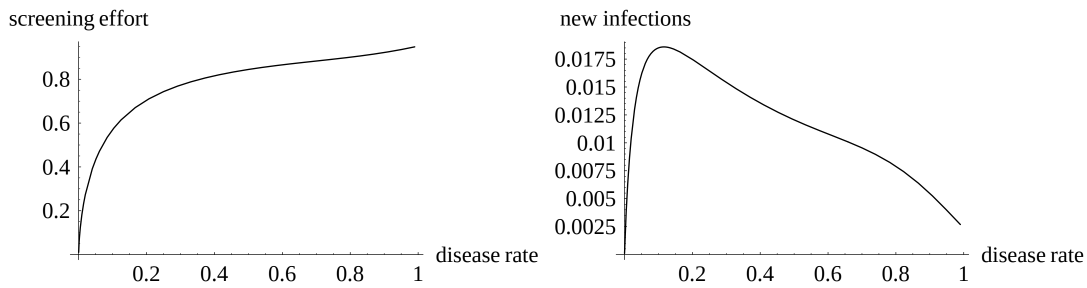

Figure 4: **Collective Bads.** This illustrates the screening effort and resulting infections with cost convexity $r = 2.5$ and cost parameter $c = 0.1$. With a constant screening effort, new infections will peak at $\gamma = 0.5$, when $\gamma(1-\gamma)$ is maximized. However, in our behavioral model, individuals increase their effort levels in response to greater γ, and accordingly the peak infection level is at a lower γ, here just over 0.1.

adjusting up or down the marginal cost c of effort at $\bar{e} = 1$. Returning to Proposition 3, the combination of a sufficiently dangerous and common disease leads to the perverse infection monotonicity.

4.3 The Disease Dynamics

Assume now an exit rate δ of those with the contagious characteristic χ. For a disease, exits may be accomplished by deaths or recovery. The infections now constitutes entrants to the disease pool. Modifying the standard disease dynamics, we find that

$$\dot{\gamma} = I(\gamma) - \delta\gamma = [1 - \bar{e}(\gamma)]^2\gamma(1-\gamma) - \delta\gamma = [1 - (\gamma/c)^{\frac{1}{r-2}}]^2\gamma(1-\gamma) - \delta\gamma$$

The steady-state level of χ clearly satisfies $\dot{\gamma} = 0$, or

$$[1 - (\gamma/c)^{\frac{1}{r-2}}]^2(1-\gamma) = \delta \tag{6}$$

Proposition 4 *Assume a collective bad. The steady-state disease prevalence γ rises in c for $r > 2$ and falls for $r < 2$.*

Proof: The left side of (3) is rising in c for $r > 2$ and falling when $r < 2$. In order to maintain equality, γ is rising and falling in these respective parameter ranges. □

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