The Comparative Statics of Sorting[*]

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Abstract

In the marriage model, Becker (1973) found that positive (or negative) sorting is efficient with supermodular (or submodular) match payoffs. But characterizing the optimal matching with general production remains unsolved decades later.

Rather than tackle this difficult open problem, we instead ask when match sorting optimally increases. To do this, we first argue that the *positive quadrant dependence* (PQD) stochastic order on bivariate cdf's captures an economically meaningful notion of increasing sorting — e.g. a higher correlation of partners.

Our theory turns on *synergy*: the local cross partial difference or derivative. A natural guess fails: *increasing synergies need not raise sorting.* But sorting rises if (1) synergy either everywhere increases or proportionately upcrosses through zero, *and* (2) cross-sectionally, synergy is upcrossing or downcrossing in types.

Our proof develops and exploits new monotone comparative statics methods. It proceeds by induction with finitely many types, and secures the continuum type results by taking limits. Our main results are easy to apply. We illustrate all theorems, applying them to the major post 1990 marriage model papers.

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1 Introduction

Assortative matching is the allocational theme in the vast literature on decentralized matching. This finding by Becker (1973) has seen application in marriage, employment, partnerships, optimal assignment, and pairwise trade. The power of this conclusion is also its weakness — higher "men" match with higher "women," without exception. Since perfectly assortative matching is an ideal, how should we understand deviations from it? Shimer and Smith (2000) asked if these can be seen as evidence of search frictions. In this frictional lens, they found that weak positive sorting — individuals match with lattice type sets — only holds under very strong complementarity assumptions. Also, their matching set must be centered about Becker's frictionless sorting partner.

And while Becker's driving premise of *supermodular* output is easy to formulate, it is intuitively very restrictive: a globally positive cross partial difference or cross partial derivative. Chade, Eeckhout, and Smith (2017) explore many natural and some well-cited economic matching settings where supermodularity fails, as we summarize in §3. But a general theory of who matches with whom remains open. This void has greatly limited the analytic reach of the matching literature in economics, except in some closed form solvable cases, and focused excessive attention on the perfect sorting case.

We develop a tractable general theory of increasing sorting in the frictionless pairwise matching model with either finitely many or a continuum of types and *transferable utility* (TU). By using old and new methods for monotone comparative statics, we make predictions without ever solving the planner's problem. Notably, we offer predictions for the matching papers in §3 that have most influenced economics since Becker (1973).

We first introduce a partial order on matching measures to capture the notion of increasingly assortative. The *positive quadrant dependence* (PQD) partial order ranks bivariate measures by the probability mass in the southwest quadrant. According to stochastic dominance theory, the expectation of any supermodular function increases in the PQD order. So equipped, we derive three economically practical measures of increased sorting in this sense: the average distance between matched types falls in PQD, while the correlation of matched types, and regression coefficients of women on their partners' types rise in PQD (**Lemma 2**). In other words, our sorting comparative statics conclusions are thus of direct empirical relevance in economics.

To illustrate the PQD order, consider the six possible complete matchings among three men a, b, c and three women A, B, C (Figure 1). One can verify that each man matches with a weakly closer partner in PAM than in NAM1 or NAM3, in turn each closer than in PAM2 or PAM4, and finally than in NAM. We have thus a partial order:

$$\text{PAM} \succ_{PQD} [\text{NAM1, NAM3}] \succ_{PQD} [\text{PAM2, PAM4}] \succ_{PQD} \text{NAM} \tag{1}$$

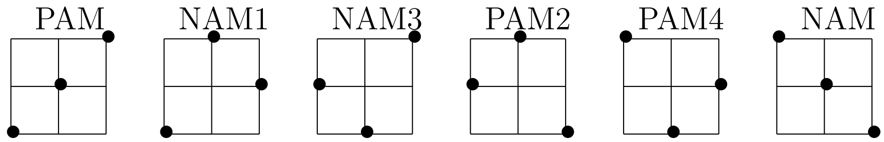


Figure 1: **All 1-1 Matchings with Three Types.** In addition to negative and positive assortative matching (NAM and PAM), there is negative assortative matching in quadrants 1 and 3 (NAM1 and NAM3), and positive assortative matching in quadrants 2 and 4 (PAM2 and PAM4). Sorting is partially ranked according to (1).

Our assumptions on production functions rely on a local complementarity measure: *Synergy* is the cross partial difference of production with finitely many types, and with continuous types, the cross partial derivative, if it exists. Becker (1973) finds that globally positive synergy induces positive sorting, and globally negative synergy induces negative sorting. We consider intermediate cases, where synergy is sometimes positive and sometimes negative. One might conjecture that sorting is higher with a production function with globally higher synergy. But the example in Figure 2 dashes any such hope, and underscores the subtlety of the problem, as the matching oscillates between NAM1 and NAM3 as synergy rises. For since neither NAM1 nor NAM3 is more assortative for arbitrary weights on men and women, sorting is not monotone.

To begin piecing together our logic, we first uncover a new formula for total match output — it only depends on the matching via the dot product of synergy and the cumulative match distribution (**Lemma 1**). Our formula yields Becker's Result at once by corollary, and shows how production only impacts match output via synergy. This formula delivers a single crossing property linking synergy and sorting. But since matching measure are not a lattice in the PQD order, existing monotone comparative statics cannot imply that sorting is monotone. Nevertheless, sorting is *nowhere decreasing* over time if synergy globally increases in time, or is a linear function of time (**Proposition 1**). The conclusion that sorting never falls is consistent with the matching oscillation in Figure 2 as synergy increases. But as synergy rises, match partners could on average move farther apart, or some match partner regression coefficient could fall. This inconvenient truth highlights the need for a new approach altogether.

In our pursuit of an increasing sorting conclusion, we then shift to an inductive approach. We work in the finite type world, and secure results for the continuum type model by taking limits. The counterexample in Figure 2 also underscores the need for more discipline on synergy. For notice that synergy rises in the woman's types for the least man, but falls in her type for the next man. We track the *sorting premium* on type rectangles — namely, the net payoff change from negatively to positively sorting any two women $x_1 < x_2$ matched to any two men $y_1 < y_2$. We assume that the sorting

Match Payoffs

	x_1	x_2	x_3
y_3	9	**14**	18
y_2	5	2	**14**
y_1	**1**	5	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	16	**24**
y_2	**5**	3	16
y_1	1	**5**	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	**20**	30
y_2	5	6	**20**
y_1	**1**	5	9

$\rightarrow$

	x_1	x_2	x_3
y_3	9	22	**36**
y_2	**5**	7	22
y_1	1	**5**	9

Cross Partial Differences of Match Payoffs

	x_1x_2	x_2x_3
y_2y_3	8	-8
y_1y_2	-7	8

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	9	-5
y_1y_2	-6	9

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	10	-4
y_1y_2	-3	10

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	11	-1
y_1y_2	-2	11

Figure 2: **Sorting Need Not Rise in Synergy.** In the top row, the unique most efficient matchings alternates between NAM1 and NAM3. In the next row, all four match synergies — or the cross differences of match payoffs — strictly increase as we move right. So it is not true that increasing synergy leads to more sorting.

premium is strictly upcrossing in types, or as the type rectangle shifts to the northeast.[1] We restrict to this class of production functions, as it precludes the example in Figure 2. We then find that sorting increases if synergy is monotone — and more generally, if the *total synergy* on all sets of potential couples is upcrossing (**Lemma 4**). This is our core finding, and is proven by induction on the number of types. The rest of the paper develops tractable local conditions that allow us to apply this finite type sorting result.

Our first local approach posits that synergy is either linear or monotone in time (**Proposition** 2), which happens in more than one cited paper. Then with finitely many types, sorting increases if the sorting premium is upcrossing or downcrossing in types. But with a continuum of types, we need a marginal cross-sectional condition: the increase in the x-marginal product over any interval of y types is monotone in x. Our second purely local approach unifies time series and cross-sectional conditions. A *proportional upcrossing* function of types and time obeys an inequality that ensures that positive synergy increases proportionately more than absolute negative synergy. This property guarantees upcrossing total synergy on all sets — as we prove in a multi-dimensional aggregation extension of Karlin and Rubin's 1956 classic upcrossing preservation result (**Theorem 2**). All told, sorting is monotone when synergy is up- or downcrossing in types, as well as proportionately upcrossing (**Proposition 3**).

Our last major finding deduces comparative statics predictions for type distribution shifts (**Corollary 1**). Our proof exploits an equivalence with productive shifts.

Our theory greatly expands the predictive reach of matching theory. For instance, with 100 men and 100 women, Becker (1973) makes predictions for just two possible

[1]Loosely, a function is *upcrossing* if it crosses the horizontal axis at most once, and if once, from below, and *downcrossing* if it crosses the horizontal axis at most once, and if once, from above.

synergy sign combinations. Our cross-sectional single crossing synergy encompasses a total of $2 \cdot 99^2$ sign combinations — and ones that specifically arise in applications.[2]

Literature Review. Becker's work sparked a vast economic literature on the transferable utility matching paradigm. For he offered a quick way to check or identify in data whether matching was perfectly assortative. Since complementarity is quite economically intuitive, it was inevitable that models would arise without this property. We offer comparative statics for these papers, and numerically illustrate the optimal matchings; these plots reflect subtle and surprising global optimality considerations.

Kremer and Maskin (1996) was an early work that made a strong case for the marriage model without complements. In this motivated twist on Becker, they proposed a partnership model with defined roles. Match output was therefore the maximum of two supermodular functions — one for each role assignment. They claim "there is a body of work within the labor economics literature that assumes such imperfect substitutability. There is also empirical evidence to justify the assumption".

Others soon highlighted the importance of matching without supermodularity. Legros and Newman (2002) noted that in the presence of imperfect credit constraints, supermodular production does not induce supermodular match payoff functions. Our nowhere decreasing theory subsumes their production function. But we instead focus on Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending with adverse selection — for which our stronger increasing sorting theory applies.

Another motivated twist on matching that undermines supermodularity is moral hazard. Serfes (2005) investigates a pairwise matching model of principals and agents, and shows that negative sorting — more risk averse agents with safer projects — arises with a low disutility of effort, but positive sorting emerges for high disutility of effort.

Finally, even with supermodular static payoffs, Anderson and Smith (2010) show that dynamic models with Bayesian updating need not inherit supermodularity. In our subsequent work with evolving human capital (Anderson and Smith, 2012), we show that preservation of supermodularity is highly exceptional. For general transition functions of old types into new types, the dynamic match values are rarely supermodular.

Becker's sorting result follows from standard monotone comparative statics results for supermodular functions (pursued at length in §3.2 in Topkis (1998)). Our insights hail from new results in monotone comparative statics, including two new ones. First, our nowhere decreasing theory owes to a comparative static result for partially ordered sets that are not lattices — the very character of bivariate matching distributions that obey adding up conditions. **Theorem 1** summarizes our key findings here.

All long proofs and new monotone comparative statics results are in the Appendix.

[2]For our upcrossing assumption, a sign change can occur after any of 99 men and 99 women.

2 The Marriage Model

A. The Marriage Model with Global Complements or Substitutes.

Assume pairwise matching by a continuum of individuals either from two groups (men and women, firms and workers, buyers and sellers) or the same set (partnerships). In the general matching model, "women" and "men" have respective *types* $x, y \in [0, 1]$ with cdfs G and H. The matching market is balanced, with unit mass $G(1) = H(1) = 1$.

We capture in parallel two models: absolutely continuous type distributions G and H, and finitely many types, when G and H are discrete measures with equal weights on female types $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ and male types $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$. We then relabel women as $i \in \mathbb{Z}_n \equiv \{1, 2, \ldots, n\}$ and men as $j \in \mathbb{Z}_n$.

We assume a C^2 production function $\phi > 0$, so that types x and y jointly produce $\phi(x, y)$. In the finite type model, the output for match (i, j) is $f_{ij} = \phi(x_i, y_j) \in \mathbb{R}$. Production is *supermodular* or *submodular* (SPM or SBM) for all $x' < x''$ and $y' < y''$ if:

$$\phi(x', y') + \phi(x'', y'') \geq (\leq)\ \phi(x', y'') + \phi(x'', y') \tag{2}$$

Strict supermodularity (respectively, strict SBM) asserts strict inequality in (2).

Like Becker's, our theory does not explore an extensive margin whether to match. A *matching* is a bivariate cdf $M \in \mathcal{M}(G, H)$ on $[0, 1]^2$ with marginals G and H. In the finite type case, G and H put equal unit weight on $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. A finite matching is a nonnegative matrix $[m_{ij}]$, with cdf $M_{i_0 j_0} = \sum_{1 \leq i \leq i_0, 1 \leq j \leq j_0} m_{ij}$, and unit marginals $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0 j}$ for all men i_0 and women j_0. In a *pure matching*, $[m_{ij}]$ is a matrix of 0's and 1's, with everyone matched to a unique partner.

There are two perfect sorting flavors. In *positive assortative matching* (PAM), any woman type of x at quantile $G(x)$ pairs with a man of type y at the same quantile $H(y)$, and thus the match cdf is $M(x, y) = \min(G(x), H(y))$. In *negative assortative matching* (NAM), complementary quantiles match, and so $M(x, y) = \max(G(x) + H(y) - 1, 0)$. Matched types are *uncorrelated* given uniform matching, and so $M(x, y) = G(x)H(y)$.

The *partnership* (or unisex) model is a special case where types x and y share a common distribution, $G = H$, the production function ϕ is symmetric ($\phi(x, y) = \phi(y, x)$), and so too is the optimal matching distribution $M(x, y) \equiv M(y, x)$. In this case, PAM is therefore the matching $y = x$.

A social planner maximizes total match output, namely $\sum_{i=1}^n \sum_{j=1}^n f_{ij}(\theta) m_{ij}$ with finite types, or more generally $\int_{[0,1]^2} \phi(x, y|\theta) M(dx, dy)$, where we index output $\phi(x, y|\theta)$ by a (*often suppressed*) *state* $\theta \in \Theta$, a partially ordered set. Solving for optimal matchings:

$$\mathcal{M}^*(\theta) = \arg \max_{M \in \mathcal{M}(G,H)} \int_{[0,1]^2} \phi(x, y|\theta) M(dx, dy) \tag{3}$$

Gretsky, Ostroy, and Zame (1992) deduce existence of $\mathcal{M}^*$, and decentralize it as a competitive equilibrium (which we exploit later in §6.2). We prove uniqueness in §6.

Maximization (3) has been solved in three cases: Every feasible matching is optimal with modular production, while Becker solved the PAM and NAM extremes:[3]

Becker's Result. *Given SPM (SBM) production ϕ, the optimal matching exists and is PAM (NAM). Given strict SPM (SBM), these pairings are uniquely optimal.*

Proof: Assume finitely many types. Existence is immediate. To see uniqueness, assume women $x' < x''$ and men $y' < y''$ are negatively sorted into matches (x', y'') and (x'', y'). Then output is not maximal, since SPM production (2) implies a higher payoff to the matches $(x', y') < (x'', y'')$. We offer a general proof of Becker's Result early in §4. $\square$

This paper derives comparative statics with neither SPM or SBM, when the optimal matching is not PAM or NAM. We'll see that this commonly arises in economics.

B. Production Synergy and Total Match Output.

Assume first finitely many types. We call the cross partial difference match *synergy*:

$$s_{ij}(\theta) = f_{i+1j+1}(\theta) + f_{ij}(\theta) - f_{i+1j}(\theta) - f_{ij+1}(\theta)$$

With a type continuum, we call the cross partial derivative $\phi_{12}(x, y|\theta)$ match *synergy*.

Production is SPM when synergy is globally nonnegative. To understand matching with signed synergy, we doubly sum or doubly integrate match output by parts.

Lemma 1 (Match Output)**.** *Fix and suppress θ. Given n types of men and women:*

$$\textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} \left[f_{nj+1} - f_{nj}\right] j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij} M_{ij}$$

Given men's and women's types $[0, 1]$, if $\mathcal{I} \equiv [0, 1]$ and $\mathcal{J} \equiv (0, 1]$, then:

$$\textstyle\int_{\mathcal{I}^2} \phi(x, y) M(dx, dy) = \int_{\mathcal{I}} \phi(x, 1) G(dx) - \int_{\mathcal{J}} \phi_2(1, y) H(y) dy + \int_{\mathcal{J}^2} \phi_{12}(x, y) M(x, y) dx dy$$

Lemma 1 highlights that the matching distribution only impacts total match output via synergy. Any two production functions with the same synergies have the same matching, if everyone matches. For instance, synergy vanishes if production is linear in types. In tis case, all match distributions yield the same output.

We do not solve the optimization (3), but instead ask how optimizers $\mathcal{M}^*(\theta)$ change as the state θ increases. Throughout, a *time series* property relates production to the state (as in Figure 2), and a *cross-sectional property* relates production to the types.

[3] Koopmans and Beckmann (1957) decentralize the solution as a competitive equilibrium assuming TU. Legros and Newman (2007) show that some NTU models can be mapped into the TU paradigm.

3 Economic Applications of the Marriage Model

We now explore some illustrative or celebrated economic applications of the marriage model not explained by Becker's Result — for production is neither SPM nor SBM.

(*a*) Quadratic Production. We start with an instructive matching example. Since empirical work often ventures quadratic production, posit $\phi(x,y) = \alpha xy + \beta(xy)^2$. Then synergy $\phi_{12}(x,y) = \alpha + 4\beta xy$ is increasing in α and β. By Becker's Result, PAM is optimal when $\alpha, \beta \geq 0$, uniquely so if also $\alpha + \beta > 0$. Likewise, NAM is optimal when $\alpha, \beta \leq 0$, and uniquely so with $\alpha + \beta < 0$. But with either of $\alpha \lesseqgtr 0 \lesseqgtr \beta$, SPM and SBM fail, as synergy can be positive and negative; Becker's Result is inapplicable.

(*b*) Principal-Agent Matching with Moral Hazard. Serfes (2005) explores a pairwise matching model of principals and agents. Project variances $y \in [\underline{y}, \overline{y}]$ vary across principals, while agents differ by their risk aversion parameter $x \in [\underline{x}, \overline{x}]$.

When agents share a common scalar dis-utility of effort $\theta > 0$, Serfes derives (in his equation (1)) the expected output and synergy of an (x,y) match:

$$\phi(x,y|\theta) = \frac{1}{2\theta\,(1+\theta xy)} \quad \Rightarrow \quad \phi_{12}(x,y|\theta) = \frac{\theta xy - 1}{2\,(1+\theta xy)^3} \tag{4}$$

Serfes applies Becker's Result to deduce NAM for $\theta < \underline{\theta}$ and PAM for $\theta > \overline{\theta}$. But he is silent about all intermediate disutility of efforts, where $1/[\overline{y}\,\overline{x}] = \underline{\theta} < \overline{\theta} = 1/[\underline{y}\underline{x}]$.

(*c*) Group Lending with Adverse Selection. We consider Guttman's (2008) dynamic extension of Ghatak's (1999) model of group lending with adverse selection. Borrowers vary by their project success chance x; a success pays π and a failure nothing. Pairs of borrowers sign lending contracts, and project outcomes are independent.

After seeing the project outcome, a borrower either repays the loan, or defaults. Each pays $d > 1$ if both repay. If only one defaults, the other repays $c + d > d$. Assume $\pi \geq c + d$, so that borrowers repay when their project succeeds. If both default, each loses access to credit markets. Borrowers discount future payoffs by $\delta < 1$, and default if the project fails. The discounted payoff to the success chance pair (x,y) obeys:

$$\phi(x,y) = x((\pi - d) - (1-y)c) + y((\pi - d) - (1-x)c) + \delta(1 - (1-x)(1-y))\phi(x,y) \tag{5}$$

One can check that synergy ϕ_{12} is globally positive if $\delta \leq \delta^* \equiv c/[c + (\pi - d)]$. But with more patience, $\delta > \delta^*$, synergy is positive for low (x,y) and negative for high (x,y).[4]

[4] Legros and Newman (2002) explore group borrowing to finance a *joint project*. In their model, expected output is $\phi = (xy - q)\mathbb{1}_{XY \geq \kappa}$, where q is the cost of capital and $\kappa \geq q$ a financing constraint. Output is globally SPM when $\kappa = q$, but is neither globally SPM nor globally SBM when $\kappa > q$.

(d) A PARTNERSHIP MODEL WITH CAPITAL. For a match by worker types (x, y), let $\ell(x, y) = (x^\eta + y^\eta)^{1/\eta}$ be the *effective labor*. Inspired by Krusell, Ohanian, Ríos-Rull, and Violante (2000), production depends on current technology via a capital index κ:

$$\phi(x, y) = (\ell(x, y)^\rho + \kappa^\rho)^{1/\rho} \quad \Rightarrow \quad \phi_{12}(x, y) \propto (\rho - \eta)\kappa^\rho + (1 - \eta)\ell(x, y)^\rho \tag{6}$$

Assume there is greater complementarity between partner types than between labor and capital, $\rho < \eta < 1$. Then synergy is negative $\phi_{12} < 0$ for low types x, y, and positive for high types when $\rho > 0$. But if instead, $\rho < 0$, then synergy is positive for low types and negative for high types. In either case, Becker's Result does not apply.

(e) PRODUCTION WITH DEFINED ROLES. In an early and influential paper, Kremer and Maskin (1996) assume that agents can be assigned to the manager or deputy roles, where $x^\theta y^{1-\theta}$ is output when x is the manager and y the deputy, and $\theta \in [0, 1/2]$.[5] As a unisex model, match output is then equal to the maximum of two SPM functions:

$$\phi(x, y|\theta) \equiv \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\} \tag{7}$$

But SPM is preserved by the minimum operator, and not the maximum operator, and so this function is neither SPM nor SBM. Indeed, consider any match (x, y) for $0 < x < y$. If $z = y/x$, the positive sorting minus negative sorting payoff difference is:

$$\phi(y, y|\theta) + \phi(x, x|\theta) - 2\phi(x, y|\theta) = y + zy - 2(zy)^\theta y^{1-\theta} \gtreqless 0 \quad \text{as } \theta \gtreqless \theta^*(z)$$

where $\theta^*(z) = (\log(1 + z) - \log(2))/\log(z)$ is an increasing function from $(0, 1)$ onto $(0, 1/2)$. That is, PAM beats NAM among the types $\{x, y\}$ when types are far apart (small z), while NAM beats PAM when types are close together (z near 1).

(f) DYNAMIC MATCHING WITH EVOLVING TYPES.[6] Assume pairwise matching in periods one and two. Production is the symmetric, increasing and SPM function $\phi^0(x, y)$. But types evolve: If types x and y match in period one, they enter period two as type x' and y' with chances $\tau(x'|x, y)$ and $\tau(y'|y, x)$. Given SPM output, PAM is statically optimal in period two. But in period one, the social planner weights output by $(1 - \delta, \delta)$, so that the payoff to an (x, y) match is:

$$\phi(x, y) = (1 - \delta)\phi^0(x, y) + \tfrac{\delta}{2}\left[\int \phi^0(x', x')\tau(x'|x, y)dx' + \int \phi^0(y', y')\tau(y'|x, y)dy'\right] \tag{8}$$

Becker's Result lacks bite: ϕ need not be SPM even if ϕ^0 is increasing and SPM.

[5] More generally, we could allow for the production $\max\{g(x, y|\theta), g(y, x|\theta)\}$.
[6] This is based on Anderson and Smith (2012), which explored matching with evolving types.

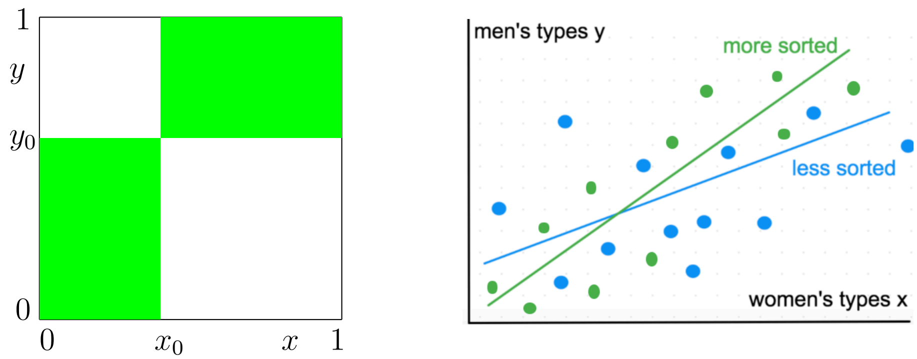


Figure 3: **PQD Order.** PQD increases for cdfs on $[0,1]^2$ raise the probability mass on all lower left rectangles with corner vertices $(0,0)$ and (x_0, y_0), and so on all upper right rectangle with corner vertices (x_0, y_0) and $(1,1)$. The right panel depicts Lemma 2(c).

4 How to Measure Increasing Sorting

Positive quadrant dependence (PQD) partially orders bivariate probability distributions $M_1, M_2 \in \mathcal{M}(G,H)$. We call M_2 *PQD higher than* M_1, or $M_2 \succeq_{PQD} M_1$, if $M_2(x,y) \geq M_1(x,y)$ for all x, y. So M_2 puts more weight than M_1 on all lower (southwest) orthants. Since M_1 and M_2 share marginals, M_2 puts more weight than M_1 on all upper (northeast) orthants too. Easily, all match cdf's are sandwiched by NAM and PAM: $\max(G(x)+H(y)-1, 0) \leq M(x,y) \leq \min(G(x), H(y))$ (the *Fréchet Bounds*). As (1) notes, PQD only partially orders the six possible pure matchings on three types.

A key known result is that the PQD and SPM orders coincide in $\mathbb{R}^2$, i.e. *increases in the PQD order increase (reduce) the total output for any SPM (SBM) function* ϕ:[7]

$$M_2 \succeq_{PQD} M_1 \quad \Leftrightarrow \quad \int \phi(x,y) M_2(dx,dy) \geq \int \phi(x,y) M_1(dx,dy) \qquad \forall \phi \text{ SPM} \tag{9}$$

Observe that, *by Lemma 1 and the Fréchet Bounds, Becker's Result follows from* (9). For since SPM implies globally nonnegative synergy, $\phi_{xy} \geq 0$, output is highest when the match cdf $M(x,y)$ is maximal — namely, PAM, as it dominates all other matchings in the PQD order. Similarly, SBM implies globally nonpositive synergy, $\phi_{xy} \leq 0$, and thus output is highest when the match cdf $M(x,y)$ is minimal, namely, for NAM.

The PQD sorting measure shows up in some economically relevant measures:

Lemma 2. *Fix increasing functions u and v. Given a PQD order upward shift:*
(a) the average geometric distance $E[|u(X)-v(Y)|^\gamma]$ for matched types falls, if $\gamma \geq 1$;
(b) the covariance $E_M[u(X)v(Y)] - E[u(X)]E[v(Y)]$ across matched pairs rises;
(c) the coefficient in a linear regression of $v(y)$ on $u(x)$ across matched pairs rises.

[7] Lehmann (1973) introduced the PQD order. See 9.A.17 in Shaked and Shanthikumar (2007).

Lemma 2 illustrates that the PQD order is scale invariant. To wit, if we claim that educational sorting rises in the PQD order, then it does so regardless of whether it is measured in highest degree attained, years of schooling, log years of schooling, etc.

Proof of (a): By inequality (9) it suffices that $|u(x) - v(y)|^\gamma$ is SBM for all $\gamma \geq 1$. Since $-g(u-v)$ is SPM for all convex g, by Lemma 2.6.2-(b) in Topkis (1998), we have $-|u-v|^\gamma$ SPM for all $\gamma \geq 1$. Thus, $|u(x) - v(y)|^\gamma$ is SBM for all increasing u and v.

Proof of (b): Since the marginal distributions on X and Y is constant for all $M \in \mathcal{M}(G, H)$, and $u(x)v(y)$ is supermodular for all increasing u and v, the covariance $E_M[XY] - E[X]E[Y]$ between matched types increases in the PQD order by (9).

Proof of (c): The coefficient $c_1 = cov(u(X)v(Y))/var(v(X))$ in the univariate match partner regression $v(y) = c_0 + c_1 u(x)$ increases in the PQD order, by part (b). $\square$

When the optimal matchings $M^*(\theta_1)$ and $M^*(\theta_2)$ are each unique, we say *sorting is higher* at θ_2 than θ_1 if $M^*(\theta_2) \succeq_{PQD} M^*(\theta_1)$. We often just say *sorting increases*.

For a useful counterpoint, posit a uniform type distribution on $[0, 1]$. Assume that every $x \leq 1/2$ matches with $x+1/2$. Since it is increasing *on the domain of larger match partners*, Legros and Newman (2002) call this matching "monotone". Notice that this matching maximizes the average distance between partners. To wit, it *minimizes* total match output for the supermodular production function $f(x, y) = 1 - |x - y|$.

5 Nowhere Decreasing Sorting

Given the extreme NAM and PAM cases of Becker's Result under SBM and SPM, we first ask what happens if synergy increases everywhere. One might conjecture that sorting increases, but Figure 2 refutes this conjecture — the uniquely optimal matching oscillates back and forth between NAM1 and NAM3 as synergy increases.

By Lemma 1, the optimal matching maximizes $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta) M_{ij}$ with finitely many types and $\int \phi_{12}(x, y|\theta) M(x, y) dx dy$ for continuous types. Were total output (3) quasi-supermodular in M, and the constraint set a lattice, a single-crossing property in (M, θ) would imply that the set of maximizers $\mathcal{M}^*(\theta)$ increases in the *strong set order* (SSO) (Milgrom and Shannon, 1994).[8] But it is known that matching cdf's are not a lattice,[9] and so quasi-supermodularity must fail. A new method of attack is required.

[8] $\mathcal{M}_2 \succeq \mathcal{M}_1$ in the SSO if $M_1 \vee M_2 \in \mathcal{M}_2$ and $M_1 \wedge M_2 \in \mathcal{M}_1$ for all $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$. Given M_1, M_2, the *join* $M_1 \vee M_2$ is their supremum, and their infimum is the *meet* $M_1 \wedge M_2$.

[9] By (1), NAM1 and NAM3 are both upper bounds for PAM2 and PAM4, but there is no pure least upper bound. More strongly, PQD does not induce a lattice, as there is no least mixed least upper bound, M for PAM2 and PAM4. As shown in Proposition 4.12 in Müller and Scarsini (2006): If M dominates PAM2 and PAM4, then $M(2, 1) \geq 1/3$ and $M(1, 2) \geq 1/3$, but $M(1, 1) = 0$ if NAM1 and NAM3 dominate M. So then $M(2, 2) = 2/3$, but then NAM1 cannot PQD dominate M.

Theorem 1 in Appendix B.1 is a monotone comparative statics result for partially ordered sets given only a single crossing property in (M, θ). For our case, the optimal matching distribution cannot fall in the PQD order as θ rises, but might incomparably shift. We say that *sorting is nowhere decreasing* in θ if for all $\theta_2 \succeq \theta_1$, whenever $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked $M_1 \succeq_{PQD} M_2$, we have $M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$. A *nowhere increasing* correspondence is analogously defined.

We say that *weighted synergy is upcrossing*[10] in θ if the following is upcrossing in θ:

- $\int \phi_{12}(x, y|\theta)\lambda(x, y)dxdy$ for all nonnegative (measurable)[11] functions λ on $[0, 1]^2$
- $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\lambda_{ij}$ with types in $\mathbb{Z}_n$, for all positive weights $\lambda \in \mathbb{R}_+^{(n-1)^2}$

Proviso (★): *Synergy ϕ_{12} or s_{ij} is nondecreasing or linear in θ with negative intercept.*

Proposition 1 (Nowhere Decreasing Sorting). *Sorting is nowhere decreasing in θ if weighted synergy is upcrossing in θ — and therefore if Proviso (★) holds.*

Proof: First, $M' \succeq_{PQD} M$ iff $\lambda \equiv M' - M \geq 0$. Since weighted synergy is upcrossing:

$$\begin{array}{c} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M_{ij}) \geq (>)\ 0 \Rightarrow \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M_{ij}) \geq (>)\ 0 \\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta)(M' - M) \geq (>)\ 0 \ \Rightarrow\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta')(M' - M) \geq (>)\ 0 \end{array} \tag{10}$$

Match output is single crossing in (M, θ) in the finite and continuous cases, by Lemma 1. Then the optimal matching $\mathcal{M}^*(\theta)$ (in the space of feasible matchings $\mathcal{M}(G, H)$) is nowhere decreasing in the state θ, by Theorem 1 in Appendix B. So monotone synergy yields upcrossing weighted synergy, as does linear synergy, by Claim 1 in §C.1. □

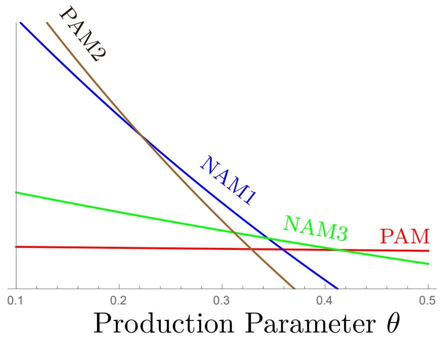
Application: Production with Defined-roles. Now return to Kremer and Maskin (1996) in §3. Their production function is not differentiable, and so is not subject to our theory. But we can smoothly approximate their production function by:

$$\phi(x, y) = x^\theta y^\theta \left(x^\varrho + y^\rho\right)^{\frac{1-2\theta}{\varrho}} \quad \to \quad \max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\} \quad \text{as } \varrho \to \infty \tag{11}$$

In Appendix C.6, we deduce PAM iff $\varrho \leq (1 - 2\theta)^{-1}$. In the $\varrho \to \infty$ limit (Kremer and Maskin, 1996), ϕ is never SPM, nor does PAM arise. Without solving for the optimal matching, we sign the sorting comparative statics when $\varrho > (1 - 2\theta)^{-1}$. In Appendix C.6, we prove that weighted synergy is upcrossing in θ and downcrossing in ϱ, and so *sorting is nowhere decreasing in θ and nowhere increasing in ϱ*. See Figure 4.

[10] Let Z be a partially ordered set. The function $\sigma : Z \mapsto \mathbb{R}$ is *upcrossing* if $\sigma(z) \geq (>)0$ implies $\sigma(z') \geq (>)0$ for $z' \succeq z$, *downcrossing* if $-\sigma$ is upcrossing. Similarly, σ is strictly upcrossing if $\sigma(z) \geq 0$ implies $\sigma(z') > 0$ for all $z' \succ z$, with strictly downcrossing defined analoguously.

[11] To save space, we henceforth assume measurable sets for integrals whenever needed.



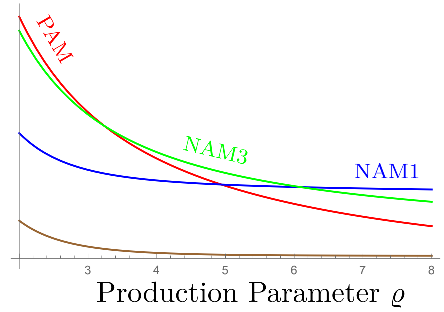


Figure 4: **Kremer-Maskin Payoffs.** We plot the payoffs for three type $(x, y) \in \{1, 2, 6.5\}$ matchings for (11) (NAM is dominated by PAM2=PAM4). On the left, matching is nowhere decreasing, from PAM2=PAM4 (brown), to NAM1 (blue), to NAM3 (green), to PAM (red) as θ rises ($\rho = 100$). On the right, matching is nowhere increasing, from PAM (red), to NAM3 (green), to NAM1 (blue), as ϱ rises ($\theta = 0.32$).

6 Increasing Sorting and Production Changes

6.1 Increasing Sorting with Finitely Many Types

We first pursue an increasing sorting theory for finitely many types $i, j = 1, 2, \ldots, n$.

Lemma 3. *An optimal matching is generically unique and pure for finite types.*

Proof: The optimal matching is generically unique, by Koopmans and Beckmann (1957). A non-pure matching M is a mixture $M = \sum_{\ell=1}^{L} \lambda_\ell M_k$ over $L \leq n+1$ pure matchings $M_1, \ldots, M_n$, with $\lambda_\ell > 0$ and $\sum_\ell \lambda_\ell = 1$.[12] As the objective function (3) is linear, if the non-pure matching M is optimal, so is each pure matching M_ℓ. $\square$

We now introduce a cross-sectional assumption. The *sorting premium* $S(r|\theta)$ is the total synergy on the lattice *rectangle* $r = (i_1, j_1, i_2, j_2)$, i.e. all couples $\{(i,j) \in \mathbb{Z}_n^2 : (i_1, j_1) \leq (i,j) \leq (i_2, j_2)\}$. This is also the equally weighted synergy sum. More simply, we have:

$$S(r|\theta) \equiv f_{i_1 j_1}(\theta) + f_{i_2 j_2}(\theta) - f_{i_1 j_2}(\theta) - f_{i_2 j_1}(\theta)$$

By (2), production is SPM (SBM) if all sorting premia are nonnegative (non-positive).[13]

Rectangle r dominates r' in the northeast order, written $r \succeq_{NE} r'$, if all coordinates are weakly higher and $r \succ_{NE} r'$ if at least one is strictly higher. The *sorting premium is upcrossing (downcrossing) in types* if $S(r|\theta)$ is upcrossing (downcrossing) in r, for all θ. As we can reverse-order types, we just develop our theory for the upcrossing case.

Next, we consider a time series assumption weaker than monotone synergy. The *total synergy* on a set of couples $K \subseteq \mathbb{Z}_n^2$ is the sum of synergies s_{ij} for all $(i,j) \in K$.

[12] This follows from Carathéodory's Theorem. It says that non-empty convex compact subset $\mathcal{X} \subset \mathbb{R}^n$ are weighted averages of extreme points of $\mathcal{X}$. The extreme points here are the pure matchings.

[13] By Proposition 1, sorting is nowhere decreasing if weighted synergy is upcrossing in θ. The sorting premium uses a weighting function that places unit density on a rectangle, and zero weight elsewhere.

Match Payoffs: $f_{ij}(\theta') \to f_{ij}(\theta'')$

	x_1	x_2	x_3
y_3	6	6	**11**
y_2	**4**	6	6
y_1	0	**4**	6

$\to$

	x_1	x_2	x_3
y_3	**7**	6	11
y_2	4	**6**	6
y_1	0	4	**7**

Synergy: $s_{ij}(\theta') \to s_{ij}(\theta'')$

	x_1x_2	x_2x_3
y_2y_3	-2	5
y_1y_2	-2	-2

$\to$

	x_1x_2	x_2x_3
y_2y_3	-3	5
y_1y_2	-2	-3

Figure 5: **Falling Matching with an Upcrossing Sorting Premium in Types and θ.** Left: The unique efficient matching falls from NAM3 to NAM as θ' shifts up to θ''. Right: The sorting premium S is upcrossing in rectangles r for each θ, and the signs of $S(r|\theta')$ and $S(r|\theta'')$ coincide for all r; thus, S is upcrossing from θ' to θ''. But Lemma 4 does not apply, as the total synergy on $\mathbb{Z}_3^2 \setminus \{(1,1)\}$ falls from 1 to -1.

Total synergy is upcrossing in θ if the synergy sum is upcrossing for all sets $K \subseteq \mathbb{Z}_n^2$.

Lemma 4. *Posit finitely many types and a generic production function with a unique optimal matching at $\theta_2 \succ \theta_1$. Sorting is PQD higher at θ_2 than θ_1 if total synergy is upcrossing in θ and the sorting premium is one-crossing in types.*

To see the necessity of the cross-sectional assumption, consider the example in §3(e) from Kremer and Maskin (1996). In Appendix C.6, we show that its weighted synergy is upcrossing in θ — and thus, so too is total synergy. But its sorting premium is not one-crossing in types.[14] And as seen in Figure 4, sorting is not increasing in θ.

Lemma 4 relaxes the time series assumption of Proposition 1 — as total synergy is a special case of weighted synergy. To compensate for this weaker premise, it introduces a cross-sectional restriction that the sorting premium is one-crossing. Could we impose all assumptions on the sorting premium, positing $S(r|\theta)$ upcrossing in r and θ? No. In Figure 5, the sorting premium is upcrossing in r and θ but sorting falls as θ rises.

We prove Lemma 4 — our theoretical core in C.2 — by induction on the number of types. Let's see how time series and cross sectional assumptions jointly force increasing sorting with $n = 3$ types. First, the optimal matching cannot fall in the PQD order — like from PAM4 to NAM. As woman 1 is matched to man 3 in NAM and PAM4, the PAM4 payoff exceeds the NAM payoff by the sorting premium among women 2, 3 and men 1, 2, i.e. the synergy $s_{21}(\theta)$. If NAM and PAM4 are uniquely optimal respectively at $\theta'' \succ \theta'$, then $s_{21}(\theta'') < 0 < s_{21}(\theta')$, violating upcrossing total synergy.

To see that non-PQD comparable shifts cannot happen requires both time series and cross sectional reasoning. Figure 6 traces the logic for the two possible non-PQD comparable transitions: both NAM1 to NAM3, and PAM2 to PAM4 (recall (1)).

[14] As $(x, y) \mapsto x^\theta y^{1-\theta}$ and $(x, y) \mapsto x^{1-\theta} y^\theta$ are supermodular, the sorting premium is positive for rectangles above or below the diagonal. But it is positive for small rectangles straddling the diagonal: If $y > x$ and $0 < \theta < 1/2$, the sorting premium is $f(x, x|\theta) + f(y, y|\theta) - 2f(x, y|\theta) = x + y - 2x^\theta y^{1-\theta} < 0$ for $1 < y/x < 1 + \varepsilon$. For it vanishes at $y = x$, and its y derivative $1 - 2(1-\theta)(x/y)^\theta < 0$ for small $\varepsilon > 0$.

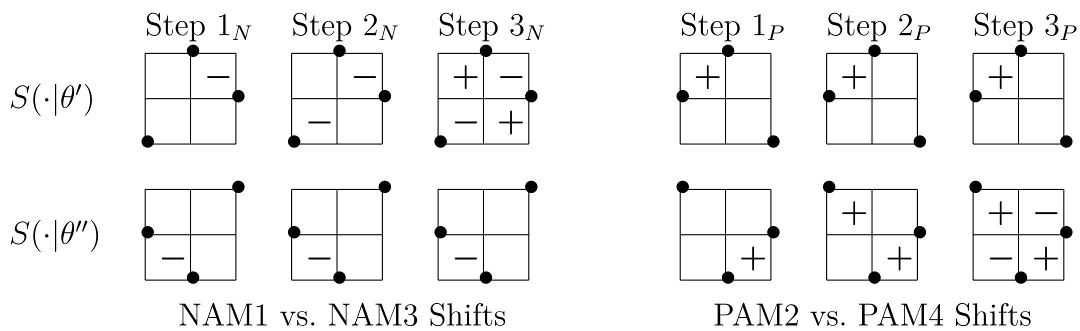


Figure 6: **Precluding Unranked Shifts with $n = 3$ and Nonzero Synergies.** NAM1 at θ' and NAM3 at θ'' is impossible, as is PAM2 at θ' and PAM4 at θ''. The synergy signs in Steps 1_N and 1_P reflect local optimality. Step 2_N deduces $s_{11}(\theta') < 0$ via upcrossing synergy from θ'' to θ'. Given PAM on rectangles $r = (1, 1, 2, 3), (1, 1, 3, 2)$ at θ', local optimality implies $S(r|\theta') > 0$. As the sorting premium is the sum of synergies, the synergy signs in Step 3_N follow — ruling out $S(r|\theta')$ one-crossing in r, a contradiction. Next, Step 2_P deduces $s_{12}(\theta'') > 0$ via upcrossing synergy from θ' to θ''. Given NAM on rectangles $r = (1, 1, 2, 3), (1, 1, 3, 2)$ at θ', local optimality implies $S(r|\theta') < 0$. Since the sorting premium is the sum of synergies, we can fully sign s_{ij}. This sign pattern in Step 3_P violates $S(r|\theta'')$ one-crossing in r, a contradiction.

Application to Dynamic Matching with Evolving Types. Herkenhoff, Lise, Menzio, and Phillips (2018) [HLMP] explore a new finite type infinite horizon frictional matching model. Here, we consider the analogous frictionless two period model in §3(f). HLMP assume a production function $f^0_{ij} = i + j + (i^p + j^p)^{1/p}$. If types i and j match, then type i increments to $i+1$ in the next period with probability $\alpha_{ij} \equiv \alpha + \bar{\alpha}\max(0, j-i) - \underline{\alpha}\max(0, i-j)$. Altogether, the time-0 payoff function (8) is:

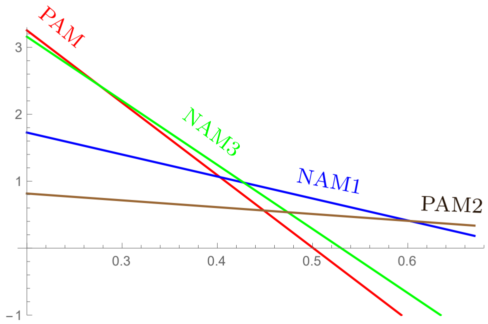
$$\begin{aligned} f_{ij} &= (1-\delta)f^0_{ij} + \delta\left(\alpha_{ij}\tfrac{1}{2}f^0_{i+1,i+1} + (1-\alpha_{ij})\tfrac{1}{2}f^0_{ii} + \alpha_{ji}\tfrac{1}{2}f^0_{j+1,j+1} + (1-\alpha_{ji})\tfrac{1}{2}f^0_{jj}\right) \\ &= (1-\delta)f^0_{ij} + \delta(1 + 2^{\frac{1-p}{p}})\left(i + j + 2\alpha + (\bar{\alpha} - \underline{\alpha})|i-j|\right) \end{aligned} \tag{12}$$

where the payoff function reflects how PAM emerges in period two (HLMP empirically find f SPM, via $p < 1$), and thus the continuation payoff at type i is $f_{ii} = i(2 + 2^{1/p})$.

Since the dynamic term $|i - j|$ is SBM, dynamic considerations favor NAM when $\bar{\alpha} > \underline{\alpha}$, as HLMP empirically estimate. Synergy, and so weighted synergy, is falling in δ and $\bar{\alpha} - \underline{\alpha}$. Proposition 1 implies that sorting is nowhere increasing in δ and $\bar{\alpha} - \underline{\alpha}$.

Next, consider Lemma 4. Its cross sectional premise fails, since the sorting premium falls approaching the diagonal from above or below — as already shown for the example of §3(e).[15] Specifically, sorting can be non-monotone in δ and $\bar{\alpha} - \underline{\alpha}$ (as in Figure 7).

[15] For while the productive sorting premium f^0_{ij} is increasing in types, the dynamic sorting premium vanishes for rectangles wholly above or below the diagonal (since $|i - j|$ is linear $i_1 < i_2 \leq j_1 < j_2$ or $j_1 < j_2 \leq i_1 < i_2$), but is strictly negative for any rectangle straddling the diagonal (where $|i - j|$ is SBM). Altogether, whenever the sorting premium is not globally positive, it is positive above and below the diagonal, but negative for small rectangles straddling the diagonal.



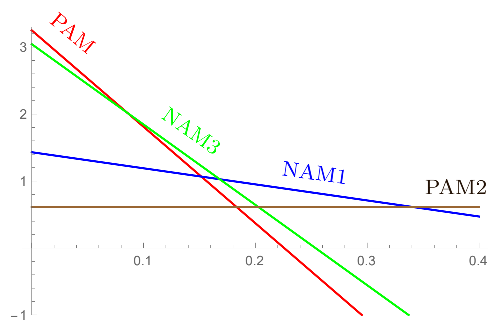


Figure 7: **HLMP Match Payoffs.** We plot the payoff premium over NAM for all three type $x, y \in \{1,2,7\}$ matchings for (12) with $\rho = 1/2$, as δ rises (left) and $\bar{\alpha}-\underline{\alpha}$ rises (right). In both graphs, the optimal matching changes from PAM (red), to NAM3 (green), to NAM1 (blue), to PAM2 = PAM4 (brown), as the x- axis parameter rises.

This nonmonotonicity is a robust feature of models whose dynamic term is a convex function of the type difference $g(i-j)$. For then $-g''(i-j)<0$ cannot be co-monotone in i and j, except in the knife-edged quadratic case $g'''=0$.

But in an ad hoc model with falling dynamic synergy, as with a transition chance $\alpha_{ij} = \alpha + j - ij$, since static synergy falls for the CES production f^0, the sorting premium is downcrossing in types. In this case, sorting falls in δ, by Lemma 4.

6.2 Increasing Sorting with Linear or Monotone Synergy

We now add to Lemma 4 with the cross-sectional strength of Proviso (★). Define the sorting premium[16] on rectangles $(x_1, y_1, x_2, y_2) \in [0,1]^4$, where $x_1 < x_2$ and $y_1 < y_2$:

$$S(x_1, x_2, y_1, y_2|\theta) = \phi(x_1, y_1|\theta) + \phi(x_2, y_2|\theta) - \phi(x_1, y_2|\theta) - \phi(x_2, y_1|\theta)$$

In terms of the x-*marginal product increments* $\Delta_x(x|y_1, y_2, \theta) \equiv \phi_1(x, y_2|\theta) - \phi_1(x, y_1|\theta)$:

$$S(x_1, x_2, y_1, y_2|\theta) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y|\theta) dx dy = \int_0^1 \Delta_x(x|y_1, y_2, \theta) \mathbb{1}_{x \in [x_1, x_2]} dx \tag{13}$$

Any indicator function $\mathbb{1}_{x \in [x_1, x_2]}$ is a log-supermodular function of (x, x_1) and (x, x_2).[17] By Karlin and Rubin's classic 1956 result, if $\Delta_x(x|y_1, y_2, \theta)$ is upcrossing in x, then the sorting premium last integral in (13) is upcrossing in x_1 and x_2, and so in (x_1, x_2). The sorting premium is also the integrated *y-marginal product increment* $\Delta_y(y|x_1, x_2, \theta) \equiv \phi_2(x_2, y|\theta) - \phi_2(x_1, y|\theta)$, and so upcrossing in (y_1, y_2). All told, the sorting premium is upcrossing in types if both x- and y-marginal product increments are upcrossing.

[16]For simplicity, we still use the S notation for the sorting premium with a continuum of types.

[17]$\phi(x, y) \geq 0$ is *log-supermodular* (LSPM) if $\phi(x', y')\phi(x'', y'') \geq \phi(x', y'')\phi(x'', y')$ for all $x' \leq x''$ and $y' \leq y''$. Easily, we can check that the indicator is LSPM: If $x \in [x_1, x_2]$ and $x' \in [x_1', x_2']$ then $\max(x, x') \in [\max(x_1, x_1'), \max(x_2, x_2')]$ and $\min(x, x') \in [\min(x_1, x_1'), \min(x_2, x_2')]$.

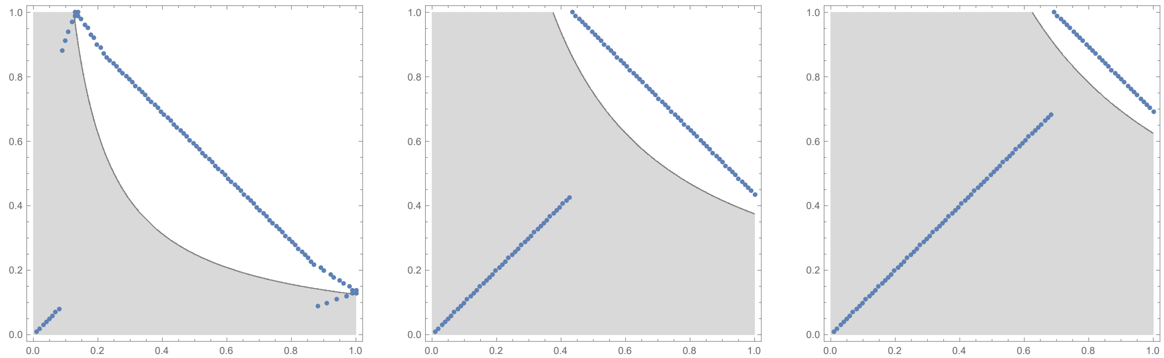


Figure 8: **Increasing Sorting with Quadratic Production.** We depict positive synergy (shade) and optimally matched pairs (blue dots) for a uniform distribution on 100 types. Sorting increases left to right in (α, β). The left plot is drawn for $(\alpha, \beta) = (0.5, -1)$, the middle for $(\alpha, \beta) = (1.5, -1)$, and the right for $(\alpha, \beta) = (1.5, -0.6)$.

Proposition 2. *Posit finitely many types and a one-crossing sorting premium in types, or a continuum of types, with absolutely continuous G (or H) and strictly one-crossing*[18] *x- and y-marginal product increments. Sorting is increasing in θ, given Proviso ($\bigstar$).*

Finite Type Proof: By Proviso ($\bigstar$), total synergy is upcrossing in θ (Claim 1), while the sorting premium is assumed one-crossing in types. Sorting increases in θ, by Lemma 4.

Continuum Types Proof Sketch: Fix $\theta_2 \succeq \theta_1$. In §C.3, we perturb payoffs for a sequence of finite type models to secure, by Lemma 4, unique PQD-ranked optimal matchings $M^n(\theta_2) \succeq_{PQD} M^n(\theta_1)$. Also, these matchings converge to limit matchings $M^*(\theta_1)$ and $M^*(\theta_2)$ optimal in the continuum model. By continuity, $M^*(\theta_2) \succeq_{PQD} M^*(\theta_1)$.

Next, strictly upcrossing marginal product increments implies unique optimal matchings, given our absolute continuity assumptions. Assume G absolutely continuous and $\Delta_x(x|y_1, y_2)$ strictly upcrossing in x. For insight into uniqueness, decentralize the optimal matching by competitive wage functions $v(x)$ and $w(y)$. Then x and y match if $x = \arg\max_{x'} \phi(x', y) - v(x')$ and $y' = \arg\max_{y'} \phi(x, y') - w(y')$. Notice that one simple nonuniqueness cannot occur, when both sortings of two women $x_1 < x_2$ and men $y_1 < y_2$ into couples are optimal. In this case, an optimal partner for y obeys two sets of FOC: $v'(x_1) = \phi_1(x_1, y_2) = \phi_1(x_1, y_1)$ and $v'(x_2) = \phi_1(x_2, y_1) = \phi_1(x_2, y_2)$. But this contradicts $\Delta_x(x|y_2, y_1) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x. □

Application to Quadratic or Cubic Production. For quadratic production $\phi(x, y) = \alpha xy + \beta(xy)^2$, match synergy $\phi_{12} = \alpha + 4\beta xy$ is strictly increasing in α and β. Synergy is also strictly increasing in types when $\beta > 0$, and decreasing in types when $\beta < 0$. So sorting is increasing in α and β for all $\beta \neq 0$, by Proposition 2 (Figure 8).[19]

[18] $\Upsilon : \mathbb{R} \mapsto \mathbb{R}$ is *strictly upcrossing* if $\Upsilon(x) \geq 0 \Rightarrow \Upsilon(x') > 0$, for all $x' > x$. Easily, it is upcrossing.

[19] The left plot suggests that the unique continuum matching is not always pure, but fortunately, none of our continuum model sorting results require purity.

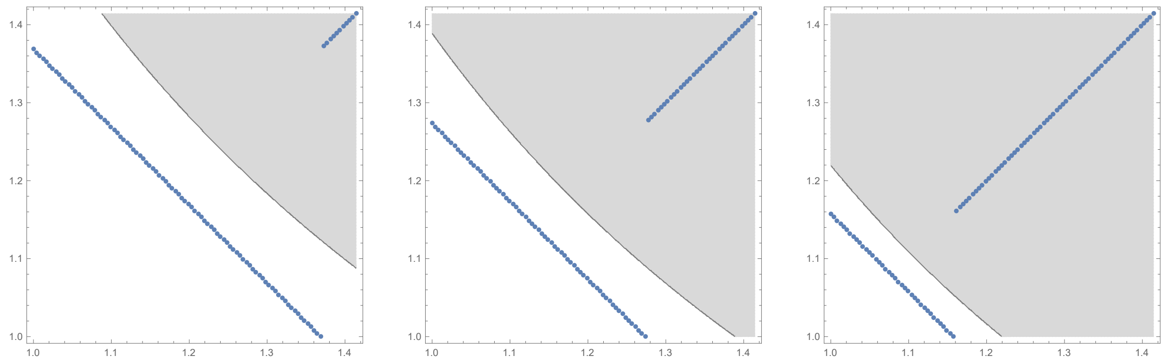


Figure 9: **Increasing Sorting in the Principal-Agent Model.** NAM is optimal for low dis-utility of effort θ, PAM for high θ, and the optimal matching is mixed for intermediate θ. These graphs depict optimally matched pairs (blue dots) for a discrete uniform distribution on 100 types of principals and agents. The left plot is drawn for $\theta = 0.65$, the middle for $\theta = 0.72$, and the right for $\theta = 0.82$. In all plots, the matching obeys local optimality — if the matching slopes up, then synergy is positive (shaded). In all plots, the reverse implication fails due to subtle global optimality considerations.

With cubic production $\phi = \alpha xy + \beta(xy)^2 + \gamma(xy)^3$, the analysis is more nuanced. Synergy $\phi_{12} = \alpha + 4\beta xy + 9\gamma(xy)^2$ is increasing in α, β, and γ; and thus, sorting is nowhere decreasing in all parameters, by Proposition 1. Also, synergy falls in types when $\beta, \gamma < 0$, and rises in types when $\beta, \gamma > 0$ — so that Proposition 2 predicts sorting increases in α, β, and γ. But when $\beta\gamma < 0$, synergy need not be one-crossing in types, and sorting is nowhere decreasing, but not generally monotone in α, β, or γ.

Application to the Principal-Agent Matching with Moral Hazard. We show that sorting is monotone in θ in the Serfes model, provided extremal types obey $\bar{x}\bar{y} \leq 2\underline{xy}$ (‡). Assume $\theta' > \theta$. If $\theta\bar{x}\bar{y} < 1$, then synergy (4) is globally negative at θ, and so NAM uniquely optimal. If $\theta'\underline{xy} > 1$, then synergy is globally positive at θ', and so PAM uniquely optimal. In both cases, sorting is weakly higher at θ' then θ.

Assume $\theta'\underline{xy} \leq 1 < \theta\bar{x}\bar{y}$. Then by (‡) we have $\theta'\bar{x}\bar{y} \leq 2$, and since the function $(1-t)/(1+t)^3$ is increasing for $t \in (0, 2]$, synergy (4) is increasing in the product θxy. Thus, synergy is both proportionately upcrossing and strictly upcrossing in (x, y, θ). Altogether, NAM obtains for $\theta \leq (\bar{x}\bar{y})^{-1}$, PAM for $\theta \geq (\underline{xy})^{-1}$, and sorting is increasing in θ between these two extremes, by Proposition 2, as seen in Figure 9.

Application to Group Lending with Adverse Selection. The production function (5) obeys the premise of Proposition 2. Indeed, differentiating (5) yields:

$$\phi_1(x, y) = \frac{(\pi - c - d)(1 - \delta y^2) + 2cy}{(1 - \delta x - \delta y + \delta xy)^2} > 0 \tag{14}$$

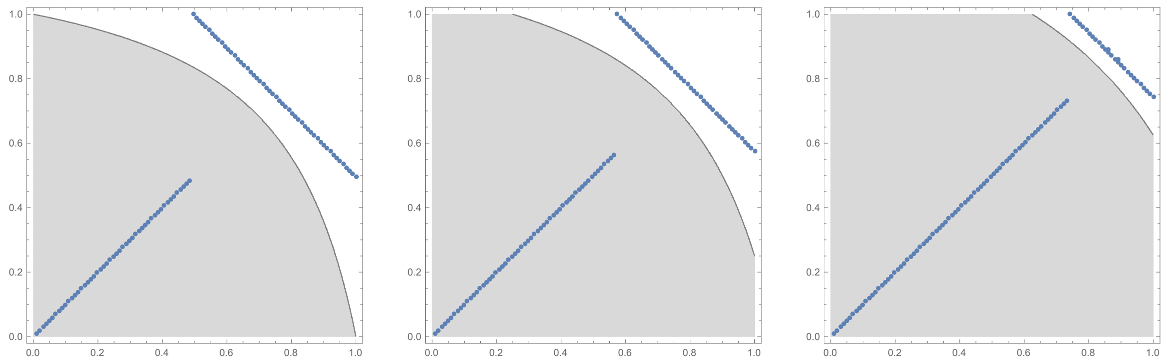


Figure 10: **Increasing Sorting in the Group Lending Model.** If parameters obey $\delta < c/(c+\pi-d)$, the optimal is PAM; otherwise, the optimal matching is mixed. These graphs depict optimally matched man-woman pairs (blue dots) assuming a uniform distribution on 100 types for $\delta = 0.8$. The left matching is drawn for $(\pi, c, d) = (10, 0, 2)$, the middle for $(\pi, c, d) = (10, 2, 2)$, and the right for $(\pi, c, d) = (4, 2, 2)$.

As $\partial[\phi_1(x, y_2)/\phi_1(x, y_1)]/\partial x < 0$ for all $y_2 > y_1$,[20] the function $\phi_1(x, y_2)/\phi_1(x, y_1) - 1$ is downcrossing. As $\phi_1 > 0$, the x-marginal product increment is strictly downcrossing in x. Symmetrically, the y-marginal product increment is strictly downcrossing in y.

Next consider synergy as a function of $\theta = (c, d, \pi)$. Differentiating (14) yields:

$$\phi_{12}(x, y|\theta) = ca(x, y) + (\pi - d)b(x, y)$$

for functions $a(x, y) > 0$ and $b(x, y) \in \mathbb{R}$. Synergy is increasing in c and linear in $\pi - d$; and thus, obeys Proviso ($\bigstar$). Altogether, sorting rises in repayments (c, d) and falls in net payoff $\pi - d$, by Proposition 2 as illustrated in Figure 10.

On the other hand, sorting is not monotone in the discount factor δ. By the derivative of (14), PAM obtains with enough impatience $\delta \le c/(c + \pi - d)$ *and* perfect patience $\delta = 1$,[21] as the sorting premium is globally positive. For intermediate $\delta \in (c/(c+\pi-d), 1)$, the sorting premium is not globally positive and PAM is not optimal.[22]

6.3 Increasing Sorting & Proportionately Upcrossing Synergy

Once more, we build on Lemma 4. We now develop a fully local approach to synergy aggregation that simultaneously secures the needed cross-sectional condition and time

[20]Indeed, $D_x[\log(\phi_1(x, y_2)/\phi_1(x, y_1))] = \frac{2\delta(1-\delta)(y_1-y_2)}{[1-\delta(x+y_1(1-x))][1-\delta(x+y_2(1-x))]}$. The numerator is negative by $\delta \in (0, 1)$ and $y_1 < y_2$, and the denominator is positive since x, y_1, and y_2 are probabilities.

[21]Payoffs are well defined when the implicit discount factor $\delta(1 - (1 - x)(1 - y)) < 1$, where $(1 - x)(1 - y)$ is the chance that both projects fail, resulting in the borrowing partnership defaulting.

[22]Indeed, when $\delta \in (c/(c + \pi - d), 1)$, the symmetric synergy function $\phi_{12}(x, x)$ is strictly negative for x close to 1. Thus, cross matching types x and $x + \varepsilon$ beats sorting them, for high x and low ε.

series conditions — a sorting premium one-crossing in types, and total synergy upcrossing. Notice that for averages on very small sets, upcrossing total synergy is *necessary* for upcrossing total synergy. We now seek a condition that renders it sufficient.

Our theory here exploits a new joint cross-sectional and time series upcrossing aggregation result in §B.2. Given a real or integer lattice $(Z, \succeq)$ and a poset $(\mathcal{T}, \succeq)$,[23] the function $\sigma : Z \times \mathcal{T} \to \mathbb{R}$ is *proportionately upcrossing* if for all $z, z' \in Z$ and $t' \succeq t$:

$$\sigma^-(z \wedge z', t)\sigma^+(z \vee z', t') \geq \sigma^-(z, t')\sigma^+(z', t) \tag{15}$$

Next, say that *synergy is proportionately upcrossing* if $\phi_{12}(x, y|\theta)$ is proportionately upcrossing in $z = (x, y)$ and $t = \theta$ with the usual vector order on $Z = [0, 1]^2$ or the reverse vector order; namely, $(x, y) \succeq (x', y')$ *iff* $(x, y) \leq (x', y')$. Analogous definitions apply to the finite type case with $\sigma(i, j, t) = s_{ij}(t)$ and $z = (i, j) \in Z \equiv \mathbb{Z}_n^2$.

Consider the cross-sectional implications of (15). With three types, synergies $s_{11} = -1, s_{12} = 2, s_{21} = -4, s_{22} = 3$ are strictly upcrossing in i and j. But the associated sorting premium is not upcrossing in types — for instance, $s_{11} + s_{12} = 1 > -1 = s_{21} + s_{22}$. And indeed, it is not proportionately upcrossing, since at $z = (2, 1)$ and $z' = (1, 2)$, we have $s^-_{z\wedge z'} s^+_{z\vee z'} = 3 < 8 = s^-_z s^+_{z'}$. Intuitively, moving up in (i, j) space, the proportionate gain in negative synergy swamps that in positive synergy $s_{21}/s_{11} = 4 > 3/2 = s_{22}/s_{12}$. In general, positive synergy proportionately rises more than the negative synergy rises.[24]

Next, consider the time series implications of (15). Easily, any monotonic synergy function is proportionately upcrossing, since $(z \vee z', \theta') \succeq (z', \theta)$ implies $\sigma^+(z \vee z', \theta') \geq \sigma^+(z', \theta)$, and $(z, \theta') \succeq (z \wedge z', \theta)$ implies $\sigma^-(z \wedge z', \theta) \geq \sigma^-(z, \theta')$. Yet monotonicity is not implied — proportionately upcrossing synergy need not be upcrossing, but only *weakly upcrossing* in (z, θ); namely, $\sigma(z, \theta) > 0$ implies $\sigma(z', \theta') \geq 0$ for all $(z', \theta') \succeq (z, \theta)$.[25]

Proportional upcrossing (15) survives multiplication by any nonnegative LSPM function. In §B.3 we also show that *smoothly LSPM* functions, namely, whose pairwise cross-derivatives $\sigma(x, y, \theta) = \phi_{12}(x, y|\theta)$ obey $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$, are proportional upcrossing. For instance, given the production function $\phi(x, y|\theta) = e^{\theta(x-1)y}$, the synergy function $\phi_{xy} = (x - 1)\theta[2 + y\theta]\phi$ is smoothly LSPM, and therefore proportionately upcrossing.

Proposition 3. *If synergy is one-crossing in types, upcrossing in θ, and proportionately upcrossing, then sorting increases in θ in generic finite type models, or if G (or H) is absolutely continuous, and synergy is strictly one-crossing in types.*

[23] Denote by $f^+ \equiv \max(f, 0)$ and $f^- \equiv -\min(f, 0)$ the positive and negative parts of a function f.

[24] Assume negative synergy at couple z, and positive at a higher couple $z' = z \vee z' \succeq z \vee z' = z$. Then (15) says that: $\sigma^+(z', \theta')/\sigma^-(z, \theta') \geq \sigma^+(z', \theta)/\sigma^-(z, \theta)$.

[25] Fix $\theta = \theta'$ and suppress θ. If $z' \succeq z$, inequality (15) is an identity. If $z \succ z'$, inequality (15) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

The proof in §C.4 shows that total synergy is upcrossing in θ and that the sorting premium is upcrossing in types. By Lemma 4, sorting increases in θ for generic finite type models. The continuum model logic parallels that given after Proposition 2.

Proposition 3 also sheds light on two economic applications.

Application to the Partnership Model with Capital. How do capital changes impact sorting? Assume partners are more complementary than capital and labor, but not much so: $1/2 \le \rho < \eta < 1$ in (6). Synergy $\phi_{12}(x,y|\kappa)$ is the product of:

$$\varsigma(x,y|\kappa) \equiv (xy)^{\eta-1}\ell(x,y)^{\rho-2\eta}\phi(x,y)^{1-2\rho} \text{ and } \chi(x,y|\kappa) \equiv (\rho-\eta)\kappa^{\rho} + (1-\eta)\ell(x,y)^{\rho}$$

Then synergy $\phi_{12}(x,y|\kappa)$ is falling in κ, $\varsigma(x,y|\kappa) \ge 0$ is LSPM in (x,y), and $\chi(x,y|\kappa)$ is increasing in (x,y). Applying these properties in sequence, for all $\kappa \ge \kappa'$:

$$\begin{aligned}
\phi^-_{12}(z \wedge z'|\kappa)\phi^+_{12}(z \vee z'|\kappa') &\ge \phi^-_{12}(z \wedge z'|\kappa')\phi^+_{12}(z \vee z'|\kappa) \\
&\ge \varsigma(z,\kappa')\chi^-(z \wedge z'|\kappa')\varsigma(z',\kappa)\chi^+(z \vee z'|\kappa) \\
&\ge \varsigma(z,\kappa')\chi^-(z|\kappa')\varsigma(z',\kappa)\chi^+(z'|\kappa) \\
&= \phi^-_{12}(z|\kappa')\phi^+_{12}(z'|\kappa)
\end{aligned}$$

As synergy proportionately downcrosses in κ, sorting falls in κ (see Figure 11).

Trading Application. Our last pairwise matching application reaches outside the standard realm of TU sorting results in economics, and inquires about a classic unit trade model. We build on Shapley and Shubik (1971), and assume trading by house sellers $j = 1, \dots n$ with costs $c_1, \dots, c_n$ and potential buyers $i = 1, \dots, n$ with values v_{ij}. Higher index buyers and sellers are more motivated — e.g. higher index houses are in better condition, and higher index buyers are willing to pay more. Specifically, $v_{i'j} > v_{ij}$, for all $i' > i$, and $v_{ij} - c_j$ rises in j, for every buyer i. Thus, match surplus $f_{ij} = \max(v_{ij} - c_j, 0)$ increases in i and j, when positive. We allow for an extensive trading margin just this one time: buyers and sellers trade if and only if they have indices $i, j \ge k^*(\theta)$. For if a pair (i,j) traded, but either buyer $i' > i$ or seller $j' > j$ did not trade, then aggregate production rises if i' trades with j (or j' trades with i).

Fix $\theta'' \succeq \theta'$ with the same trade volume: $k^*(\theta'') = k^*(\theta')$. Assume uniquely optimal trading assignments $\hat{M}(\theta')$ and $\hat{M}(\theta'')$ among $k^*, \dots, n$ — as generically holds. Since $f_{ij} = v_{ij} - c_j > 0$ for all trading buyers and sellers, synergy for such types is only a function of v_{ij}. Thus, if "buyer synergy" $v_{i+1j+1}(\theta) + v_{ij}(\theta) - v_{i+1j}(\theta) - v_{ij+1}(\theta)$ is one-crossing and proportionately upcrossing, then $\hat{M}(\theta'') \succeq_{PQD} \hat{M}(\theta')$ by Proposition 3.

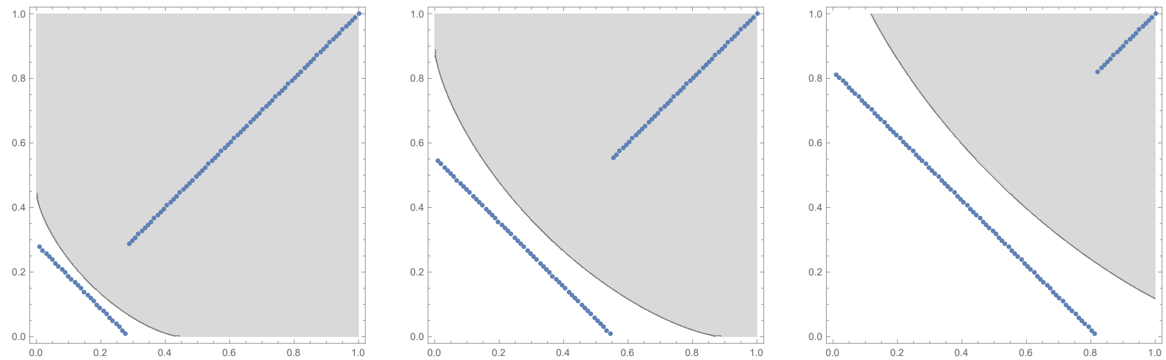


Figure 11: **Sorting in the Partnership Model with Capital.** We assume $\rho = 0.5$, and $\eta = 0.7$; capital rises left to right $k = 1, 2, 3$. Positive synergy is grey. We depict the support of the optimal matching for a discrete uniform distribution on types (blue dots).

7 Increasing Sorting and Type Distribution Shifts

Distributional shifts can be formally embedded in production functions, and thus allow us to use our comparative statics theory to deduce sorting predictions for changes in the type distributions $G(\cdot|\theta)$ and $H(\cdot|\theta)$. We say that X *types shift up (down)* in θ if $G(\cdot|\theta)$ stochastically increases (decreases) in θ, i.e. $G(\cdot|\theta') \leq G(\cdot|\theta)$ if $\theta' \succeq \theta$. Similarly, Y *types shift up (down)* in θ if $H(\cdot|\theta)$ stochastically increases (decreases) in θ.

PQD only ranks matching distributions with the same marginals. But a type change generally impacts the marginals. To enable our theory, we consider the associated bivariate *copula* $C(p,q) = M(X(p,\theta), Y(q|\theta))$, for *quantiles* $p = G(X(p,\theta)|\theta)$ and $q = H(Y(q,\theta)|\theta)$. If the matching cdfs M' and M'' share the same marginals, then *quantile sorting increases* if the associated copulas are ranked $C'' \succeq_{PQD} C'$.

While the production function $\phi(x,y)$ does not now depend on θ, the *quantile production function* $\varphi(p,q|\theta) \equiv \phi(X(p,\theta), Y(q,\theta))$ does; and so too quantile synergy:

$$\varphi_{12}(p,q|\theta) = \phi_{12}(X(p,\theta), Y(q,\theta)) X_p(p,\theta) Y_q(q,\theta) \tag{16}$$

In the Appendix, we apply Lemma 4 and Proposition 2 to prove our comparative static:

Corollary 1. *Quantile sorting increases if types shift up (down):*
(a) generically with finite types, if synergy is non-decreasing (non-increasing) in types;
(b) given G and H absolutely continuous, if synergy is increasing (decreasing) in types.

Applications to Examples in §6.3. With cubic production, synergy rises in types when $\beta, \gamma > 0$ and falls in types when $\beta, \gamma < 0$. Hence, quantile sorting increases if types shift up (down) when $\beta, \gamma > 0$ ($\beta\gamma < 0$), as illustrated in Figure 12.

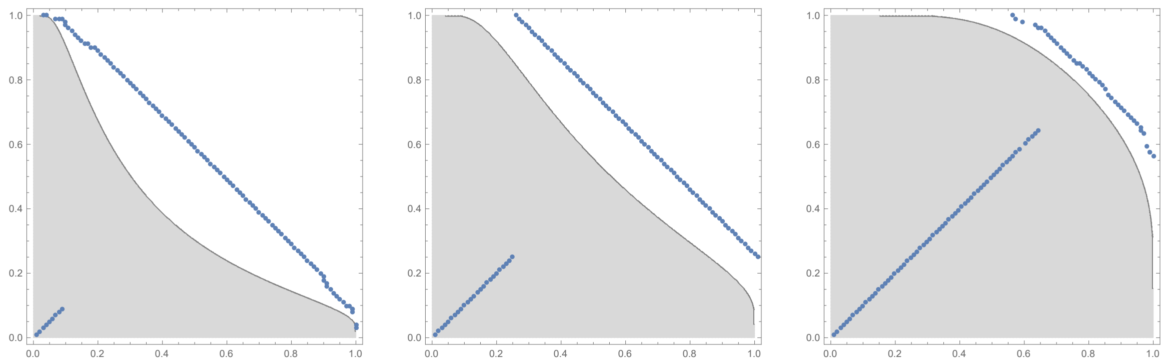


Figure 12: **Increasing Sorting with Type Shifts for Quadratic Production.** These graphs depict optimally matched quantile pairs (blue dots) assuming an exponential distribution on types $G(x|\theta) = 1 - e^{-x/\theta}$ and $H(y|\theta) = 1 - e^{-y/\theta}$ with quadratic production $xy - (xy)^2$. Since synergy is decreasing in types, Corollary 1 predicts quantile sorting will increase as θ falls, left ($\theta = 1$) to middle ($\theta = 2/3$) to right ($\theta = 1/3$).

In the moral hazard model of §3(b), given $\bar{x}\bar{y} \leq 2\underline{x}\underline{y}$ (‡), synergy increases in types if PAM is suboptimal. So quantile sorting increases when types shift up, by Corollary 1.

8 Conclusion

Becker's insight that supermodularity yields positive sorting sparked a huge literature on matching. But an impassable wall of mathematical complexity has prevented any general theory for non-assortative matching since 1973. Nonetheless, as evidenced by the many motivated models without perfect sorting, economists want to understand this more general model. We provide a missing general theory for comparative statics for such settings. We build on a economically-motivated notion of increasing sorting. Bypassing the solution of the optimal matching, we answer when the match sorting increases given monotone shifts in productivity or shifts in the type distributions.

The analysis succeeds, even though we first note that globally more synergistic matching need not lead to more sorting. Rather, we can only conclude that sorting does not fall. Our theory therefore requires cross-sectional discipline. We show that if synergy one-crosses as types increase, then sorting increases if either (i) the total synergy on all sets of couples increases, or (ii) synergy linearly or monotonely changes, or (iii) synergy upcrosses through zero, and proportionately so, in a sense we define.

We revisit the beacons of the matching literature since 1990, quickly deriving and strengthening their findings. Our paper offers a tractable foundation for future theoretical and empirical analysis of matching. A subtle and valuable direction for future work is a multidimensional extension of our theory (see Lindenlaub (2017)).

A Match Output Reformulation: Proof of Lemma 1

Finitely Many Types. Summing $\sum_{i=1}^{n}\left[\sum_{j=1}^{n} f_{ij}m_{ij}\right]$ by parts in j and then i yields:

$$\begin{aligned}
&\sum_{i=1}^{n}\left[f_{in}\sum_{j=1}^{n} m_{ij} - \sum_{j=1}^{n-1}[f_{i,j+1}-f_{ij}]\sum_{k=1}^{j} m_{ik}\right] \\
= \; &\sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\sum_{i=1}^{n}[f_{i,j+1}-f_{ij}]\sum_{k=1}^{j} m_{ik} \\
= \; &\sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left([f_{n,j+1}-f_{n,j}]\sum_{\ell=1}^{n}\sum_{k=1}^{j} m_{\ell k} - \sum_{i=1}^{n-1} s_{ij}\sum_{\ell=1}^{i}\sum_{k=1}^{j} m_{\ell k}\right) \\
= \; &\sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1}\left([f_{n,j+1}-f_{nj}]\, j - \sum_{i=1}^{n-1} s_{ij}M_{ij}\right)
\end{aligned}$$

Continuum of Types. If ψ is C^1 on $[0,1]$ and Γ is a cdf on $[0,1]$, integration by parts yields:

$$\int_{[0,1]} \psi(z)\Gamma(dz) = \psi(1)\Gamma(1) - \int_{(0,1]} \psi'(z)\Gamma(z)dz \tag{17}$$

where the interval $(0,1]$ accounts for the possibility that Γ may have a mass point at 0. Since $M(dx,y) \equiv M(y|x)G(dx)$ for a conditional matching cdf $M(y|x)$, we have:

$$M(x,y) \equiv \int_{[0,x]} M(y|x')G(dx') \tag{18}$$

Applying Theorem 34.5 in Billingsley (1995) and then in sequence (17), (18) and Fubini's Theorem, (17), the objective function $\int_{[0,1]^2} \phi(x,y)M(dx,dy)$ in (3) equals:

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]} \phi(x,y)M(dy|x)G(dx) \\
= \; &\int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]}\int_{(0,1]} \phi_2(x,y)M(y|x)dyG(dx) \\
= \; &\int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]}\int_{[0,1]} \phi_2(x,y)M(dx,y)dy \\
= \; &\int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]}\left[\phi_2(1,y)M(1,y) - \int_{(0,1]} \phi_{12}(x,y)M(x,y)dx\right]dy
\end{aligned}$$

which easily reduces to the expression in Lemma 1, using $M(1,y) = H(y)$.

B New Results in Monotone Comparative Statics

B.1 Nowhere Decreasing Optimizers

We now show that nowhere decreasing sorting is the appropriate partial order on sets of maximizers of single-crossing functions on general partially ordered sets (*posets*).[26]

Let Z and Θ be posets. The correspondence $\zeta : \Theta \to Z$ is *nowhere decreasing* if $z_1 \in \zeta(\theta_1)$ and $z_2 \in \zeta(\theta_2)$ with $z_1 \succeq z_2$ and $\theta_2 \succeq \theta_1$ imply $z_2 \in \zeta(\theta_1)$ and $z_1 \in \zeta(\theta_2)$.

Theorem 1 (Nowhere Decreasing Optimizers)**.** *Let $F : Z \times \Theta \mapsto \mathbb{R}$, where Z and Θ are posets, and let $Z' \subseteq Z$. If $\max_{z\in Z'} F(z,\theta)$ exists for all θ and F is single crossing in (z,θ), then $\mathcal{Z}(\theta|Z') \equiv \arg\max_{z\in Z'} F(z,\theta)$ is nowhere decreasing in θ for all Z'. If $\mathcal{Z}(\theta|Z')$ is nowhere decreasing in θ for all $Z' \subseteq Z$, then $F(z,\theta)$ is single crossing.*

This result removes the assumption in Milgrom and Shannon (1994) that F is quasisupermodular in $z \in Z$, and that the domain Z is a lattice.

($\Rightarrow$): If $\theta_2 \succeq \theta_1$, $z_1 \in \mathcal{Z}(\theta_1)$, $z_2 \in \mathcal{Z}(\theta_2)$, and $z_1 \succeq z_2$, optimality and single crossing give:

$$F(z_1,\theta_1) \geq F(z_2,\theta_1) \quad \Rightarrow \quad F(z_1,\theta_2) \geq F(z_2,\theta_2) \quad \Rightarrow \quad z_1 \in \mathcal{Z}(\theta_2)$$

Now assume $z_2 \notin \mathcal{Z}(\theta_1)$. By optimality and single crossing, we get the contradiction:

$$F(z_1,\theta_1) > F(z_2,\theta_1) \quad \Rightarrow \quad F(z_1,\theta_2) > F(z_2,\theta_2) \quad \Rightarrow \quad z_2 \notin \mathcal{Z}(\theta_2)$$

($\Leftarrow$): If F is not single crossing, then for some $z_2 \succeq z_1$ and $\theta_2 \succeq \theta_1$, either: (i) $F(z_2,\theta_1) \geq F(z_1,\theta_1)$ and $F(z_2,\theta_2) < F(z_1,\theta_2)$; or, (ii) $F(z_2,\theta_1) > F(z_1,\theta_1)$ and $F(z_2,\theta_2) \leq F(z_1,\theta_2)$. Let $Z' = \{z_1,z_2\}$. In case (i), $z_2 \in \mathcal{Z}(\theta_1|Z')$ and $z_1 = \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_2 \notin \mathcal{Z}(\theta_2|Z')$. In case (ii), $z_2 = \mathcal{Z}(\theta_1|Z')$ and $z_1 \in \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_1|Z')$. $\square$

For completeness, we note that Theorem 1 has a converse claim with implications in our matching model: *If $\mathcal{M}^*(\theta)$ is nowhere decreasing in θ for all type distributions G, H, then $S(R|\theta)$ is upcrossing in θ.* For assume a two type model with women (x_1,x_2) and men (y_1,y_2). Assume that $S(R|\theta)$ is not upcrossing in θ, i.e. for some $\theta'' \succeq \theta'$ and rectangle $R = (x_1,y_1,x_2,y_2)$, we have $S(R|\theta'') \leq 0 \leq S(R|\theta')$ with one inequality strict. These inequalities respectively imply that NAM is optimal at θ'' and PAM is optimal at θ'. Since one inequality is strict, either NAM is uniquely optimal at θ'' or PAM is uniquely optimal at θ'. Either case precludes nowhere decreasing sorting.

[26] This may be a known result. We include it for completeness, and as we cannot find any reference.

B.2 Integral Preservation of Upcrossing Functions on Lattices

Given a real or integer lattice[27] $Z \subseteq \mathbb{R}^N$ and poset $(\mathcal{T}, \succeq)$, the function $\sigma : Z \times \mathcal{T} \to \mathbb{R}$ is proportionately upcrossing if it obeys inequality (15) $\forall z, z' \in Z$ and $t' \succeq t$.[28]

Theorem 2. *Let $\sigma(z,t)$ be proportionately upcrossing. Then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is weakly upcrossing in t, and upcrossing in t if $\sigma(z,t)$ is upcrossing in t.*[29]

This generalizes a key information economics result by Karlin and Rubin (1956): *If $\sigma_0(z)$ is upcrossing in $z \in \mathbb{R}$, and $\sigma_1 \geq 0$ is LSPM, then $\int \sigma_0(z)\sigma_1(z,t)d\lambda(z)$ is upcrossing.* Our result subsumes theirs when $n = 1$ and $\sigma = \sigma_0\sigma_1$ is proportional upcrossing.

Proof: Karlin and Rinott (1980) prove the following: *If functions $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ obey $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for $z \in Z \subseteq R^N$, then for all positive measures λ:*

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z) \tag{19}$$

Now, if $t' \succeq t$, then (15) reduces to $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for the functions:

$$\xi_1(z) \equiv \sigma^+(z,t), \quad \xi_2(z) \equiv \sigma^-(z,t'), \quad \xi_3(z) \equiv \sigma^+(z,t'), \quad \xi_4(z) \equiv \sigma^-(z,t)$$

Thus, by (19):

$$\int \sigma^+(z,t')d\lambda(z) \int \sigma^-(z,t)d\lambda(z) \geq \int \sigma^+(z,t)d\lambda(z) \int \sigma^-(z,t')d\lambda(z) \tag{20}$$

This precludes $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$ and $\int \sigma^+(z,t')d\lambda(z) < \int \sigma^-(z,t')d\lambda(z)$, simultaneously. And thus, $\Sigma(t) > 0$ implies $\Sigma(t') \geq 0$, proving weakly upcrossing.

We now argue Σ upcrossing. First assume $\Sigma(t) > 0$. Then $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$. By (20), either $\int \sigma^+(z,t')d\lambda(z) > \int \sigma^-(z,t')d\lambda(z)$, or $\int \sigma^+(z,t')d\lambda(z) = \int \sigma^-(z,t')d\lambda(z) = 0$. But the latter is impossible, since $\int \sigma^+(z,t')d\lambda(z) = 0$ implies $\int \sigma^+(z,t)d\lambda(z) = 0$, as $\sigma(z,t)$ is upcrossing in t — contradicting $\Sigma(t) > 0$. So $\Sigma(t') > 0$.

Next, posit $\Sigma(t) = 0$, then $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z)$. By (20), either $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, and so $\Sigma(t') \geq 0$. Or, we have $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z) = 0$, whereupon $\int \sigma^-(z,t')d\lambda(z) = 0$ — as $\sigma(z,t)$ is upcrossing in t, and so $\sigma^-(z,t)$ is downcrossing. Thus, $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, or $\Sigma(t') \geq 0$. $\square$

[27] We prove a stronger than needed result, as it applies to general lattices; we just need it for $\mathbb{R}^2$.

[28] This result is related to Theorem 2 in Quah and Strulovici (2012). They do not assume (15). Rather, they assume σ is upcrossing in (z, θ), and a time a series condition: signed ratio monotonicity. Our results are independent, but overlap more closely for our smoothly LSMP condition in §B.3.

[29] The proof for the integer lattice requires that λ be a counting measure. Also true: if λ does not place all mass on zeros of σ, then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is upcrossing in t.

B.3 Proportionately Upcrossing and Logsupermodularity

We now introduce a sufficient condition for (15) that emphasizes the link between log-complementarity and proportional upcrossing. Let $\theta \in \mathbb{R}$, and call $\sigma(z,\theta)$ *smoothly signed log-supermodular (LSPM)* if its derivatives obey the inequality $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$.

Theorem 3. *If $\sigma(z,\theta)$ is upcrossing and smoothly signed LSPM, then σ obeys* (15)*.*

Step 1: Ratio Ordering. Abbreviate $w = (z,\theta) \in \mathbb{R}^{N+1}$. Assume $\hat{w} \geq w$, sharing the i coordinate $w_i = \hat{w}_i$, with $\sigma(\bar{x}, w_{-i}) < 0 < \sigma(\hat{w})$ for some $\bar{x} > w_i$. Then we prove:

$$\sigma_i(x, w_{-i})\sigma(x, \hat{w}_{-i}) \geq \sigma_i(x, \hat{w}_{-i})\sigma(x, w_{-i}) \quad \forall\ x \in [w_i, \bar{x}] \tag{21}$$

Since σ is upcrossing, $\sigma(x, w_{-i}) < 0 < \sigma(x, \hat{w}_{-i})$ for all $x \in [w_i, \bar{x}]$. If (21) fails, then for some $x' \in [w_i, \bar{x}]$:

$$\frac{\sigma_i(x', w_{-i})}{\sigma(x', w_{-i})} > \frac{\sigma_i(x', \hat{w}_{-i})}{\sigma(x', \hat{w}_{-i})}$$

This contradicts smoothly LSPM, as $(\sigma_i/\sigma)_j \geq 0$ for all $\sigma \neq 0$ and $i \neq j$. So (21) holds. Given $\sigma(x, \hat{w}_{-i}) \neq 0$, the ratio $\sigma(x, w_{-i})/\sigma(x, \hat{w}_{-i})$ is non-decreasing in x on $[w_i, \bar{x}]$, so that:

$$\frac{\sigma(w)}{\sigma(\hat{w})} \leq \frac{\sigma(\bar{x}, w_{-i})}{\sigma(\bar{x}, \hat{w}_{-i})} \tag{22}$$

Step 2: σ obeys (15). By assumption $\theta' \geq \theta$ (now a real). So if $(z,\theta') \leq (z \wedge z', \theta)$, we have $z \leq z'$ and $\theta' = \theta$, in which case (15) is an identity. If not $(z,\theta') \leq (z \wedge z', \theta)$, then let $i_1 < \cdots < i_K$ be the indices with $(z,\theta')_{i_k} > (z \wedge z', \theta)_{i_k}$ for $k = 1, \ldots, K$. Let's change $w^0 \equiv (z \wedge z', \theta)$ into $w^K \equiv (z, \theta')$ in K steps, $w^0, \ldots, w^K$, one coordinate at a time, and likewise $\hat{w}^0 \equiv (z', \theta)$ into $\hat{w}^K \equiv (z \vee z', \theta')$, changing coordinates in the same order. Notice that $w^{k-1}_{i_k} = \hat{w}^{k-1}_{i_k} = (z', \theta)_{i_k} < (z, \theta')_{i_k}$ and $\hat{w}^k \geq w^k$ for all k.

Now, inequality (15) holds if its RHS vanishes. Assume instead the RHS of (15) is positive for some $\theta' \geq \theta$, so that $\sigma(z, \theta') < 0 < \sigma(z', \theta)$; and so, replacing $\hat{w}^0 = (z', \theta)$ and $w^K = (z, \theta')$, we get $\sigma(w^K) < 0 < \sigma(\hat{w}^0)$. But then since the sequences $\{w^k\}$ and $\{\hat{w}^k\}$ are increasing and σ is upcrossing, we have $\sigma(w^k) < 0 < \sigma(\hat{w}^{k-1})$ for all k. Altogether, we may repeatedly apply inequality (22) to get:

$$\frac{\sigma(z \wedge z', \theta)}{\sigma(z', \theta)} \equiv \frac{\sigma(w^0)}{\sigma(\hat{w}^0)} \leq \frac{\sigma(w^k)}{\sigma(\hat{w}^k)} \leq \cdots \leq \frac{\sigma(w^K)}{\sigma(\hat{w}^K)} \equiv \frac{\sigma(z, \theta')}{\sigma(z \vee z', \theta')}$$

So given $\sigma(z \wedge z', \theta), \sigma(z, \theta') < 0 < \sigma(z', \theta), \sigma(z \vee z', \theta')$, inequality (15) follows from:

$$\frac{\sigma^-(z \wedge z', \theta)}{\sigma^+(z', \theta)} \geq \frac{\sigma^-(z, \theta')}{\sigma^+(z \vee z', \theta')} \qquad \square$$

C Omitted Proofs

C.1 One Crossing Weighted Synergy via Linear Synergy

Claim 1 (Linear Synergy)**.** *If synergy is linear in a parameter θ, say $\phi_{12}(x,y|\theta) = A(x,y) + \theta B(x,y)$ or $s_{ij}(\theta) = A_{ij} + \theta B_{ij}$, and A is globally positive (negative), then weighted synergy is strictly downcrossing (upcrossing) in θ, and so also is total synergy.*

Proof: Assume the case with $A > 0$ globally. Then for all $\theta'' > \theta' \geq 0$ and $\lambda \geq 0$:

$$\int A(x,y)\lambda(x,y) + \theta' \int B(x,y)\lambda(x,y) \leq 0 \;\Rightarrow\; \int A(x,y)\lambda(x,y) + \theta'' \int B(x,y)\lambda(x,y) < 0$$

as $A > 0$ and $\theta > 0$, together imply $\int B(x,y)\lambda(x,y) < 0$. Symmetric logic establishes the finite type case and weighted synergy strictly upcrossing in θ when $A < 0$. $\square$

C.2 Finite Type Increasing Sorting: Proof of Lemma 4

(a) We restrict to the generic case with unique optimal pure matchings μ, described by men partners $(\mu_1, \ldots, \mu_n)$ of women, or women partners $\omega = (\omega_1, \ldots, \omega_n)$ of men.

(b) The *total synergy* $\mathcal{S}^n(K|\theta) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in K}$ on a *couple set* $K \subseteq \mathbb{Z}^2_{n-1}$. The sorting premium $S^n(r|\theta)$ is the total synergy on rectangle r in $\mathbb{Z}^2_{n-1}$.

(c) We consider the *total synergy dyad* $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ for generic $\theta'' \succeq \theta'$. Let domain $\mathcal{D}_n$ be the space of total synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ that are each upcrossing in K on rectangles $\mathcal{R}$ and upcrossing in θ on $\{\theta', \theta''\}$ for any $K \in \mathcal{R}$. The domain $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n$ further insists that they be upcrossing in θ for all sets of couples K. Lemma 4 assumes that total synergy dyads are in $\hat{\mathcal{D}}_n$ for all n.

(d) *Removing couple* (i,j) from an n-type market *induces sorting premium* S^{n-1}_{ij} among the remaining $n-1$ types, satisfying the formula:

$$S^{n-1}_{ij}(r|\theta) \equiv S^n(r + \mathcal{I}_{ij}(r)|\theta) \quad \text{for} \quad \mathcal{I}_{ij}(r) = (\mathbb{1}_{r_1 \geq i}, \mathbb{1}_{r_2 \geq j}, \mathbb{1}_{r_3 \geq i}, \mathbb{1}_{r_4 \geq j}) \tag{23}$$

where $\mathcal{I}_{ij}(r)$ increments by one the index of the women $i' \geq i$ and men $j' \geq j$, *where the type indices refer to the original model whenever removing types* henceforth.

(e) To avoid ambiguity when changing the number n of types, we denote by (i_n, j_n) the ith highest woman and the jth highest man. Now, consider the sequence models with $\kappa = n+k, n+k-1, \ldots, n$ types induced by removing couple (i'_κ, j'_κ) at θ' and (i''_κ, j''_κ) at θ'' from the κ type model. We say the sequence of couples has *higher partners at θ' than θ''* if $(i'_\kappa, j'_\kappa) \geq (i''_\kappa, j''_\kappa)$ and $i'_\kappa = i''_\kappa$ or $j'_\kappa = j''_\kappa$.

(f) Let domain $\mathcal{D}_n^*$ be the set of total synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ induced by sequentially removing k optimally matched couples with higher partners at θ' than θ'' from total synergy dyads $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$, for some $k \in \{0, 1, \ldots\}$.

A. Key Properties of our Domains and Pure Matchings.

Fact 1. *Given a total synergy dyad in $\mathcal{D}_{n+1}^*$, removing couple (i', j') at θ' and (i'', j'') at θ'' induces a total synergy dyad in $\mathcal{D}_n^*$ if $(i', j') \geq (i'', j'')$ and $i' = i''$ or $j' = j''$.*

Fact 2. *Given a total synergy dyad in $\mathcal{D}_{n+1}$, removing couple (i', j') at θ' and (i'', j'') at θ'' induces a total synergy dyad in $\mathcal{D}_n$ if $\langle i' = i''$ and $j' \geq j''\rangle$ or $\langle j' = j''$ and $i' \geq i''\rangle$.*

Proof: We prove this for $i' = i''$ and $j' \geq j''$. For any fixed θ, the sorting premium $S_{ij}^n(r|\theta)$ is upcrossing in r, as fewer inequalities are needed. To see that total synergy is upcrossing in θ on rectangular sets in $\mathbb{Z}_{n-1}^2$, assume $S_{ij'}^n(r|\theta') \geq (>) 0$ for some r. Then

$$\begin{aligned} S^{n+1}(r + \mathcal{I}_{ij'}(r)|\theta') \geq (>) 0 \quad &\Rightarrow \quad S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta') \geq (>)\, 0 \\ &\Rightarrow \quad S^{n+1}(r + \mathcal{I}_{ij''}(r)|\theta'') \geq (>)\, 0 \\ &\Rightarrow \quad S_{ij''}^n(r|\theta'') \geq (>)\, 0 \end{aligned}$$

respectively, as (i) $S^{n+1}(r|\theta)$ is upcrossing for rectangles r, non-increasing $\mathcal{I}_{ij}$ in j, and $j'' \leq j'$, and (ii) $S^{n+1}(r|\theta)$ is upcrossing in θ for rectangles r, and (iii) by (23). □

Fact 3. *The domains are nested: $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n^* \subseteq \mathcal{D}_n$.*

Proof: Trivially, $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n^*$, since we may set $k = 0$ in the definition of $\mathcal{D}_n^*$.

To get $\mathcal{D}_n^* \subseteq \mathcal{D}_n$, pick any $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta'')) \in \mathcal{D}_n^*$. This dyad is induced by removing k optimally matched couples with higher partners at θ' than θ'' from a dyad $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k} \subseteq \mathcal{D}_{n+k}$, for some $k \geq 0$. For $\ell = 1, \ldots, k$, induce total synergy dyads $(\mathcal{S}^{n+k-\ell}(K|\theta'), \mathcal{S}^{n+k-\ell}(K|\theta''))$ by sequentially removing optimally matched couples. Then $(\mathcal{S}^{n+k-\ell}(K|\theta'), \mathcal{S}^{n+k-\ell}(K|\theta'')) \in \mathcal{D}_{n+k-\ell}$ for $\ell = 1, \ldots, k$, as removed couples are ordered, as Fact 2 needs. So $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta'')) \in \mathcal{D}_n$. □

Fact 4. *If $M \neq \hat{M}$ are pure n-type matchings, $\hat{\mu}_i > \mu_i$ at some i and $\hat{\omega}_j > \omega_j$ at some j.*

Proof: Since $M \neq \hat{M}$, there is a highest type man j matched with woman $\hat{\omega}_j > \omega_j$. Logically then, woman $i = \hat{\omega}_j$ is matched to a lower man under M, i.e. $j = \hat{\mu}_i > \mu_i$. □

Adding a couple (i_0, j_0) to a matching μ creates a new matching $\hat{\mu}$ with indices of women $i \geq i_0$ and men $j \geq j_0$ renamed $i + 1$ and $j + 1$, respectively. Equivalently, this means inserting a row i and column j into the matching matrix m — with all 0's except 1 at position (i, j) — and shifting later rows and columns up one.

Fact 5. *Adding respective couples* $(1,\hat{m}) \le (1,m)$*, or* $(\hat{w},1) \le (w,1)$*, to the* n*-type matchings* $\hat{\mu} \succeq_{PQD} \mu$ *preserves the PQD order for the resulting* $n+1$ *type matchings.*

Proof: We just consider adding couples $(1,\hat{m}) \le (1,m)$, as the analysis for $(\hat{w},1) \le (w,1)$ is similar. For pure matchings μ, let $C^{\mu}(i_0,j_0)$ count matches by women $i \le i_0$ with men $j \le j_0$, and so call $C^{\mu}(0,j) = C^{\mu}(i,0) = 0$. So $\hat{\mu} \succeq_{PQD} \mu$ *iff* $C^{\hat{\mu}} \ge C^{\mu}$.

By adding a couple $(1,m)$, the new count is:

$$\mathcal{C}^{\mu}_{m}(i,j) \equiv C^{\mu}\left(i-1, j-\mathbb{1}_{j\ge m}\right) + \mathbb{1}_{j \ge m} \quad \text{for all } i,j \in \{1,2,\ldots,n+1\}$$

To prove the step, we must show that if $\hat{\mu} \succeq_{PQD} \mu$, then $\mathcal{C}^{\hat{\mu}}_{\hat{m}} \ge \mathcal{C}^{\mu}_{m}$ for all $\hat{m} \le m$.

By assumption $\hat{\mu} \succeq_{PQD} \mu$ and thus, $C^{\hat{\mu}} \ge C^{\mu}$. So since $\hat{m} \le m$:

$$.\mathcal{C}^{\hat{\mu}}_{\hat{m}}(i,j) - \mathcal{C}^{\mu}_{m}(i,j) = \begin{cases} C^{\hat{\mu}}(i-1,j) - C^{\mu}(i-1,j) & \ge \;\; 0 \;\; \text{for } j < \hat{m} \\ C^{\hat{\mu}}(i-1,j-1) + 1 - C^{\mu}(i-1,j) & \ge \;\; 0 \;\; \text{for } \hat{m} \le j < m \\ C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) & \ge \;\; 0 \;\; \text{for } j \ge m \end{cases}$$

To understand the middle line, note that this match count can be written as

$$C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) - \left[C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) - 1\right]$$

As $C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) \le 1$, this is at least $C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) \ge 0$. □

B. The Induction Proof. We use induction on the number of types. Let M'_n and M''_n be uniquely optimal n type matchings at θ' and θ''. Lemma 4 assumes total synergy dyads in $\hat{\mathcal{D}}_n$. We prove the result on the larger domain $\mathcal{D}^*_n$:

Induction Premise $\mathcal{P}_n$: *If the total synergy dyad is in* $\mathcal{D}^*_n$*, then* $M''_n \succeq_{PQD} M'_n$*.*

Step 1. *Base Case* $\mathcal{P}_2$ *holds: If the total synergy dyad is in* $\mathcal{D}^*_2$*, then* $M''_2 \succeq_{PQD} M'_2$*.*

Proof: If not, then NAM is uniquely optimal at θ'' and PAM at θ'. Since $\mathcal{D}^*_2 \subseteq \mathcal{D}_2$ by Fact 3, the sorting premium is upcrossing in θ. This precludes a negative sorting premium at θ'' (NAM) and a positive sorting premium at θ' (PAM). □

- A *pair* refers to two *couples*, such as (i_1,j_1) and (i_2,j_2).
- A pair is a *PAM pair* is $(i_1,j_1) < (i_2,j_2)$, and a *NAM pair* is $i_1 < i_2$ and $j_1 > j_2$.

Step 2. *If the total synergy dyad is in* $\mathcal{D}^*_{n+1}$*, then neither* M'_{n+1} *nor* M''_{n+1} *includes a matched NAM pair that exceeds a matched PAM pair.*

Proof: By Fact 3, $\mathcal{D}^*_{n+1} \subseteq \mathcal{D}_{n+1}$. So $S^{n+1}(r|\theta)$ is upcrossing in rectangles r for θ' and θ''. Also, PAM (NAM) is optimal for a pair *iff* $S^{n+1}(r|\theta) \ge (\le) 0$ on rectangle r. As the optimal matching is unique, $S^{n+1}(r|\theta) \ne 0$ for all optimally matched pairs. □

Steps 3–8 impose premises $\mathcal{P}_2, \ldots, \mathcal{P}_n$, but not $\mathcal{P}_{n+1}$, and arrive at a contradiction:

(‡‡): *In a model with total synergy dyads in $\mathcal{D}^*_{n+1}$, the uniquely optimal matchings at $\theta'' \succ \theta'$ are not ranked $\mu'' \succeq_{PQD} \mu'$ ($\omega'' \succeq_{PQD} \omega'$).*

Step 3. *At states θ' and θ'', the matchings obey $\mu''_1 = \mu'_1 + 1 \geq 2$ and $\omega''_1 = \omega'_1 + 1 \geq 2$.*

We establish the first relationship. Symmetric steps would prove the second.

Proof of $\mu''_1 > \mu'_1$: If not, then $\mu''_1 \leq \mu'_1$. In this case, remove couple $(1, \mu'_1)$ at θ', and couple $(1, \mu''_1)$ at θ''. The remaining matching is PQD higher at θ'', by Induction Premise $\mathcal{P}_n$ and Fact 1. By Fact 5, if we add back the optimally matched pairs $(1, \mu'_1)$ and $(1, \mu''_1)$, then the PQD ranking still holds with $n+1$ types, given $\mu''_1 \leq \mu'_1$, namely $\mu'' \succeq_{PQD} \mu'$. This contradiction to (‡‡) proves that $\mu''_1 > \mu'_1$. □

Proof of $\mu''_1 < \mu'_1 + 2$. If not, then $\mu''_1 \geq \mu'_1 + 2$. By Fact 4, choose a woman $i > 1$ with $\mu''_i < \mu'_i$. Remove couples (i, μ'_i) at θ', and (i, μ''_i) at θ''. Since $\mu''_i < \mu'_i$, the resulting matching is PQD higher at θ'' than θ', by Fact 1 and Premise $\mathcal{P}_n$. In the resulting model, woman 1 is not matched to a higher man at θ'' than θ'. This is impossible if $\mu''_1 \geq \mu'_1 + 2$, as $\mu''_1 - \mu'_1$ falls by at most 1 when removing man μ_i at θ' and μ''_i at θ''. □

Step 4. *The couple (ω''_1, μ''_1) is matched at θ', namely, $\mu'_{\omega''_1} = \mu''_1$ and $\omega'_{\mu''_1} = \omega''_1$.*

Proof of $\mu'_{\omega''_1} \geq \mu''_1$ and $\omega'_{\mu''_1} \geq \omega''_1$: We prove the first inequality. If not, then $\mu'_{\omega''_1} < \mu''_1$. As man $\mu'_1 = \mu''_1 - 1$ is matched at θ' by Step 3, $\mu'_{\omega''_1} < \mu''_1 - 1 = \mu'_1$. Remove couple $(\omega''_1, \mu'_{\omega''_1})$ at θ' and $(\omega''_1, 1)$ at θ''. This new matching is PQD higher at θ'', by $\mathcal{P}_n$ and Fact 1. As man $\mu'_{\omega''_1}$ removed at θ' and man 1 removed at θ'' are below $\mu'_1 = \mu''_1 - 1$, the match count weakly below $(1, \mu'_1)$ is unchanged at θ'' and θ'. By Step 3, this count is higher at θ' than θ'', contradicting the n type matching PQD higher at θ''. □

Proof of $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$: Just one strict inequality in part (a) is impossible, as it overmatches some type: $\omega'_{\mu''_1} > \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$ or $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} > \mu''_1$. Next assume two strict inequalities in part (a). As $\mu'_{\omega''_1} > \mu''_1$, the θ' matching includes the PAM pair $(1, \mu'_1) < (\omega''_1, \mu'_{\omega''_1})$ — by Step 3 — and the higher NAM pair $(\omega''_1, \mu'_{\omega''_1})$ and $(\omega'_{\mu''_1}, \mu''_1)$. NAM pairs above PAM pairs violate Step 2 (left panel of Figure 13). □

The middle panel of Figure 13 depicts the takeout of Steps 3–4. We iteratively use this matching patter to show how (‡‡) greatly restricts the matching at θ' and θ''.

Step 5. $\mu'_1 \geq \mu'_i = \mu''_i - 1$ *for* $i = 1, \ldots, \omega'_1$ *and* $\omega'_1 \geq \omega'_j = \omega''_j - 1$ *for* $j = 1, \ldots, \mu'_1$.

Proof: We proved this for $i = 1$ and $j = 1$, and now prove the claimed ordering $\mu'_1 \geq \mu'_i = \mu''_i - 1$ for $i = 2, \ldots, \omega'_1$. By symmetry, $\omega'_1 \geq \omega'_j = \omega''_j - 1$ for $j = 2, \ldots, \omega'_1$.

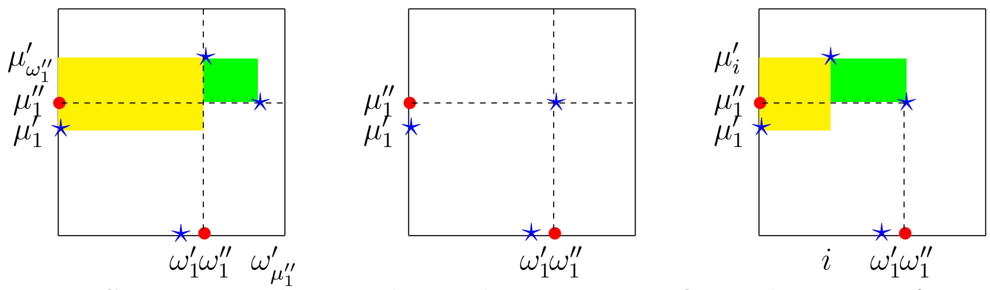


Figure 13: **Steps 4 and 5 in the Induction Proof.** In the counterfactual logic in Step 4 and 5, stars and dots denote respective proposed matched pairs at θ' and θ''. The left panel depicts the NAM pair (green) above the PAM pair (yellow) in Step 4. The middle panel depicts the conclusion of Step 4: man μ''_1 and woman ω''_1 must match under θ'. The right panel depicts the NAM pair above the PAM pair in Step 5-(a).

Part (a): $\mu'_i < \mu'_1$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu'_1$ for some $2 \leq i \leq \omega'_1$. And since $\mu'_i = \mu'_1$ entails overmatching, we have $\mu_i > \mu_1$ for $i = 2, \ldots, \omega'_1$. Thus, μ' involves a PAM pair $(1, \mu'_1) < (i, \mu'_i)$. We claim that (i, μ'_i) and (ω''_1, μ''_1) constitutes a higher NAM pair, violating the upcrossing of $S(r|\theta)$ in r, by Step 2. Indeed, $i \leq \omega'_1 < \omega''_1$ (by the premise above, and Step 3, respectively). Also, $\mu'_i > \mu''_1$, since we have assumed $\mu'_i > \mu'_1$, and deduced $\mu'_1 = \mu''_1 - 1$ in Step 3, and, in Step 4, that μ''_1 is matched to ω''_1 at θ', and we just showed $\omega''_1 > i$. (See the right panel of Figure 13.) $\square$

Part (b): $\mu'_i < \mu''_i$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu''_i$ for some $2 \leq i \leq \omega'_1$. Since $\mu'_i \geq \mu''_i$, if we remove couple (i, μ'_i) at θ' and couple (i, μ''_i) at θ'', then the resulting matching is PQD higher at θ'', by Fact 1 and $\mathcal{P}_n$. In the resulting matching, woman 1's partner is thus not higher at θ'' than θ'. But $\mu''_1 = \mu'_1 + 1$ by Step 3, and $\mu'_1 > \mu'_i \geq \mu''_i$ by part (a) and the premise of (b). Both removed men μ'_i and μ''_i are then strictly below μ'_1. So, woman 1's partner is still 1 higher at θ'' than θ'. Contradiction. $\square$

Part (c): $\mu'_i \geq \mu''_i - 1$ for $i = 2, \ldots, \omega'_1$. If not, then $\mu'_{i^*} < \mu''_{i^*} - 1$ for some $2 \leq i^* \leq \omega'_1$. Remove couple (ω''_1, μ''_1) at θ' (matched, by Step 4), and the couple $(\omega''_1, 1)$ at θ''. By Fact 1 and Assumption $\mathcal{P}_n$, the resulting matching is PQD higher at θ''.

But since $\omega''_1 > \omega'_1$ by Step 3, all women $i = 1, \ldots, \omega'_1$ remain. Each has a weakly lower partner at θ' than θ'', since we started with $\mu'_i < \mu''_i$ for $i = 1, \ldots, \omega'_1$ by Step 3 for $i = 1$, and part (b) for $i > 1$. Also, woman $i^* \leq \omega'_1$ has a strictly lower partner, as $\mu'_{i^*} < \mu''_{i^*} - 1$. The resulting matching cannot be PQD higher at θ''. Contradiction. $\square$

Step 6. *The matching μ'' is NAM among men and women at most $\omega''_1 = \mu''_1 \geq 2$.*

Proof of $\omega''_1 = \mu''_1$. By Steps 3 and 5, we get $\mu''_1 = \mu'_1 + 1 \geq \mu''_i$ for $i = 1, \ldots, w'_1 = \omega''_1 - 1$ and $\mu''_1 \geq 2 > 1 = \mu''_{w''_1}$. So in matching μ'', women $i \leq \omega''_1$ match with men $j \leq \mu''_1$. Hence, $\mu''_1 \geq \omega''_1$. Ditto, by Steps 3 and 5, $\omega''_1 \geq \omega''_j$ for $j = 1, \ldots, \mu''_1$, and in matching ω'', men $j \leq \mu''_1$ match with women $i \leq \omega''_1$. Hence, $\mu''_1 \leq \omega''_1$. Thus, $\mu''_1 = \omega''_1 \geq 2$. $\square$

Proof of $\mu''_i = \mu''_1 - i + 1$ for $1, \ldots, \omega''_1$. This is an identity at $i = 1$ and true at $i = \omega''_1$ by $\omega''_1 = \mu''_1$ (just proven) and $\mu''_{\omega''_1} = 1$. So, henceforth assume $i \in \{2, \ldots, \omega''_1 - 1\}$. We claim that for all such i, $\mu'_1 \geq \mu''_i$. Indeed, by Steps 3 and 5, $\mu''_1 = \mu'_1 + 1 \geq \mu''_i$; and since we do not over match, $\mu''_1 \neq \mu''_i$ for $i \neq 1$. Since $\mu'_1 \geq \mu''_i$, Step 5 yields equality $\omega'_j = \omega''_j - 1$ at $j = \mu''_i$, and so $\omega'_{\mu''_i} = \omega''_{\mu''_i} - 1 = i - 1$. But then since $\omega'_{\mu'_{i-1}} = i - 1$ and each woman has a unique partner, $\omega'_{\mu''_i} = i - 1$ implies $\mu''_i = \mu'_{i-1}$. As $\mu'_{i-1} = \mu''_{i-1} - 1$ by Step 5 and $i \leq \omega''_1 - 1 = \omega'_1$ (by our premise and Step 3), we have $\mu''_i = \mu''_{i-1} - 1$. $\square$

An n-type pure matching μ is NAM* if $\mu_n = n$ and $\mu_i = n - i$ for $i = 1, \ldots, n-1$, i.e. NAM among types $1, \ldots, n-1$, so that NAM* = NAM3 when $n = 3$.

Step 7. *The matching μ' is NAM* among men and women at most $\omega''_1 = \mu''_1 \geq 2$.*

Proof: Steps 3, 5 and 6 imply $\mu'_i = \mu''_i - 1 = \mu''_1 - i$ for $i = 1, \ldots, \omega'_1 = \omega''_1 - 1$. Couple (ω''_1, μ''_1) matches under μ', by Step 4. So μ' is NAM* for types $1, \ldots, \mu''_1 = \omega''_1$. $\square$

By Steps 6–7, μ'' is NAM and μ' is NAM* on types $1, \ldots, \omega''_1 = \mu''_1 \equiv k \geq 2$. Since NAM* $\succ_{PQD}$ NAM, if $k < n + 1$ then Premise $\mathcal{P}_k$ fails. Step 8 finishes the proof by showing that NAM at θ'' and NAM* at θ' is also impossible for $k = n + 1$ types.

NAM for men $\{i_1, \ldots, i_\ell\}$ and women $\{j_1, \ldots, i_\ell\}$ is $\{(i_1, j_\ell), (i_2, j_{\ell-1}), \ldots, (i_\ell, j_1)\}$. Rematching to NAM*, $\{(i_1, j_{\ell-1}), (i_2, j_{\ell-2}), \ldots, (i_\ell, j_\ell)\}$ changes payoffs by

$$\sum_{u=1}^{\ell-1}(f_{i_u,j_{\ell-u}} - f_{i_u,j_{\ell+1-u}}) + f_{i_\ell,j_\ell} - f_{i_\ell,1} = \sum_{u=1}^{\ell-1}\left[(f_{i_\ell,j_{\ell+1-u}} - f_{i_\ell,j_{\ell-u}}) - (f_{i_u,j_{\ell+1-u}} - f_{i_u,j_{\ell-u}})\right]$$

So *the payoff of NAM* less that of NAM on any subset of ℓ types equals* (suppressing the superscript on S)

$$\sum_{u=1}^{\ell-1} S(i_u, j_{\ell-u}, i_\ell, j_{\ell+1-u}) \tag{24}$$

Step 8. *NAM at θ'' and NAM* at θ' is impossible for total synergy dyads in $\mathcal{D}^*_{n+1}$.*

Part (*a*): Contradiction Assumption. For $n+1$ types, posit NAM* and NAM uniquely optimal at θ' and θ'' (Figure 14, panel 1). Total synergy dyads in $\mathcal{D}^*_{n+1}$ are induced by removing $k - 1 \geq 0$ optimally matched couples with higher partners at θ' than θ'' (building block (f)) from a total synergy dyad $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$. The θ' matching here is NAM* for men $\boldsymbol{i}' = (i'_1, \ldots, i'_{n+1})$ and women $\boldsymbol{j}' = (j'_1, \ldots, j'_{n+1})$, while the θ'' matching with these $n + k$ types is NAM for men $\boldsymbol{i}'' = (i''_1, \ldots, i''_{n+1})$ and women $\boldsymbol{j}'' = (j''_1, \ldots, j''_{n+1})$, with $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (Figure 14, panel 2).

Part (*b*): couple sets U', U'' with $\mathcal{S}^{n+k}(U''|\theta'') < 0 < \mathcal{S}^{n+k}(U'|\theta')$. For rectangles $r'_u \equiv (i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u})$ and $r''_u \equiv (i''_u, j''_{n+1-u}, i''_{n+1}, j''_{n+2-u})$ define "upper sets":

- $U' \equiv \cup_{u=1}^{n} r'_u$, the union of the grey and orange rectangles in panel 2 of Figure 14

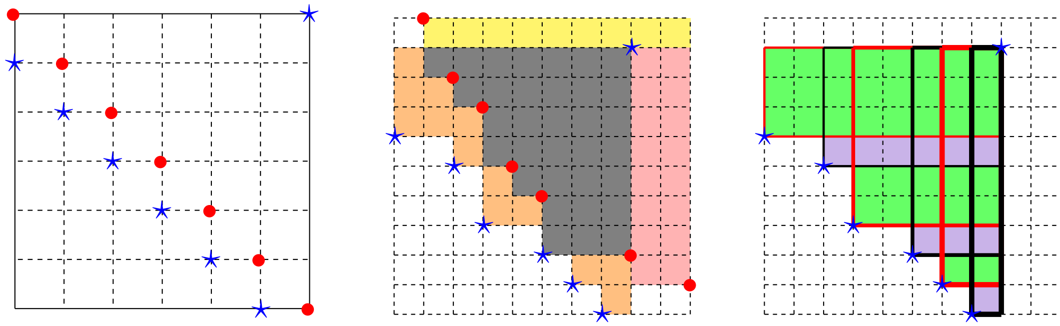


Figure 14: **Step 8 of Induction Proof.** We rule out NAM for θ'' (dots) and NAM* for θ' (stars) with $n+1$ types (left). Middle: These matches with $n+k > n+1$ types, after adding couples weakly higher at θ' than θ''. Let K^G, K^O, K^P, K^Y be the grey, orange, pink, and yellow regions. By (24), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^O|\theta') > 0$, as NAM* is optimal for θ'. But $\mathcal{S}^{n+k}(K^O|\theta') < 0$, as K^O is the union of of rectangles, each below a NAM pair for θ''. So $\mathcal{S}^{n+k}(K^G|\theta') > 0$. By (24), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^P \cup K^Y|\theta'') < 0$, negative by NAM optimal for θ''. Finally, $\mathcal{S}^{n+k}(K^Y|\theta'), \mathcal{S}^{n+k}(K^P|\theta') > 0$, as the yellow and pink rectangles are each above a PAM pair for θ'. So $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. But since $\mathcal{S}^{n+k}(K^G|\theta') > 0$, this contradicts upcrossing total synergy in θ. The right panel illustrates Step 8(c).

- $U'' \equiv \cup_{u=1}^{n} r''_u$, the union of the grey, yellow, and pink regions

Since NAM* is uniquely optimal for the subsets of men $\boldsymbol{i}'$ and women $\boldsymbol{j}'$ at θ', it payoff-dominates NAM. So linearity of total synergy at $\ell = n+1$ in (24) yields

$$\mathcal{S}^{n+k}(U'|\theta') = \textstyle\sum_{u=1}^{n+1} S^{n+k}(r'_u|\theta') = \sum_{u=1}^{n+1} S^{n+k}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u}|\theta') > 0$$

Likewise, NAM uniquely optimal for subsets $\boldsymbol{i}''$ and $\boldsymbol{j}''$ at θ'' implies $\mathcal{S}^{n+k}(U''|\theta'') < 0$.

PART (c): $\mathcal{S}^{n+k}(K^G|\theta') > 0$ FOR $K^G \equiv U' \cap U''$. First, $U' = \cup_{u=1}^{n}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+1})$, i.e., a union of rectangles with fixed upper corner (Figure 14, panel 3). Likewise, we have $U'' \equiv \cup_{u=1}^{n} r''_u$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), if $(i,j) \in U' \setminus U'' = U' \setminus K^G$ (orange shade, Figure 14, panel 2), then $(i'_{u^*}, j'_{n+1-u^*}) \leq (i,j)$, and $i \leq i''_{u^*}$ or $j \leq j''_{n+1-u^*}$, with at least one strict, at some u^*. So couple (i,j) is below the meet of the θ'' matched NAM pair $(i''_{u^*}, j''_{n+2-u^*})$ and $(i''_{u^*+1}, j''_{n+1-u^*})$. As the sorting premium is upcrossing in types, $s_{ij}(\theta'') < 0$. Then $s_{ij}(\theta') < 0$, as synergy is upcrossing in θ. Then $\mathcal{S}^{n+k}(U' \setminus K^G|\theta') < 0$, as this holds for all $(i,j) \in U' \setminus K^G$. Since total synergy is additive and $\mathcal{S}^{n+k}(U'|\theta') > 0$ (part (b)), $\mathcal{S}^{n+k}(K^G|\theta') = \mathcal{S}^{n+k}(U'|\theta') - \mathcal{S}^{n+k}(U' \setminus K^G|\theta') > 0$.

PART (d): $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), define rectangles $K^Y \equiv (i''_1, j'_{n+1}, i'_{n+1}, j''_{n+1})$ and $K^P \equiv (i'_{n+1}, j''_1, i''_{n+1}, j'_{n+1})$ (resp., yellow and pink regions, Figure 14, panel 2). Then $U'' \setminus K^G = K^Y \cup K^P$. As total synergy is linear:

$$\mathcal{S}^{n+k}(K^G|\theta) = \mathcal{S}^{n+k}(U''|\theta) - \mathcal{S}^{n+k}(K^Y|\theta) - \mathcal{S}^{n+k}(K^P|\theta) \tag{25}$$

Rectangle K^Y is above the rectangle defined by the θ' PAM pair (i'_1, j'_n) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^Y|\theta'') > 0$, as total synergy is upcrossing on rectangles and θ. Likewise, K^P is above the rectangle defined by the θ' PAM pair (i'_n, j'_1) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^P|\theta'') > 0$. Then $\mathcal{S}^{n+k}(K^G|\theta'') < 0$, since $\mathcal{S}^{n+k}(U''|\theta'') < 0$ (part (b)) and (25).

Since $\mathcal{S}^{n+k}(K^G|\theta') > 0$ (part (c)), we cannot have $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$; and thus, by part (a) we have contradicted dyads $(\mathcal{S}^{n+1}(K|\theta'), \mathcal{S}^{n+1}(K|\theta'')) \in \mathcal{D}^*_{n+1}$, and thus conclude that NAM at θ'' and NAM* at θ' is impossible. □

C.3 Increasing Sorting via Proviso (★): Proof of Proposition 2

Claim 2. *An optimal matching is unique given an absolutely continuous cdf G (or H) and the x- and y- marginal product increments strictly one crossing, and sorting is increasing in θ if also total synergy is upcrossing in θ, for example given Proviso (★).*

This claim proves Proposition 2. We prove continuum type upcrossing in Step 1. Steps 2–3 produces finite mesh approximations to the continuum, and prove sorting by Lemma 4. In Steps 4–6, we secure a PQD ordering in the unique continuum limit.

Step 1. *The continuum sorting premium is strictly upcrossing in types.*

Proof: In §6.2, we proved that $S(x_1, y_1, x_2, y_2) = \int_{x_1}^{x_2} \Delta_x(x|y_1, y_2)dx$ is upcrossing in (x_1, x_2), where $x_1 < x_2$. Since $\Delta_x(x|y_1, y_2)$ is strictly upcrossing, if $S(x'_1, y_1, x'_2, y_2) = 0$ then $\Delta_x(x'_1|y_1, y_2) < 0 < \Delta_x(x'_2|y_1, y_2)$. So $S_{x_1}(x'_1, y_1, x'_2, y_2) = -\Delta_x(x'_1|y_1, y_2) > 0$ and $S_{x_2}(x'_1, y_1, x'_2, y_2) = \Delta_x(x'_2|y_1, y_2) > 0$. Then $S(x''_1, y_1, x''_2, y_2) > 0$ for all $(x''_1, x''_2) > (x'_1, x'_2)$. By symmetric reasoning, S strictly upcrosses in (y_1, y_2). □

Step 2. *Uniquely optimal finite type matchings exist for a payoff perturbation with total synergy upcrossing in θ.*

Proof: Let $\mathcal{X}^n = \{x_1^n, \ldots, x_n^n\}$ and $\mathcal{Y}^n = \{y_1^n, \ldots, y_n^n\}$ be equal quantile increments, with $G(x_1^n) = H(y_1^n) = 1/n$ and $G(x_i^n) = G(x_{i-1}^n) + 1/n$ and $H(y_j^n) = H(y_{j-1}^n) + 1/n$. Let G^n and H^n be cdfs on $[0, 1]$, stepping by $1/n$ at $\mathcal{X}^n$ and $\mathcal{Y}^n$ (resp.). Put $f_{ij}^n(\theta) = \phi(x_i^n, y_j^n|\theta)$. The set $\mathcal{M}^n(\theta)$ of pure optimal matchings is non-empty, by Lemma 3.

Since unique optimal matchings are pure, we restrict to pure matchings. These are uniquely defined by the male partner vector $\mu = (\mu_1, \ldots, \mu_n)$. Call the pure matching $\hat{M}$ *lexicographically higher* than M *iff* its male partner vector $\hat{\mu}$ lexicographically dominates μ. Let $\bar{M}^n(\theta)$ (resp. $\bar{\mu}^n(\theta)$) be the optimal pure matching highest in the lexicographic order, and $\underline{M}^n(\theta)$ (resp. $\underline{\mu}^n(\theta)$) the lowest. Easily, each is well-defined.

Fix $\theta'' \succ \theta'$. Let $\iota(j) = \bar{\mu}_j^n(\theta') - 1$ and pick $\varepsilon > 0$. Perturb synergy down at θ':

$$s_{ij}^{n\varepsilon}(\theta') \equiv s_{ij}(\theta') - \varepsilon^j \mathbb{1}_{(i,j)=(\iota(j),j)} \tag{26}$$

We prove that $\bar{M}^n(\theta')$ is uniquely optimal at θ' for any production function with ε-perturbed synergy (26), for all small $\varepsilon > 0$. Similar logic will prove that $\underline{M}^n(\theta'')$ is uniquely optimal at θ'' with $s_{ij}^{n\varepsilon}(\theta'') \equiv s_{ij}(\theta'') + \varepsilon^j \mathbb{1}_{(i,j)=(\underline{\mu}_j^n(\theta''),j)}$ for all small $\varepsilon > 0$.

Pick a matching M that is not optimal at $\varepsilon = 0$. Since $\bar{M}^n(\theta')$ is optimal at $\varepsilon = 0$, $\bar{M}^n(\theta')$ yields a higher payoff than M for all small $\varepsilon > 0$.

As $\bar{\mu}^n(\theta')$ is the lexicographically highest optimal matching at θ', another optimal μ obeys $(\bar{\mu}_1^n(\theta'), \ldots, \bar{\mu}_{\ell-1}^n(\theta')) = (\mu_1, \ldots, \mu_{\ell-1})$, and first diverges at $\bar{\mu}_\ell^n(\theta') > \mu_\ell$, for some woman $\ell < n$. Using $M_{ij} = \sum_{k=1}^{j} \mathbb{1}_{\mu_k \leq i}$, Lemma 1, and (26), the payoff $\bar{M}^n(\theta')$ exceeds that of $M \in \mathcal{M}^n(\theta')$ by $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta')\left[\bar{M}_{ij}^n(\theta') - M_{ij}\right]$. This expands to:

$$\sum_{j=1}^{n-1} \varepsilon^j \left[M_{\iota(j)j} - \bar{M}_{\iota(j)j}^n(\theta')\right] \;=\; \varepsilon^\ell + \sum_{j=\ell+1}^{n-1} \varepsilon^j \sum_{k=\ell+1}^{j} \left[\mathbb{1}_{\mu_k \leq \iota(j)} - \mathbb{1}_{\bar{\mu}_k^n \leq \iota(j)}\right]$$

Altogether, $\lim_{\varepsilon \to 0} \varepsilon^{-\ell} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta')\left[\bar{M}_{ij}^n(\theta') - M_{ij}\right] = 1 > 0$. $\square$

Step 3. *If $\theta'' \succ \theta'$, then $\bar{M}^n(\theta'') \succeq_{PQD} \underline{M}^n(\theta')$ for all n.*

Proof: Since $S^{n\varepsilon}(r|\theta)$ is continuous in ε, there exists $\hat{\varepsilon}_n > 0$ such that, for all $r = (i_1, j_1, i_2, j_2)$ and $0 \leq \varepsilon < \hat{\varepsilon}_n$, if $S^{n0}(r|\theta) \lesseqgtr 0$ then $S^{n\varepsilon}(r|\theta) \lesseqgtr 0$. By the contrapositives:

$$S^{n\varepsilon}(r|\theta) \geq 0 \;\Rightarrow\; S^{n0}(r|\theta) \geq 0 \quad \text{and} \quad S^{n\varepsilon}(r|\theta) \leq 0 \;\Rightarrow\; S^{n0}(r|\theta) \leq 0. \tag{27}$$

We claim that $S^{n\varepsilon}(r|\theta)$ *is strictly upcrossing in r for all* $0 < \varepsilon < \hat{\varepsilon}_n$. For if not, then $S^{n\varepsilon}(r''|\theta) \leq 0 \leq S^{n\varepsilon}(r'|\theta)$ for some $r'' \succ_{NE} r'$. But then $S^{n0}(r''|\theta) \leq 0 \leq S^{n0}(r'|\theta)$ by (27), contradicting $S^{n0}(r|\theta)$ strictly upcrossing in r, as follows from Step 1.

Continuum total synergy is upcrossing in θ by assumption; and thus, finite total synergy $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta)\mathbb{1}_{(i,j)\in K}$ for all finite approximations. Then, $\Sigma^{n\varepsilon}(K|\theta) \equiv \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}^{n\varepsilon}(\theta)\mathbb{1}_{(i,j)\in K}$ is upcrossing in θ, since $s_{ij}^{n\varepsilon}(\theta')$ is non-increasing in ε and $s_{ij}^{n\varepsilon}(\theta'')$ is non-decreasing in ε by construction (26).

Altogether, for all $\varepsilon \in (0, \hat{\varepsilon}_n)$, the sorting premium $S^{n\varepsilon}(r|\theta)$ is strictly upcrossing in r and total synergy $\Sigma^{n\varepsilon}(K|\theta)$ upcrossing in θ, for all couple sets $K \subseteq \mathbb{Z}_n^2$. Given $\bar{M}^n(\theta'), \underline{M}^n(\theta'')$ uniquely optimal, $\underline{M}^n(\theta'') \succeq_{PQD} \bar{M}^n(\theta')$ for all n, by Lemma 4. $\square$

Step 4. *There exists a subsequence of matchings $\{M^{n_k}(\theta)\}$ that converges to an optimal matching in the continuum model.*

Proof: Define a step function $\phi^n(x, y|\theta) = f_{ij}^{n\varepsilon_n}(\theta)$ for $(x, y) \in [x_{i-1}^n, x_i^n) \times [y_{j-1}^n, y_j^n)$, where $\varepsilon_n = \hat{\varepsilon}_n/n$. By construction, $\{G^n\}$ and $\{H^n\}$ weakly converge to G and H as $n \to \infty$, while ϕ^n uniformly converges to ϕ. By Theorem 5.20 in Villani (2008), the associated optimal matching cdfs have a convergent subsequence $\{M^{n_k}(\theta)\}$ with limit point $M^{\infty}(\theta)$ optimal in the continuum model.[30] $\square$

Step 5. *$M^{\infty}(\theta'') \succeq_{PQD} M^{\infty}(\theta')$ for all $\theta'' \succeq \theta'$*

Proof: Fix $\theta'' \succeq \theta'$, and let $\{n_k\}$ be a subsequence along which the sequence of finite type matchings $\{M^{n_k}(\theta')\}$ converges to $M^{\infty}(\theta')$, as defined in Step 4. Now, since cdfs $\{G^{n_k}\}$ and $\{H^{n_k}\}$ weakly converge to G and H, and $\phi^{n_k}(x, y|\theta'')$ converges uniformly to $\phi(x, y|\theta'')$, there exists a subsequence $\{n_{k_\ell}\}$ of $\{n_k\}$, along which the sequence of finite type matchings $\{M^{n_{k_\ell}}(\theta'')\}$ converges to $M^{\infty}(\theta'')$ by Theorem 5.20 in Villani (2008). Further, by Step 3, $M^{n_{k_\ell}}(\theta'') \succeq_{PQD} M^{n_{k_\ell}}(\theta')$. But then, the limits must be ordered $M^{\infty}(\theta'') \succeq_{PQD} M^{\infty}(\theta')$ by Theorem 9.A.2.a in Shaked and Shanthikumar (2007). $\square$

Step 6. *An optimal matching is unique given an absolutely continuous cdf G (or H) and the x- and y- marginal product increment strictly one-crossing.*[31]

Proof: Assume G is absolutely continuous and $\Delta_x(x|y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x, for $y_2 > y_1$. By Theorem 5.1 in Ahmad, Kim, and McCann (2011), there is a unique optimal matching for absolutely continuous G if the production function ϕ is C^2, and when the critical points of (their "twist difference") $\phi(x, y_2) - \phi(x, y_1)$ include at most one local max and one local min, for all y_1, y_2. If $y_1 < y_2$, then $\Delta_x(x|y_1, y_2)$ is upcrossing in x, and any critical point of the twist difference is a global minimum. Similarly, then any critical point is a global maximum if $y_1 > y_2$. $\square$

C.4 Increasing Sorting: Proof of Proposition 3

Finite Types Proof. We verify the premise of Lemma 4. First, by Theorem 2, total synergy $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in K}$ is upcrossing in the parameter $t = \theta$. Next, to see that the sorting premium $S(r|\theta) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ is upcrossing in r, we apply Theorem 2 to the parameter $t = r \in \mathbb{R}^4$. By a similar proof to footnote 17, the indicator function $\mathbb{1}_{(i,j)\in r}$ is a non-negative LSPM function of (i, j, r), since a rectangle r is a sublattice.[32] Then $s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ obeys inequality (15) in $z = (i, j)$ and r, since $s_{ij}(\theta)$ obeys (15) for fixed θ. The sorting premium is then upcrossing in r, by Theorem 2. $\square$

[30] Namely: Fix a sequence $\{\phi_k\}$ of continuous and uniformly bounded production functions converging uniformly to ϕ. Let $\{G_k\}$ and $\{H_k\}$ be cdf sequences and M_k an optimal matching for ϕ, given G_k and H_k. If G_k and H_k weakly converge to G and H, then some subsequence of $\{M_k\}$ weakly converges to a matching M^* optimal for ϕ, G, and H.

[31] We call any function $\Upsilon : \mathbb{R} \mapsto \mathbb{R}$ *strictly upcrossing* if, for all $x' > x$, we have $\Upsilon(x) \geq 0 \Rightarrow \Upsilon(x') > 0$. Easily, a strictly upcrossing function is upcrossing.

[32] Theorem 2 assumes $t \in \mathcal{T}$, a poset. Here we exploit the fact that the space of rectangular *sets* of couples is a sublattice of $\mathbb{Z}^2$, even though the PQD order on *distributions* over couples is not a lattice.

Continuum of Types Proof. We apply Claim 2. By Theorem 2, total synergy $\int_Z \phi_{12}(x,y|\theta)dxdy$ is upcrossing in $t=\theta$. Next, the x-marginal product increment $\int \phi_{12}(x,y)\mathbb{1}_{y\in[y_1,y_2]}dy$ is strictly upcrossing in x. Let $x''>x'$. Posit for a contradiction:

$$\int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y_2]}dy \leq 0 \leq \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y_2]}dy \tag{28}$$

As synergy $\phi_{12}(x,y)$ is strictly upcrossing in x and y, by (28), there exist zeros $y',y''\in(y_1,y_2)$ such that $\phi_{12}(x',y) \lesseqgtr 0$ for $y \lesseqgtr y'$ and $\phi_{12}(x'',y) \lesseqgtr 0$ for $y \lesseqgtr y''$. Easily, these zeros are ordered $y''<y'$. But then inequalities in (28) are simultaneously impossible, for:

$$\begin{aligned} 0 &\leq \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y_2]}dy < \int \phi_{12}(x',y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy \\ \Rightarrow\quad 0 &< \int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy < \int \phi_{12}(x'',y)\mathbb{1}_{y\in[y_1,y_2]}dy \end{aligned}$$

by Theorem 2, since $\int \phi_{12}(x,y)\lambda(y)dy$ is upcrossing in $t=x$ for any non-negative $\lambda(y)$ — because $\phi_{12}(x,y)$ is proportionately upcrossing in types and upcrossing in y. □

C.5 Type Distribution Shifts: Proof of Corollary 1

Proof: Throughout we WLOG assume types shift up in the parameter θ.

Total Quantile Synergy is Upcrossing. In part (b), total quantile synergy (16) over a set of quantile pairs $Z\subseteq[0,1]^2$ is:

$$\Upsilon(\theta) \equiv \int\int \varphi_{12}(p,q|\theta)\mathbb{1}_{(p,q)\in Z}dpdq = \int\int \phi_{12}(x,y)\mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy$$

by the change of variables $x=X(p,\theta)$ and $y=Y(q,\theta)$ (equivalently, $p=G(x|\theta)$ and $q=H(y|\theta)$); and thus, $dx=X_p dp$ and $dy=Y_q dq$. Since distributions G and H fall in θ, the cdf associated with pdf $\lambda(x,y|\theta)\equiv \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}/[\int\int \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy]$ is stochastically increasing in θ. And thus, since $\phi_{12}(x,y)$ is strictly increasing:

$$0\leq\Upsilon(\theta) \Rightarrow 0 \leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta)dxdy \leq \int\int \phi_{12}(x,y)\lambda(x,y|\theta')dxdy \Rightarrow 0\leq \Upsilon(\theta')$$

Identical logic establishes average synergy upcrossing in θ in the finite type case (a).

Case (a): The Quantile Sorting Premium is Upcrossing. The sorting premium is upcrossing in types when synergy is non-decreasing in types. Because types $X(p,\theta)$ and $Y(q,\theta)$ are non-decreasing in the quantiles p and q, the *quantile sorting premium* $S(X(p_1,\theta),Y(q_1,\theta),X(p_2,\theta),Y(q_2,\theta))$ upcrosses in (p_1,q_1,p_2,q_2). Hence, the quantile sorting increases in θ by Lemma 4.

Case (b): Quantile Marginal Product Increments Strictly Upcross. Non-decreasing synergy is proportionately upcrossing; and thus $\Delta_x(x|y_1,y_2)$ strictly upcrosses in x as shown in §C.4. Given $G(x|\theta)$ absolutely continuous $X_p > 0$; and so,

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(X(p,\theta)|Y(q_1,\theta), Y(q_2,\theta))X_p(p,\theta)$$

is strictly upcrossing in p. Similarly, $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in q. All told, we've seen that quantile sorting increases in θ, by Step 1 and Claim 2. □

C.6 Nowhere Decreasing in Kremer-Maskin Example (§5)

Step 1. *PAM is not optimal if $\varrho > (1-2\theta)^{-1}$, and is uniquely optimal for $\varrho < (1-2\theta)^{-1}$.*

Proof: In a unisex model, PAM is optimal iff the symmetric sorting premium $S(x,x,y,y)$ is globally positive. Its sign is constant along any ray $y = kx$, and proportional to:

$$s(k) \equiv 2^{\frac{1-2\theta}{\varrho}}(1+k) - 2k^{\theta}(1+k^{\varrho})^{\frac{1-2\theta}{\varrho}} \tag{29}$$

Since $s(1) = s'(1) = 0$, $s''(1) \propto (1 + \varrho(2\theta - 1))$, and $\theta \in [0, 1/2]$, we have $s(k) < 0$ close to $k = 1$ precisely when $\varrho > (1-2\theta)^{-1} \geq 1$. In this case, the symmetric sorting premium is negative in a cone around the diagonal, and PAM fails.

Conversely, posit $\varrho < (1-2\theta)^{-1}$. Then $s(k) > 0$ for all $k \in [0,1]$. Since $S(x,x,y,y)$ is symmetric about $y = x$, it is globally positive and PAM is uniquely optimal. □

Step 2. *If $\varrho \geq (1-2\theta)^{-1}$ then weighted synergy is upcrossing in θ, downcrossing in ϱ.*

Proof: Change variables $y = kx$. If $\Delta(k) = \int_0^1 \lambda(x,kx)dx$, weighted synergy is

$$\int\int \phi_{12}(x,y)\lambda(x,y)dydx = 2\int_0^1\int_0^1 x\phi_{12}(x,kx)\lambda(x,kx)dkdx = \int_0^1 \sigma(k,\theta,\varrho)\Delta(k)dk$$

where $\sigma = \sigma_A\sigma_B$ for $\sigma_A \equiv 2k^{\theta-1}(1+k^{\varrho})^{\frac{1-2\theta-2\varrho}{\varrho}}$ and $\sigma_B \equiv \theta(1-\theta)(1+k^{2\varrho}) + (1-\varrho+2\theta(\theta-1+\varrho))k^{\varrho}$. As $\varrho \geq (1-2\theta)^{-1}$, $\sigma_A > 0$ is LSPM in (k,θ,ϱ), σ_B is increasing in $(\theta,-k,-\varrho)$ for $k \in [0,1]$. So $\sigma = \sigma_A\sigma_B$ is proportionately downcrossing in (k,θ) and $(k,-\varrho)$. Weighted synergy is upcrossing in θ, downcrossing in ϱ, by Theorem 2. □

Step 3. *Sorting is nowhere decreasing in θ and nowhere increasing in ϱ.*

Proof: Pick $\theta'' > \theta'$. If $\varrho < (1-2\theta'')^{-1}$, then PAM is uniquely optimal at θ'' (Step 1) and sorting increases from θ' to θ''. If $\varrho \geq (1-2\theta'')^{-1}$, then $\varrho > (1-2\theta')^{-1}$ and weighted synergy is upcrossing on $[\theta', \theta'']$ (Step 2) and sorting is non-decreasing (Proposition 1).

Now pick any θ and $\varrho'' > \varrho'$. If $\varrho' < (1-2\theta)^{-1}$, then PAM is uniquely optimal at ϱ' (Step 1) and sorting is decreasing from ϱ' to ϱ''. If, instead, $\varrho' \geq (1-2\theta)^{-1}$, then, necessarily, $\varrho'' > (1-2\theta)^{-1}$, weighted synergy is downcrossing from ϱ' to ϱ'' (Step 2) and sorting is non-increasing in ϱ, by Proposition 1. □

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