Rushes in Large Timing Games

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Fall, 2016

Wisdom of Old Dead Dudes

Natura non facit saltus. - Leibniz, Linnaeus, Darwin, Marshall

Examples:

- Tipping points in neighborhoods with “white flight”
- Bank runs
- Land run
- gold rush
- ◇ Fundamental payoff “ripens” over time peaks at a “harvest time”, and then “rots”
- This forces rushes, as in plots

Players and Strategies

- Continuum of identical risk neutral players $i \in [0, 1]$.
- Players choose stopping times τ on $[0, \infty)$
- *Anonymous* summary of actions: $Q(t) =$ the cumulative probability that a player has stopped by time $\tau \le t$.
- With a continuum of players, Q is the cdf over stopping times in any symmetric equilibrium.
- At any time t in its support, a cdf Q is either absolutely continuous or jumps, i.e. $Q(t) > Q(t-)$.
- This corresponds to *gradual play*, or a *rush*, where a positive mass stops at a time-t atom.

A Simple Payoff Dichotomy

- Payoffs depend on the stopping time t and quantile q.
- Common payoff at t is $u(t, Q(t))$ if t is not an atom of Q
- If Q has an atom at time t, say $Q(t) = p > Q(t-) = q$, then each player stopping at t earns:

$$\int_q^p \frac{u(t,x)}{p-q} dx$$

- A Nash equilibrium is a quantile function Q whose support contains only maximum payoffs.

Tradeoff of Fundamentals and Quantile

- For fixed q, payoffs u are quasi-concave in t, strictly rising from $t = 0$ ("ripening") until a *harvest time $t^*(q)$*, and then strictly falling ('rotting").
- uniquely optimal entry time!!!!
- For all times s, payoffs u are either monotone or log-concave in q, with unique *peak quantile* $q^*(s)$.
- payoff function is log-submodular, eg. $u(t,q) = \pi(t)v(q)$

$\Rightarrow$ harvest time $t^*(q)$ is a decreasing in q

$\Rightarrow$ peak quantile $q^*(s)$ is decreasing in time s.

- Stopping in finite time beats waiting forever:

$$\lim_{s\to\infty} u(s, q^*(s)) < u(t,q) \quad \forall t, q \text{ finite}$$

- To ensure *pure strategies*, label players $i \in [0, 1]$, and assume assume that i enters at time $T(i) = \inf\{t \in \mathbb{R}_+ | Q(t) \geq i\} \in [0, \infty)$, the "generalized inverse distribution function" of Q

Is Nash Equilibrium Credible?

- Because of payoff indifference, our equilibria are subgame perfect too, for suitable off-path play
  - Assume fraction $x \in [0, 1)$ of players stop by time $\tau \geq 0$.
  - induced payoff function for this subgame is:

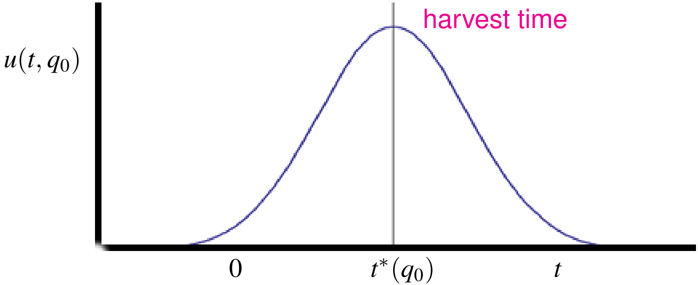
$$u_{(\tau,x)}(t, q) \equiv u(t + \tau, x + q(1 - x)).$$

  - $u_{(\tau,x)}$ obeys our assumptions if $(\tau, x) \in [0, \infty) \times [0, 1)$.

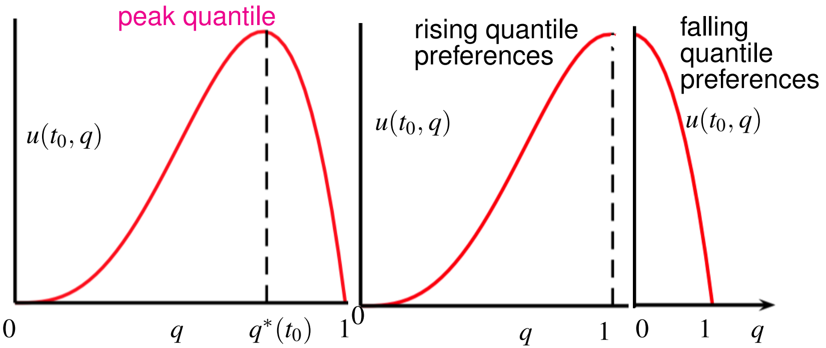
Nash Equilibrium is Strictly Credible (Nerdy)

- Our equilibria are strictly subgame perfect for a nearby game in which players have perturbed payoffs:
- As in Harsanyi (1973), payoff noise purifies strategies
  - Index players by types ε with C^1 density on $[-\delta, \delta]$
  - stopping in slow play at time t as quantile q yields payoff $u(t, q, \varepsilon)$ to type ε.
  - $\varepsilon = 0$ has same payoff function as in original model: $u(t, q, 0) = u(t, q)$, $u_t(t, q, 0) = u_t(t, q)$, $u_q(t, q, 0) = u_q(t, q)$.
  - $u(t, q, \varepsilon)$ obeys all properties of $u(t, q)$ for fixed ε, and is log-supermodular in (q, ε) and (t, ε)

  $\Rightarrow$ players with higher types ε stop strictly later
- For all Nash equilibria Q, and $\Delta > 0$, there exists $\bar{\delta} > 0$ s.t. for all $\delta \leq \bar{\delta}$, a Nash equilibrium Q_δ of the perturbed game exists within (Lévy-Prohorov) distance Δ of Q.


harvest time
$u(t, q_0)$
0
$t^*(q_0)$
t

Payoffs and Quantile



hump-shaped quantile preferences

monotone quantile preferences

- Since players earn the same Nash payoff $\bar{w}$, indifference prevails during gradual on an interval:

$$u(t, Q(t)) = \bar{w}$$

- So it obeys the gradual play differential equation:

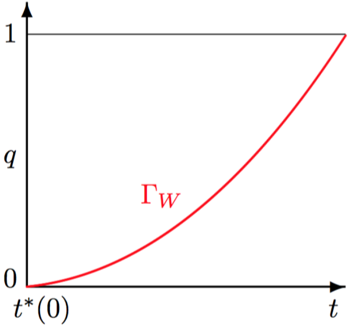
$$u_q(t, Q(t))Q'(t) + u_t(t, Q(t)) = 0$$

- The stopping rate is the marginal rate of substitution, i.e. $Q'(t) = -u_t/u_q$
- Since $Q'(t) > 0$, slope signs u_q and u_t must be mismatched in any gradual play phase (interval):
  - *Pre-emption phase:* $u_t > 0 > u_q \Rightarrow$ time passage is fundamentally beneficial but strategically costly.
  - *War of Attrition phase:* $u_t < 0 < u_q \Rightarrow$ time passage is fundamentally harmful but strategically beneficial.

Pure War of Attrition: $u_q > 0$

- If $u_q > 0$ always, gradual play begins at time $t^*(0)$.
- So the Nash payoff is $u(t^*(0), 0)$, and therefore the *war of attrition gradual play locus* Γ_W solves:

$$u(t, \Gamma_W(t)) = u(t^*(0), 0)$$



Alarm and Panic

- *running average payoffs*: $V_0(t,q) \equiv q^{-1}\int_0^q u(t,x)dx$
- *Fundamental growth dominates strategic effects if*:

$$\max_q V_0(0,q) \leq u(t^*(1),1) \quad (1)$$

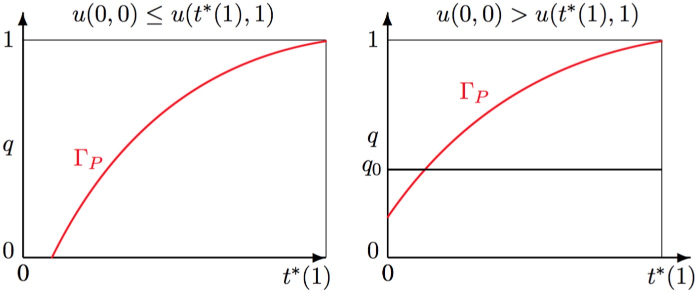
- When (1) fails, stopping as an early quantile dominates waiting until the harvest time, if a player is last.
- There are then two mutually exclusive possibilities:
  - *alarm* when $V_0(0,1) < u(t^*(1),1) < \max_q V_0(0,q)$
  - *panic* when $u(t^*(1),1) \leq V_0(0,1)$.
- Given alarm, there is a size $q_0 < 1$ alarm rush at $t = 0$ obeying $V_0(0,q_0) = u(t^*(1),1)$.

Pure Pre-Emption Game: $u_q < 0$

- If $u_q < 0$ always, gradual play ends at time $t^*(1)$.
- So the Nash payoff is $u(t^*(1), 1)$, and therefore:

$$u(t, \Gamma_P(t)) = u(t^*(1), 1)$$

- If $u(0,0) > u(t^*(1), 1)$, there is alarm or panic $\Rightarrow$ a time-0 rush of size q_0 and then an inaction period along the black line, until time t_0 where $u(q_0, t_0) = u(1, t^*(1))$.



Equilibrium Characterization

[Equilibria]

1. With increasing quantile preferences, a war of attrition starts at the harvest time in the unique equilibrium.
2. With decreasing quantile preferences, a pre-emption game ends at the harvest time in the unique equilibrium.
   - With alarm there is also a time-0 rush of size q_0 obeying $V_0(0, q_0) = u(t^*(1), 1)$, followed by an inaction phase, and then a pre-emption game ending at $t^*(1)$
   - With panic, there is a unit mass rush at time $t = 0$.

Rushes

- Purely gradual play requires that early quantiles stop later and later quantiles stop earlier: $u_t \lesseqgtr 0$ as $u_q \gtreqless 0$
- We cannot have more than one rush, since a rush must include an interval around the quantile peak
- There is exactly one rush with an interior peak quantile.
- By our logic for rushes, we deduce that equilibrium play can never straddle the harvest time.
- So all equilibria are *early*, in $[0, t^*]$, or *late*, in $[t^*, \infty)$.

Peak Rush Locus

Model
Monotone Payoffs in Quantile
Hump-Shaped Quantile Preferences
Comparative Statics
Applications

- A *terminal rush* includes quantiles $[q_1, 1]$.
- An *initial rush* includes quantiles $[0, q_0]$.
- The *peak rush locus* secures indifference between payoffs in the rush and in adjacent gradual play:

$$u(t, \Pi_i(t)) = V_i(t, \Pi_i(t))$$

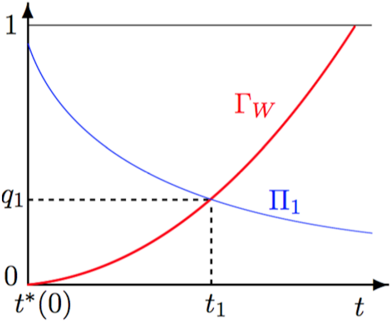
- Since "marginal equals average" at the peak of the average, we have $q_i(t) \in \arg\max_q V_i(t, q)$, for $i = 0, 1$

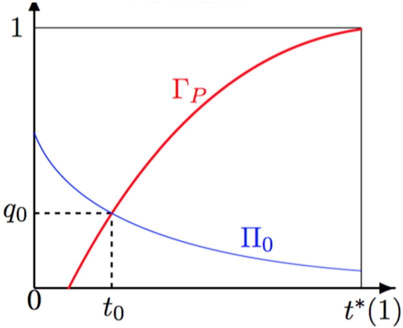
Early and Late Rushes

Peak Initial Rush

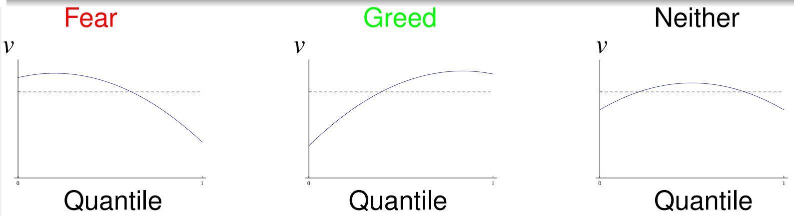
Peak Terminal Rush

Finding Equilibria using the Peak Rush Loci





Greed and Fear



- We generalize the first and last mover advantage.
- *Fear at time t* if $u(t,0) \geq \int_0^1 u(t,x)dx$.
  Extreme case: peak quantile is 0 (pure pre-emption)
- *Greed at time t* if $u(t,1) \geq \int_0^1 u(t,x)dx$.
  Extreme case: peak quantile is 01 (pure war of attrition)
- *Greed and fear at t are mutually exclusive*, because payoffs are single-peaked in q.

[Equilibria with Rushes] *For a hump-shaped quantile preferences, all Nash equilibria have a single rush. There is either:*

1. *A pre-emption equilibrium: an initial rush followed by a pre-emption phase interval ending at harvest time* $t^*(1)$ *iff there is not greed at time* $t^*(1)$.
2. *A war of attrition equilibrium: a terminal rush preceded by a war of attrition phase interval starting at harvest time* $t^*(0)$ *iff there is not fear at time* $t^*(0)$ *and no panic.*
3. *A unit mass rushes, but not at any positive time with strict greed or strict fear.*

- Recall the gradual play differential equation:

$$u_q(t, Q(t))Q'(t) + u_t(t, Q(t)) = 0$$

- Since $u_t(t^*(q), q) = 0$ at the harvest time, $Q'(t_\pi) = 0$.
- Differentiate, and substitute for Q', into:

$$Q'' = -\left[u_{tt} + 2u_{qt}Q' + u_{qq}(Q')^2\right]/u_q$$

[Stopping Rates] *If the payoff function is log-concave in t, the stopping rate $Q'(t)$ increases from 0 during a war of attrition phase, and decreases during a pre-emption game phase down to 0.* Proof if $u_t < 0$: As u is logconcave in t, logsubmodular in (t, q):

$$[\log Q'(t)]' = [\log(-u_t/u_q)]' = [\log(-u_t/u)]_t - [\log(u_q/u)]_t \geq 0 - 0$$

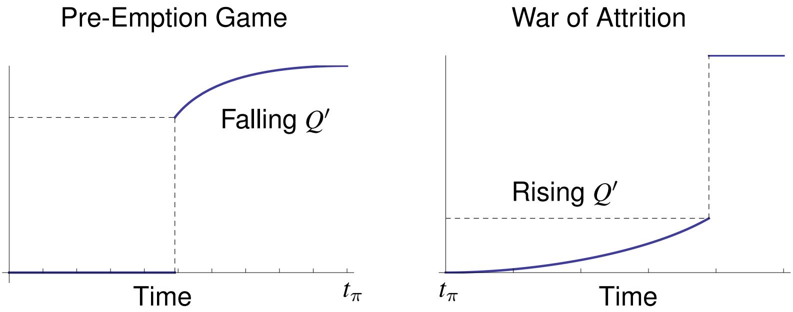


Figure: **Stopping Rates.**

- Wars of attrition: waxing exits, culminating in a rush.
- Pre-emption games begin with a rush and conclude with waning gradual exit.

Refinement: Safe Equilibria

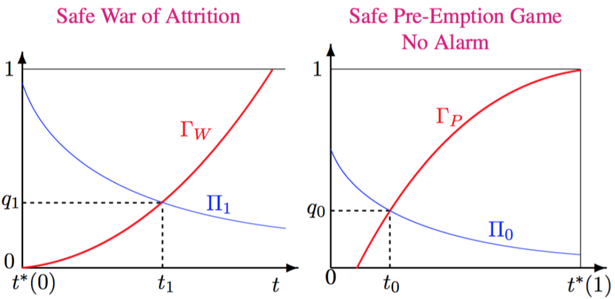
- *ε-safe equilibria* are immune to large payoff losses from ε timing mistakes, when agents have both slightly fast and slightly slow clocks.
- A Nash equilibrium is *safe* if ε-safe for all small $\varepsilon > 0$

Theorem

A Nash equilibrium Q is safe if and only if it support is non-empty time interval or the union of $t = 0$ and a later non-empty time interval.

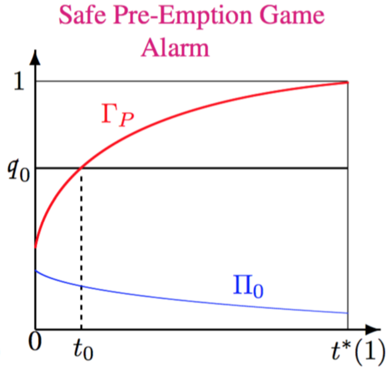
Absent fear at the harvest time $t^*(0)$, a unique safe war of attrition equilibrium exists. Absent greed at time $t^*(1)$, a unique safe equilibrium with an initial rush exists:

1. with neither alarm nor panic, a pre-emption equilibrium with a rush at time $t > 0$



[continued]

2. with alarm, a rush at $t = 0$ followed by a period of inaction and then a pre-emption phase;
3. with panic, a unit mass rush at time $t = 0$.



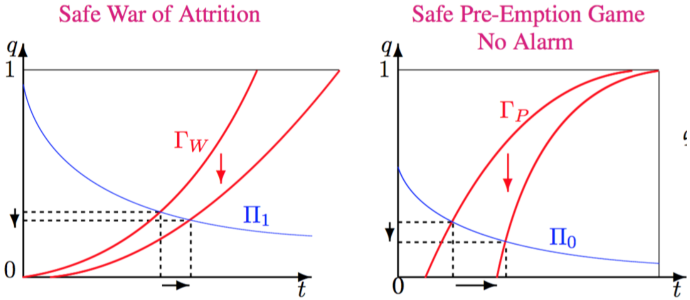
Equilibrium Characterization

- An *inaction phase* is an interval $[t_1, t_2]$ with no stopping
- There can only be one inaction phase in equilibrium, necessarily separating a rush from gradual play.
- There exist at most two safe Nash Equilibria:
  1. With strict greed, there is a unique safe equilibrium: a war of attrition equilibrium and then a rush.
  2. With strict fear, there is a unique safe equilibrium: a rush and then a pre-emption equilibrium.
  3. With neither greed nor fear, both safe equilibria exist, and no others.

- In a *harvest delay*, $u(t, q|\phi)$ is log-supermodular in (t, ϕ) and log-modular in (q, ϕ), so that $t^*(q|\phi)$ increases in ϕ

[Fundamentals] Let Q_H and Q_L be safe equilibria for $\varphi_H > \varphi_L$.

1. If Q_H, Q_L are wars of attrition, then
   - $Q_H(t) \leq Q_L(t)$
   - the rush for Q_H is later and no smaller
   - gradual play for Q_H starts later
   - $Q_H'(t) < Q_L'(t)$ in the common gradual play interval
2. If Q_H, Q_L are pre-emption equilibria, then
   - $Q_H(t) \leq Q_L(t)$
   - the rush for Q_H is later and no larger
   - gradual play for Q_H ends later
   - $Q_H'(t) > Q_L'(t)$ in the common gradual play interval



- Since the marginal payoff u is log-modular in (t, ϕ) so is the average.

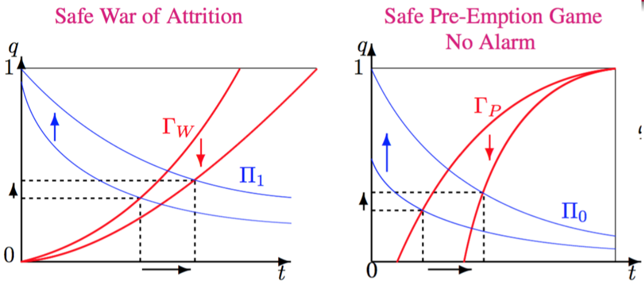
$\Rightarrow$ maximum $q_0(t) \in \arg\max_q V_0(t, q|\phi)$ is constant in ϕ.

$\Rightarrow$ the peak rush locus is unchanged by ϕ

- *Greed rises in γ* if $u(t, q|\gamma)$ is log-supermodular in (q, γ) and log-modular in (t, γ).
- So the quantile peak $q^*(t|\gamma)$ rises in γ.

[Quantile Changes] Let Q_H and Q_L be safe equilibria for $\gamma_H > \gamma_L$.

1. If Q_H, Q_L are war of attrition equilibria, then
   - $Q_H \leq Q_L$
   - the rush for Q_H is later and smaller
   - $Q'_H(t) < Q'_L(t)$ in the common gradual play interval.
2. If Q_H, Q_L are pre-emption equilibria without alarm, then
   - $Q_H \leq Q_L$
   - the rush for Q_H is later and larger
   - $Q'_H(t) > Q'_L(t)$ in the common gradual play interval.



- Define $\mathbb{I}(q,x) \equiv q^{-1}$ for $x \leq q$ and 0 otherwise
- Easily, $\mathbb{I}$ is log-supermodular in (q,x),
- So $V_0(t,q|\gamma) = \int_0^1 \mathbb{I}(q,x)u(t,x|\gamma)dx$.
- So the product $\mathbb{I}(\cdot)u(\cdot)$ is log-supermodular in (q,x,γ)
- Thus, V_0 is log-supermodular in (q,γ) since it is preserved by integration
- So the peak rush locus $q_0(t) = \arg\max_q V_0(t,q|\gamma)$ rises in γ.

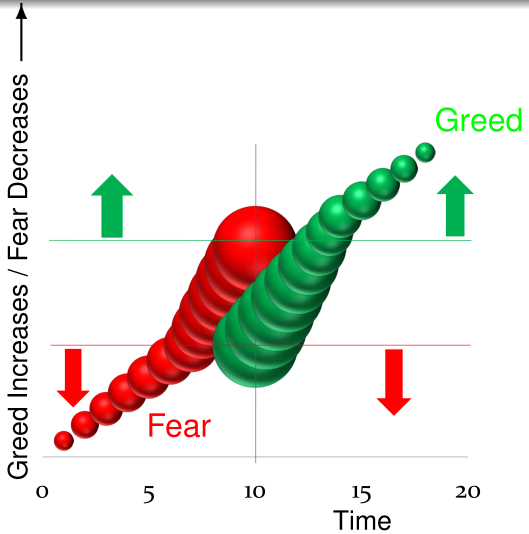
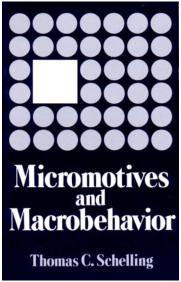


Figure: **Rush Size and Timing**

Example 1: Schelling Tipping

- Schelling (1969): Despite only a small threshold preference for same type neighbors in a lattice, myopic adjustment quickly tips into complete segregation.
- The tipping point is the moment when a mass of people dramatically discretely changes behavior, such as flight from a neighborhood
- In our model (without a lattice), the tipping point is the rush moment in a timing game with fear



Selling from an asset bubble is an *exit* timing game.

Reduced Form Model: Fundamentals

- Asset bubble price $p(t)$ increases deterministically and smoothly, until the bubble bursts; then $p = 0$.
- The exogenous bursting chance is $1 - e^{-rp(t)}$

$\Rightarrow$ Fundamental Payoff: $\pi(t) \equiv e^{-rp(t)} p(t)$

peaks at $p = 1/r$

Reduced Form Model: Quantile Effect

- After fraction q of strategic investors have sold, the endogenous burst chance is q/ℓ
  - $\ell \geq 1$ measures market *liquidity*
- "Keeping up with the Jones" effect: later ranks secure higher compensation through increased fund inflows
  - Seller q enjoys multiple $1 + \rho q$ of the selling price
  - $\rho \geq 0$ measures *relative performance concern*

$\Rightarrow$ Quantile Payoff $\quad v(q) \equiv (1 - q/\ell)(1 + \rho q)$

- v single peaked when $\rho/(1 + 2\rho) < 1/\ell < \rho$.
- $\exists$ fear with low liquidity $3\ell\rho/(3 + 2\rho) < 1$, and greed with high liquidity $3\ell\rho/(3 + 4\rho) > 1$
- Abreu and Brunnermeier (2003) assume $\rho = 0$ (so fear)

Matching (Alvin Roth, et al) turns on an *entry* decision.

- Fundamental ripens and rots because:
  - Early matching costs $\Leftarrow$ "loss of planning flexibility"
  - Penalty for late matching $\Leftarrow$ market thinness
- Equal masses of two worker types, A and B, each with a continuum of uniformly distributed qualities $q \in [0, 1]$.
- Hiring the right type of quality q yields payoff q.
- Firms learn their need at a rate $\delta > 0$ for A or B (50-50)
- The chance of choosing the right type by matching at time t is $p(t) = 1 - e^{-\delta t}/2$.
- Impatience causes a rotting effect. Altogether, the fundamental $\pi(t)$ is hill-shaped.

Reduced Form Model: Quantile Effect

- Quantile: condemnation of early match agreements
  - Assume *stigma* $\sigma(q) = \bar{\sigma}(1-q)$ of early matching
- Assume initially unit mass of workers and 2α firms
  - The best remaining worker after quantile q of firms has already chosen is $1-\alpha q$.
- The quantile function $v(q) = (1-\alpha q)(1-\sigma(q))$ is concave if σ is decreasing and convex.
- $\exists$ fear if $\bar{\sigma} < 3\alpha/(3+\alpha)$ and greed if $\bar{\sigma} > 3\alpha/(3-\alpha)$.
- Fear obtains provided stigma is not a stronger effect than market thinness.

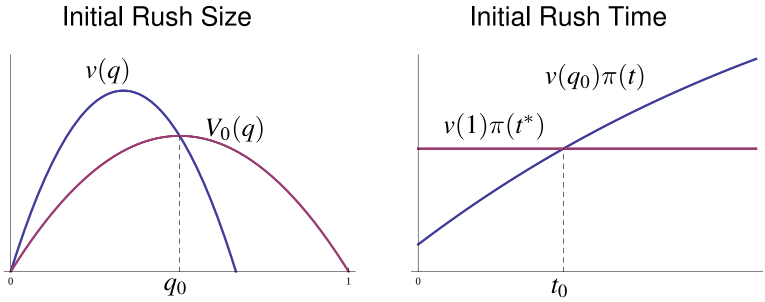


Figure: **Matching Example: Pre-Emption Construction.** With the multiplicative matching payoffs: $u(q,t) = v(q)\pi(t)$, the rush size and rush time are determined separately. At left, the crossing of v and V_0 fixes the initial rush size q_0. At right, the crossing of the rush payoff and harvest time payoff fixes the initial rush time t_0.

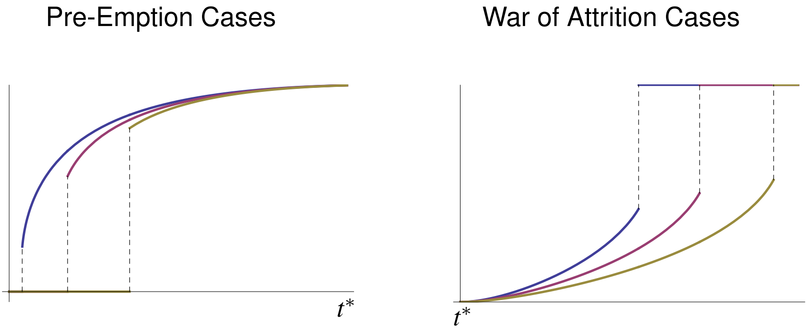


Figure: **Matching Example: Changes in Stigma.** For the safe pre-emption equilibrium, as stigma rises, larger rushes occur later and stopping rates *rise* on shorter pre-emption games. For the safe war of attrition equilibrium, as stigma rises, smaller rushes occur later and stopping rates *fall* during longer wars of attrition.

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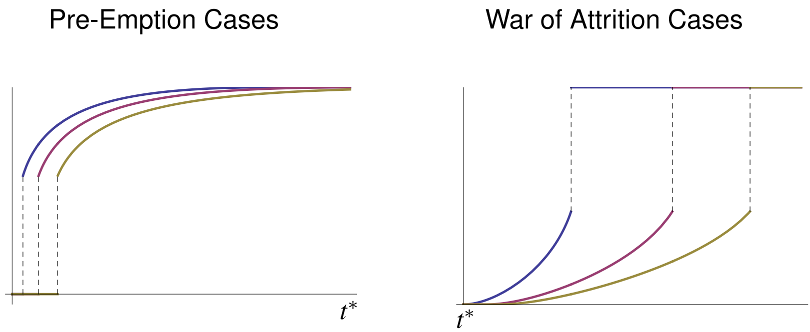

Figure: **Matching Example: Changes in r.** For the safe pre-emption equilibrium, as r falls, rushes and stopping during gradual play again occur later, but stopping rates rise. For the war of attrition equilibrium, as r falls, rushes and stopping during gradual play both occur later, and stopping rates fall.