Search at the Margin[†]

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We extend search theory to multiple indivisible units and perfectly divisible assets, solving them respectively with induction and recursion. Buyer demands and prices are random, and the seller can partially exercise orders. With divisible assets, the Bellman value function is increasing and strictly concave, and the optimal reservation price falls in the position, reflecting increasing holding costs (*opportunity cost of delaying optionality for inframarginal units*). *The marginal value exists, and is strictly convex with a falling purchase cap density. Our model is amenable to price-quantity bargaining; e.g., greater buyer bargaining power is tantamount to greater search frictions.* (*JEL* C61, C78, D25, D83, G31)

Dynamic search has so far largely been applied to wage search (McCall 1970) and the rare market environments where one seeks to trade a single unit of a good. In multi-unit trade settings, the general dynamic search model has been a bridge too far, and so has been simplified by assuming a one-period horizon, or fixing outside options. Extending dynamic search theory is important, as the single unit restriction limits its scope of application.

We develop a unified theory of search for the sale of many units of a good, or of a large divisible asset position. Of course, if each arriving buyer seeks to purchase the entire position at a constant unit price, then a reservation price rule is optimal, per usual. Our buyers have finite demands. In our motivational example, the seller holds finitely many indivisible units and a buyer arrives every period seeking a random number at a random bid price. In our primary continuously divisible units model, buyers periodically arrive with random price-quantity limit orders. In both models, the seller may partially exercise buyer requests—hence, *search at the margin*. We formulate and solve each as a dynamic programming problem.

Both choice and chance play key roles in dynamic search theory. The seller chooses a reservation price to balance the *optionality* gains of a better stopping payoff—namely, the expected surplus over the reservation price—and the sure search costs. A riskier price distribution increases optionality, and thereby encourages more ambitious search, namely, a higher reservation price. But with multi-unit search, optionality cuts two ways, as both an immediate benefit and an opportunity cost of delaying the optionality of one's inframarginal units. This latter *holding cost* is endogenous, and is a key idea underlying both parts of our paper. The holding cost penalizes ambitious search and is increasing in the position, since each unit held offers some optionality. As a result, each additional unit must offer greater optionality to compensate for the rising holding cost; therefore, the reservation price must decrease, and eventually we argue that it falls to the dividend value (see Figures 2 and 6).

Three Economic Examples.—Multi-unit search is a widespread phenomenon, and arises without an organized thick market. Market thickness is indeed a matter of the time frame. When a vast number of goods must be quickly sold—as in a liquidation sale with seasonal products, fashion clothing, or hi-tech products—retailers face a multi-unit buyer search exercise. Retailers annually liquidate thousands of stores and billions of dollars of inventory (Bitran and Mondschein 1997).

For another example, consider how an increasing share of workers hire out their weekly time piecemeal, as opportunities arrive. A lawyer or accountant just starting out may be unable to secure enough billable hours, and so may instead take side jobs working for another firm. After that, he may entertain periodically arriving demands on his increasingly limited time.

A class of examples that may best fit our assumptions and illustrate our predictions are ticket sales for timed events, such as an opera or an airline flight. If buyers seek tickets randomly, as opportunities and needs arise, then the looming deadline induces a thin market. In a key operations research paper, Lee and Hersh (1993) developed a dynamic programming search model of airline ticket pricing in which reservation prices rise as the booking capacity falls; our indivisible unit model in Section IB fully solves this exercise. The empirical work has since precisely measured this paper's capacity findings. In line with our results, Williams (2013) finds evidence that the expected revenue (i.e., value) is increasing and concave in the unsold capacity, with prices rising as seats are sold.[1]

Summary of Results.—Our motivational model in Section I with indivisible asset holdings subsumes standard wage and price search as a special case. There, the seller trades off an immediate sure gain for an uncertain future one; the reservation wage balances these concerns, accounting for the optionality. With multiple units, rather than a single Bellman value, we have a sequence of values, one for each level of holdings. The value increments govern trading behavior, acting as the seller's marginal opportunity cost. The reservation price for each level holdings

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[1] He calculates that a 20 percent fuller flight is on average 25 percent more, and a 30 percent fuller flight 35 percent pricier. Alderighi, Nicolini, and Piga (2015) find that each extra sold seat raises flight fares 3.11 percent. Escobari (2012) reports that a standard deviation increase in utilized capacity raises fares 49.09 percent.

equals the incremental value—thus leaving the seller indifferent about the last sale. So equipped, we reason by induction on the holdings, proving that the reservation price falls with more units, and that the value rises but with diminishing increments. As the units are sold off, holding costs fall, since fewer remain to be sold. The impact can be dramatic: the reservation price rises from 2.9 to 8.3 in Figure 2 as 10 units are sold. The seller's ability to partially act upon buyer demands is critical, as reservation prices need not fall when the seller cannot partially act on buyer offers.

Our primary model assumes a perfectly divisible asset position, and therefore employs standard marginal analysis. To understand the intensive margin, we focus in Section IIA on the one-stage decision problem that confronts a seller when a buyer arrives, and his value function is fixed. If the value is weakly convex, then selling is all-or-nothing. For instance, a special case is wage search, where a single reservation wage describes behavior. Partially selling a position is optimal with a concave value function, and in this case, the marginal value dictates the seller's supply curve.

So inspired, our voyage indeed finds that value functions are concave in the general model in Section IIB. We assume an impatient owner of a large position in a divisible asset. The seller discounts the future, possibly earning dividends on his holdings until any sale—for instance, he might earn a rental fee on unsold inventory. The seller lacks access to a market, and instead can only slowly sell it off to buyers randomly arriving in continuous time. A focal special case is stationary Poisson arrivals, whose predictions standardly coincide with those for discrete time. Buyers arrive with random limit orders, described by a random bid price and a random purchase cap. This is consistent with the "Name-Your-Own-Price" (NYOP) business model of Priceline, and a vast number of Third World markets.

We employ recursion logic to characterize the *option value* of search, as well as the larger Bellman value, which includes the present value of dividends. Parallel to the diminishing value increments with indivisible units in Section I, value functions are increasing and strictly concave with a perfectly divisible asset (Theorems 1 and 3). Theorem 2 uses contraction proof logic, normally reserved for value functions, to deduce that the marginal value exists. Specifically, we argue that if the value function is differentiable, then its derivative defines a contraction mapping that admits a unique bounded fixed point that is continuous.[2] As is true for indivisible units, holding costs increase in the asset position. Corollary 2 then summarizes our key search formula: the time cost of the marginal value is the sum of the dividend, and the optionality of the unit, minus the holding costs.

The trading schedule evolves as the asset position falls, reflecting the endogenous option value of a smaller position. Marginal analysis is valid because the second-order conditions follow from value concavity. The seller's ask price is the inverse marginal value, analogous to the reservation price expression in Section I.

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[2]Benveniste and Scheinkman (1979) does not apply as the optimal policy is frequently a corner solution. The similar result of Blume, Easley, and O'Hara (1982) imposes smoothness assumptions on the objective functions and constraints that do not hold in our model. This new deduction method should prove valuable in other contexts where corner solutions invalidate envelope theorem logic.

The equivalent supply function, or the maximum sold at any price, then admits a simple formula in Corollary 4 that is increasing in the price, and linearly increasing in position. All told, the seller sells more with higher bid prices, and less with a smaller position. He refuses to sell for bid prices below a *choke price*, wants to sell out (or liquidate) for bid prices above a higher *sell-all price*, and partially unwinds his position for intermediate prices. In all cases, his offer is truncated by the buyer's cap.

The marginal value is falling because of the finite buyer purchase caps. But consider our indivisible units model with unit demands. Here, holding costs are the discounted continuation values, and these rise in the holdings, but with diminishing increments. Hence, reservation prices—which are depressed by holding costs—have diminishing decrements. With a perfectly divisible asset, Theorem 4 extends this indivisible unit result by extrapolating on unit demands: with a decreasing purchase cap density, the marginal value is convex. As a result, the cost of selling an extra unit—analogous to the marginal cost function in producer theory—is convex in the quantity sold (see Figure 5). Its inverse, the supply curve, is a concave function of the price whenever it is positive (Corollary 6). So the elasticity of this supply curve tapers off at larger positions, as the buyer offers a more generous bid.

In our paper, search frictions are captured by the buyer arrival rates, the purchase cap distribution, and the seller's impatience. The value function and trading behavior optimally adjust when these frictional measures change—e.g., a thinner or more sporadic market at one extreme (low arrival rates) or a fire sale at the opposite (extreme impatience). Intuitively, with greater frictions, search optionality drops, and the value and marginal value of assets fall too (Theorem 5). For an intuition, assume that buyers arrive less often. Then the seller trades more in each meeting, and therefore is less price sensitive; accordingly, his value function falls and flattens, as seen in Figure 6. Our most surprising prediction concerns purchase caps, which play a central role. Whereas the seller standardly profits from greater price risk, we find that he is harmed by a riskier purchase cap distribution, due to the value concavity (Theorem 6).

The seller adjusts his behavior when he grows more impatient, when buyers arrive more frequently, or when the price or purchase cap distribution changes. He adjusts on both intensive and extensive margins: his willingness to sell in each trading opportunity, and his willingness to pass up trades altogether, and thereby adjust the waiting time between sales. For instance, a more impatient seller, or one with a smaller dividend, acts as if his asset position is larger—the mean time to trade falls and supply both increase (Theorems 7 and 8).

As is well known, search models capture settings with temporal market power, and thus where prices may be bargained—as holds in many markets where buyers interact directly with the seller. Our default limit order model makes the standard assumption of search theory, and assumes that buyers venture their willingness to pay, and sellers optimally respond.[3] Intuitively, this assigns all bargaining power to the seller. But our theory remains tractable and our results robust in

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[3] Our limit order model is a one-round version of the Walrasian bargaining model of Yildiz (2003).

Section III when Nash bargaining fixes the trade quantity and price. We uncover a general principle—that *greater bargaining power for buyers is formally equivalent to increased search frictions*; it raises the supply, reduces the negotiated price, improving the ease of trade. All told, the price and quantity move oppositely. With bargaining, for instance, the supply still rises in the position, but the negotiated price falls. Greater bargaining power for buyers also lowers the waiting times between sales. In an application, we show that if two buyers A and B share the same reservation price, but B wishes to acquire a larger share of the seller's position, then B will pay a higher negotiated price. Crucially, seller B does not get a quantity discount.

Literature.—Since our paper develops a dynamic theory of multi-unit search, it helps to document the economic settings where this need is most pressing. In industrial organization, price search literature has been limited by the existing theory, and has only explored single unit search.[4]

But trade models are the primary potential application of search at the margin, since goods, assets, and money are naturally divisible.[5] Yet, these models have conspicuously avoided dynamic search exercises with a perfectly divisible asset. Monetary theory is the largest body of work here (Lagos, Rocheteau, and Wright 2017). For instance, Lagos and Wright (2005) deliver a tractable monetary random matching model with divisible assets and a divisible good. Note that they assume that search takes one period ("day"), and is followed by an unchanging frictionless centralized "night" market that fixes the outside option value of money.

Closer to us, Molico (2006) removes the access to the "night market" and *numerically* finds a concave value of money holdings in a dynamic equilibrium model. He assumes a strictly concave payoff function and an increasing and convex marginal production cost. Our theory applies if we swap the roles of buyers and sellers, and think of limit orders as random convex cost functions. We conjecture that even with linear stage payoffs, value concavity would still arise, as in our model, due to the convex cost functions.

Periodic access to a market is the search friction underlying Lagos and Rocheteau (2009), who assume divisible assets in an equilibrium search model of an over-the-counter market. Investors randomly meet dealers, who trade in a competitive market. Unlike our paper, investors do not face a purchase cap restriction, and thus the asset optionality is constant. In other words, while the investor's selling strategy depends forwardly on the expected market price, it is *independent* of his asset position—the essence of our search at the margin. Our value function is concave for a dynamic reason due to the random purchase caps, whereas theirs inherits the assumed concavity of the static utility of consumption.

Since our paper lacks a close prequel, we next modify the benchmark McCall wage search model in Section I, assuming finitely many units. In this standalone section, we link diminishing value increments and falling reservation prices. For the perfectly divisible asset model in Section II, we characterize the value function

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[4]Classics here (e.g., Rothschild 1974 and Weitzman 1979) are hardwired to unit demands.
[5]For instance, Smith (1994) explores how search frictions affect trade with indivisible heterogeneous goods and unit demands where "beauty is in the eye of the beholder."

and supply schedule in Section IIC, and do sensitivity analysis in Section IID. We analyze the seller's behavior and supply in Section IIE, and introduce bargaining in Section III. Our indivisible units section (Section I) relies on the interplay of values and reservation prices, and logically proceeds via induction proofs. By contrast, the divisible asset analysis in Sections II and III depends on value functions and supply schedules, and uses recursion contraction logic.

I. Sequential Search for Multiple Indivisible Units

We first characterize the optimal gradual sale of many indivisible goods, subsuming existing single unit search theory as a special case. In the first case, arriving buyers have unit demands, but in the second case, they have random multi-unit demands. This model is in discrete time.

A. *Single Unit Sales*

Consider a home builder who can rent out n unsold homes for a rental (*dividend*) $\kappa > 0$ per period, or a ticket seller who earns nothing ($\kappa = 0$) on held inventory.[6] A new home or ticket buyer arrives every period with a take-it-or-leave-it *random price* P from cumulative distribution function (CDF) F. The price P has positive variance on the support $[\underline{p}, \overline{p}]$, and exceeds the present value of dividends $\kappa/(1 - \beta)$ with positive probability. The seller has discount factor[7] $\beta < 1$.

We start with a key insight on how holdings impact the optimal reservation price. The per-period return on the policy value $\overline{V}_n(\mathcal{R})$ of holding n units given reservation price $\mathcal{R}$ is[8]

$$(1 - \beta)\,\overline{V}_n(\mathcal{R}) = n\kappa + \beta(1 - F(\mathcal{R}))[E(P - \mathcal{R}|P \geq \mathcal{R}) - (\overline{V}_n - \overline{V}_{n-1} - \mathcal{R})]$$

given continuation values $\overline{V}_{n-1}$ and $\overline{V}_n$. If the reservation price $\mathcal{R}$ rises, the falling optionality $(1 - F(\mathcal{R}))E(P - \mathcal{R}|P \geq \mathcal{R})$ balances[9] rising opportunity costs $(1 - F(\mathcal{R}))(\overline{V}_n - \overline{V}_{n-1} - \mathcal{R})$. But with multiple unit search, when holdings rise, the continuation value $\overline{V}_{n-1}$ rises, and the marginal gains to searching fall—the optimal reservation price $\mathcal{R}$ falls.

The (optimal) *value function* V_n of the n units obeys the Bellman equation ($\Diamond$): $V_n = n\kappa + \beta E(\max\{P + V_{n-1}, V_n\})$. The *optimal reservation price* $\mathcal{R}_n$ for the nth unit—the least acceptable bid price one—leaves one indifferent about selling ($\heartsuit$): $\mathcal{R}_n + V_{n-1} = V_n$. Then standard search theory is the $n = 1$ case, i.e., $V_1 = \kappa + \beta E(\max\{P, V_1\})$, since $V_0 = 0$. The *optionality* of the nth unit is the expected excess of the price over the reservation price ($\clubsuit$): $\Omega_n = E(\max\{P - \mathcal{R}_n, 0\})$. Finally, we focus on the search *option value*, or the

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[6] We subsume Dixit's fun (2012) paper that lacks general results. Ours also is a theory of the six bullet dynamic optimization in the 1971 movie *Dirty Harry*, whose last bullet was most valuable.

[7] For computing expected payoffs, the timing is as follows: trades happen in the morning, values are computed at noon, dividends are received in the afternoon, and discounting happens at midnight.

[8] We rewrite $\overline{V}_n(\mathcal{R}) = n\kappa + \beta[(1 - F(\mathcal{R}))(E(P|P \geq \mathcal{R}) + \overline{V}_{n-1}) + F(\mathcal{R})\overline{V}_n]$ assuming $\overline{V}_n(\mathcal{R}) = \overline{V}_n$.

[9] Indeed, the optionality $\int_{\mathcal{R}}(p - \mathcal{R})\,dF(p)\,dp$ has $\mathcal{R}$ derivative $F(\mathcal{R}) - 1 < 0$, which is negative of the $\mathcal{R}$ derivative of $(1 - F(\mathcal{R}))(\mathcal{R} + \overline{V}_{n-1} - \overline{V}_n)$ at the optimum $\mathcal{R} = \overline{V}_n - \overline{V}_{n-1}$. See also footnote 28.
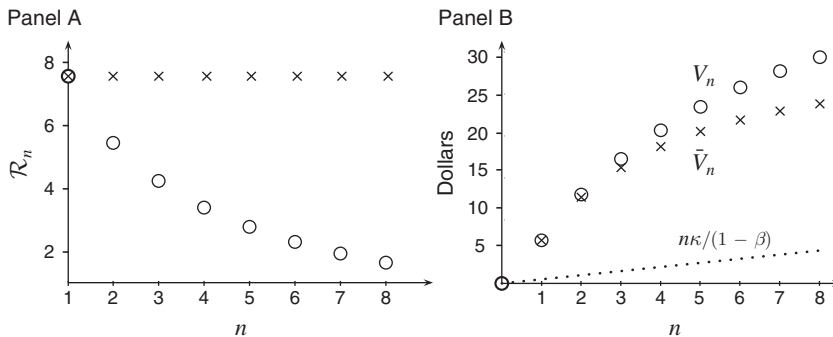
FIGURE 1. SEARCH WITH FINITELY MANY INDIVISIBLE UNITS

Notes: The optimal reservation prices $\mathcal{R}_n$ (left circles) and the optimal values V_n (right circles) when $P \sim \Gamma(4, 2)$, $\kappa = 0.1$, and $\beta = 0.8$. Our effects are large: reservation prices $\mathcal{R}_n$ fall from 7.6 to 1.6, with diminishing steps $|\mathcal{R}_n - \mathcal{R}_{n-1}|$. A reservation price fixed at $\mathcal{R}_1$ (left ×s) yields a lower policy value $\bar{V}_n$ (right ×s).

excess of the value over the dividend present value (♠): $W_n \equiv V_n - n\kappa/(1 - \beta)$. As seen in Figure 1 the following result holds.

PROPOSITION 1: *As asset holdings n increase, the option value W_n increases, and the optimal reservation price $\mathcal{R}_n$ decreases, but with diminishing steps* $|\mathcal{R}_n - \mathcal{R}_{n-1}|$. *Its formula is*

$$(1) \qquad (1 - \beta)\mathcal{R}_n = \kappa + \beta(\Omega_n - \Omega_{n-1}).$$

PROOF:

Write (◇) as $V_n = n\kappa + \beta(E(\max\{P, \mathcal{R}_n\}) + V_{n-1}) = n\kappa + \beta(\mathcal{R}_n + \Omega_n + V_{n-1})$, using (♡) and (♣). Subtract $V_{n-1} = (n - 1)\kappa + \beta V_{n-1} + (1 - \beta)W_{n-1}$, which reworks (♠), to get the reservation price formula (★): $\mathcal{R}_n = \kappa + \beta(\mathcal{R}_n + \Omega_n) - (1 - \beta)W_{n-1}$. To wit, one expects to secure the dividend immediately, and the gross price $\mathcal{R}_n + \Omega_n$ starting next period. For example, $\mathcal{R}_1 = \kappa + \beta(\mathcal{R}_1 + \Omega_1)$ in standard wage search. The last term in (★) does not appear in standard search theory, as it reflects the multiple unit sales. It is the *holding cost*, or *the time cost of delaying the continuation search option value* with one fewer unit $(1 - \beta)W_{n-1}$. Lastly, using (♠), then (◇) and (♡), and then (♣), we conclude

$$(2) \quad (1 - \beta)W_n = (1 - \beta)V_n - n\kappa = [n\kappa + \beta E(\max\{P - \mathcal{R}_n, 0\})] - n\kappa = \beta\Omega_n,$$

thereby proving (1). Hence, the reservation price $\mathcal{R}_n$ falls in the holdings n, as it moves inversely to the optionality $\beta\Omega_{n-1}$, which increases in the option values W_{n-1}, by (2). ∎

For historical context, in their study of airline ticket pricing, Lee and Hersh (1993) studied this discrete-time dynamic programming model with a finite price mesh. They find a weakly falling reservation price. We allow for dividends and deduce *strictly falling* reservation prices for if possible prices sometimes exceed the present value of dividends.
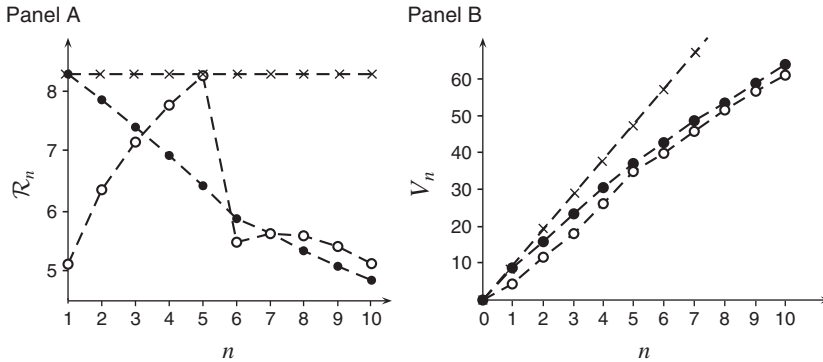
FIGURE 2. RESERVATION PRICES AND INDIVISIBILITY

Notes: The optimal reservation prices $\mathcal{R}_n$ (at left) and value functions V_n (at right) given prices $P \sim \Gamma(4, 2)$, dividend $\kappa = 0.5$, and discount factor $\beta = 0.8$. With no purchase caps, optimal reservation prices are constant ($\times$): one sells or rejects a price. But with five equilikely purchase caps $\ell = \{1, 2, 3, 4, 5\}$, optimal reservation prices fall dramatically. Reservation prices are nonmonotone with all-or-nothing sell decisions (white circles), but fall in one's holdings if unit trades are allowed (black circles). At right, the values are all strictly higher given optimal exercise of this option.

B. *Multiple Unit Sales and Limit Orders*

Now, assume a buyer arrives every period wishing $\ell = 1, 2, \ldots$ units with *arrival chance* $\alpha_\ell > 0$, where $\alpha_1 + \alpha_2 + \cdots = 1$. Define the running arrival chance $A_n \equiv \alpha_1 + \cdots + \alpha_n$. Then $\alpha_1 = 1$ in Section IA. We assume that the buyer arrives with a *limit order* demands of the form (p, ℓ), namely, for the purchase of at most ℓ units at a price p. Lee and Hersh (1993) attack this multiple unit demands problem only numerically, assuming finitely many prices. They give an example with a nonmonotone reservation price. We offer our own simple example in Figure 2. We recover a dramatically falling reservation price by adding an intensive margin optimization, allowing the seller to partially meet the limit order.

For a fixed price, the seller is indifferent across all buyer demands that liquidate his holdings, given n units. So the relevant buyer arrival chances $\alpha^n \in \mathbb{R}^n$ obey $\alpha_\ell^n = \alpha_\ell$ for all $\ell \leq n - 1$ and $\alpha_n^n = 1 - A_{n-1}$. Given the vector of past option values $(W_1, \ldots, W_{n-1}) \equiv W^{n-1}$, the next search option value W_n is a fixed point $W_n = F_n(W^{n-1}, W_n | \alpha^n)$, where

$$(3) \qquad F_n(W^n | \alpha^n) \equiv \beta \sum_{\ell=1}^{n} \alpha_\ell^n E\left[\max_{0 \leq i \leq \ell}\left((P - \kappa/(1 - \beta))\,i + W_{n-i}\right)\right].$$

A feasible strategy given n units is to employ the selling policy optimal for $n - 1$ units, but incremented by 1 if $P > \kappa/(1 - \beta)$. Since this option is immediately useful with chance $\alpha_n^n > 0$, this policy yields extra payoff of at least $\beta \alpha_n^n E(\max\{P - \kappa/(1 - \beta), 0\}) > 0$ if one liquidates whenever it is myopically profitable. Hence, $W_1 < W_2 < \cdots < W_n < \cdots$. Define the difference operator $\Delta W_n = W_n - W_{n-1}$.

PROPOSITION 2: *Option value increments are positive and falling:* $\Delta W_1 > \cdots > \Delta W_n > 0$.

The sales policy now must dictate a quantity to sell for every limit order (p, ℓ). The seller uses *search at the margin* if he sells the maximal $i \leq \ell$ units for which $p \geq \mathcal{R}_{n-i+1}$, namely, a reservation price that only depends on the final holdings $n - i$, and not the sales quantity or original holdings. Search at the margin is the natural extension of the reservation price rule of Section IA, and is formally a discrete first-order condition (FOC). Proposition 2 supplies the discrete second-order condition that justifies its optimality. Intuitively, we intersect supply and demand curves, and choose the demand curve price.

We next argue that the reservation price falls in the holdings, and increases in supply, and therefore supply is monotone in the bid price.

COROLLARY 1: *Search at the margin is optimal. The reservation price for selling i of n units equals* $\mathcal{R}_{n-i+1} = \Delta V_{n-i+1}$, *and falls in the final holdings $n - i$. The reservation price solves*

$$(4) \qquad\qquad (1 - \beta)\,\mathcal{R}_n = \kappa + \beta\left(\Omega_n - \sum_{j=1}^{n-1} \alpha_{n-j}\Omega_j\right).$$

This formula subsumes (1), with $\alpha_1 = 1$ and $\alpha_j = 0$ for $j > 1$. The time cost of the reservation price $(1 - \beta)\,\mathcal{R}_n$ is the dividend κ plus the discounted optionality $\beta\Omega_n$, minus the *holding cost* $\beta\sum_{j=1}^{n-1} \alpha_{n-j}\Omega_j$, or discounted foregone surplus of holding an extra unit.[10]

Proposition 2 proves that $\Delta \mathcal{R}_n = \Delta^2 W_n < 0$. Figure 3 depicts the domain for the optionality integral. The claim in Proposition 1 that $\Delta^2 \mathcal{R}_n > 0$ does not extend here, since that relied on the now inapplicable identity $(1 - \beta)W_n = \beta\Omega_n$. In robust numerical examples, the reservation price falls at an increasing or decreasing rate. For sure, as seen in Section IA, reservation prices fall at a decreasing rate for the arrival chances $\alpha^n = (1, 0, \ldots, 0)$.

We next turn to perfectly divisible assets, and thereby shift focus in dynamic search from (sequences of) Bellman values to Bellman value functions.

II. Dynamic Search for Continuously Divisible Assets

A. *Foretaste of Search at the Margin*

We now explore the intensive margin—how much to sell—in our model with a perfectly divisible asset. For simplicity, consider a two-period setting without discounting, fixing the continuation value function u. In the first period, a seller holds a single unit of an asset and meets a buyer, proposing some price $p > 0$. The seller chooses how much $y \in [0, 1]$ to sell. In period two, he derives utility $u(1 - y)$. We assume $u(0) = 0$ and that u is increasing. This trade opportunity has optimal value $v(p) = \max_{y \in [0, 1]} (py + u(1 - y))$.

If the utility function $u(a)$ is convex, then the value function $v(p) = \max\{p, u(1)\}$ is piecewise convex, as in panel A of Figure 4. The outside option is worth $u(1)$ and

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[10] The holding cost $\beta\sum_{j=1}^{n-1} \alpha_{n-j}\Omega_j$ in (4) is no longer simply the time cost $(1 - \beta)\,W_{n-1}$ on the value with one less unit. Generalizing our deduction (2), we derive $(1 - \beta)W_n = \beta\sum_{j=1}^{n} (1 - A_{n-j})\,\Omega_j$ in Section IB.
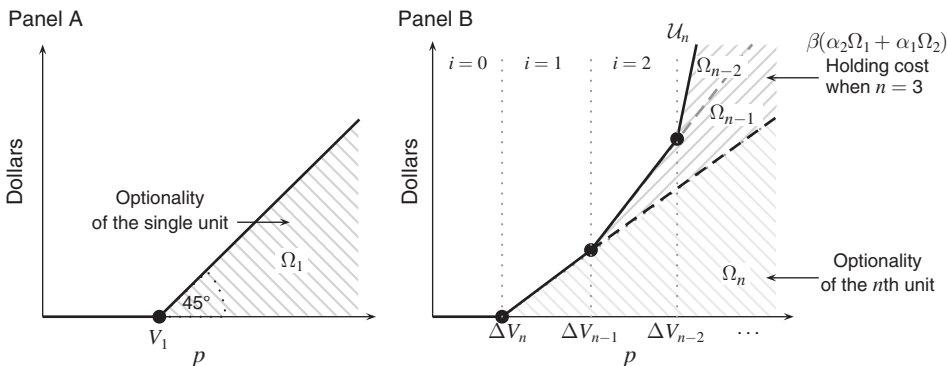
FIGURE 3. SEARCH OPTIONALITY

Notes: In the search Bellman equation $V_1 = \kappa + \beta E \max\{P, V_1\}$ with one unit at left, the max term is V_1 plus the expected *ex post surplus* $E\max\{p - V_1, 0\}$, namely, the optionality Ω_1 in ♣. In a transaction with limit order (p, ℓ), one may sell ℓ of $n \geq 1$ units at price p, the analogous surplus is the upper envelope $\mathcal{U}_n(p, \ell) = \max_{0 \leq i \leq \ell}(pi + V_{n-i} - V_n)$, as defined in equation (21) of Appendix B. Wage search is the $\ell = n = 1$ special case (as $V_1 = \Delta V_1$), seen at left. Its plot at right kinks upward at $p = \Delta V_{n-i+1}$ for sales of $i = 1, \ldots, \ell$ units. Then $E[\mathcal{U}_n(P, \ell)] = \sum_{j=n+1-\ell}^{n} \Omega_j$ is the optionality on the sale. The option value formula in Proposition 2 accounts for the chances of demands $\ell = 1, \ldots, n$.
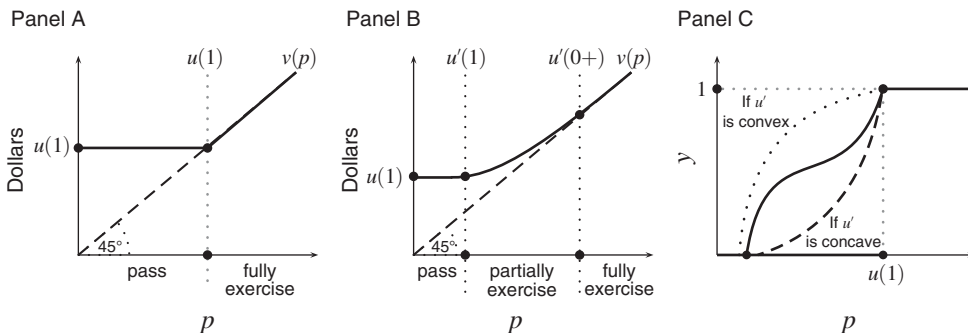

FIGURE 4. SEARCH WITH INDIVISIBLE VERSUS PERFECTLY DIVISIBLE UNITS

Notes: Panels A and B depict the dynamic solution of the wage search problem. The optimal value $v(p)$ is convex. When the utility function $u(a)$ is convex (panel A), stopping is all or nothing. When the utility function $u(a)$ is strictly concave (panel B), one may partially exercise the stopping option. The optimal supply $y^*(p)$ solves the FOC $p - u'(1 - y) = 0$ (panel C) and its shape requires a characterization of the marginal continuation value $u'(a)$.

the inside option p. *In a stationary wage search model*, $u(1)$ *is the reservation wage and p the current wage offer* (McCall 1970). The seller fully exercises his option for high enough prices $p \geq u(1)$; otherwise, he sells nothing. In this case, the divisibility assumption is irrelevant: the same outcome arises when assets are both indivisible ($y = 0, 1$) or divisible ($y \in [0, 1]$).

Assume next a strictly concave and differentiable continuation value $u(a)$ on $[0, 1]$. As seen in Figure 4 (panel B), he sells nothing if $p < u'(1)$, and liquidates for all high prices $p \geq u'(0+)$. For intermediate prices $u'(1) \leq p < u'(0+)$, he partially liquidates his position, and the FOC $p - u'(1 - y) = 0$ fixes the optimal supply $y^*(p) = 1 - (u')^{-1}(p)$. So a strictly concave value $u(a)$ yields a positive and increasing supply (the solid line in panel C of Figure 4). Because the slope of

the ex ante value $v(p)$ is the sales $y(p)$, to understand the trading behavior requires characterizing the marginal continuation value.[11]

We recursively prove that, despite linear dividend payoffs, the utility function u is differentiable and strictly concave, and the marginal utility u' strictly convex. We derive predictions for the value function and resulting trading behavior.

B. *The Model*

Time is now assumed to be continuous on $[0, \infty)$. An infinitely lived seller owns a large but finite *asset position* $a < \infty$ of a perfectly divisible asset (analogous to his holdings in Section I). The asset pays a constant *flow dividend* $k \in [0, \infty)$ per unit share, discounted at the interest rate $r > 0$.

We posit two standard ingredients of search models, that the seller periodically meets a buyer with a random offer. Our arrival process of buyers is quite general: at any time $t \geq 0$, the waiting time τ until the next arrival is a random variable with CDF $\Gamma_t(\tau)$, a differentiable function of t and τ. To capture *positive search frictions*, we assume some delay: $\Gamma_t(0) < 1$. If Γ_t has support $[0, \overline{T} - t]$, there is a deadline $\overline{T} < \infty$, after which arrivals cease. For instance, ticket markets shut down after an event or airline flight. A tractable and common special case is the time-stationary Poisson model, where $\Gamma_t(\tau) = 1 - e^{-\rho\tau}$ for all $t \geq 0$, and $\rho > 0$. Notably, this continuous time model is behaviorally equivalent to a discrete time model with buyers arriving every period, discount factor $\beta \equiv \rho/(r + \rho) < 1$, and per-period dividend $\kappa = k(1 - E(e^{-r\tau}))/r = k/(r + \rho)$, as with indivisible units, in Section I.

In our key model twist, each buyer specifies a limit order *offer* (p, x), namely, not only a *bid price* $p > 0$, but also a *purchase cap* $x > 0$, or the maximum desired quantity.

The price and cap are possibly dependent random variables P, X, and are *independent of the arrival times* τ.[12] We assume a CDF $F(p, x)$ with bounded continuous density $f(p, x)$, weakly falling in x, and marginals $g(p), h(x) > 0$ on $(0, \infty)$. Denote expectations over P, X, τ by E_t. To ensure a well-defined search problem and binding purchase caps, we assume that $E[P] < \infty$ and $E[X] < \infty$. Also, the marginal $h(x)$ and conditional expected price $E[P|x]$ are uniformly bounded: $h(x) \leq \overline{h} < \infty$, and $E[P|x] \leq \overline{p} < \infty$ for all $x > 0$.

Given the buyer's offer (p, x), the seller can then elect any *supply* $y \in [0, \min\{x, a\}]$. This arises, e.g., whenever buyer (p, x) derives quasilinear utility $p \cdot \min\{y, x\}$ minus his costs, where y is the supply. In Section III, we explore a richer alternative model in which the buyer has a reservation price $w > 0$ and a purchase cap $x > 0$, with density $f(w, x)$. In that case, the *terms of trade*—price and supply (p, y)—arise from Nash bargaining.

After selling $y \leq a$ at price p, search continues with the new position $a - y$, and a cash inflow of py. The seller maximizes his expected present value $V_t(a)$ of cash flows

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[11] In the equilibrium model of Lagos and Rocheteau (2009), an investor has a concave utility function over fruit from held assets $a \in \mathbb{R}$ (fruit-bearing trees). His value function $u(\cdot)$ over end of period assets is concave, as it is the sum of a linear trading value and a concave utility of consumption. Given that p is the asset price, the trade size is $y \leq a$, his optimization is $\max_{y \leq a}(py + u(a - y))$.

[12] We use the standard probability protocol that uppercase (e.g., P, X) are random variables, and lowercase their realizations (p, x). We standardly shorten conditional expectations $E[P|X = x]$ to $E[P|x]$.

from dividends and sales at time t, namely, $V_t(a) = ak/r + E_t \sum_{i=1}^{\infty} e^{-r\tau_i} y_i (P_i - k/r)$ where the expectation is taken with respect to the random sequence of trade times τ_i, bid prices P_i, and supplies $y_i \leq X_i, i = 1, 2, \ldots$, where (P_i, X_i) and (τ_i) are governed by F and Γ_t.

For expositional ease, we restrict to stationary Poisson arrivals when we flesh out proofs in Section IIC, as well as for the analysis in Section IID and then in Section III. Proofs for $\tau \sim \Gamma_t$ are in the Appendix.

C. *The Value Function and Selling Strategy*

When meeting a buyer, the seller optimally decides whether and how much to exploit the proposed terms of trade. In so doing, he trades off a sure immediate gain for the option value of future trades. Since one available policy is never to trade, we have $V_t(a) \geq ak/r$. As the right side is an unbounded function of a when $k > 0$, we instead focus on the net-of-dividend *option value function* $W_t(a) = V_t(a) - ak/r \geq 0$.

We solve the problem recursively, using a dynamic programming model in which the state variable is the position $a \geq 0$. The option value discounts until the first buyer arrives, if he does so before the deadline. The trade surplus is scaled by the *expected discounted factor* $B_t = e^{-r\tau} \mathbf{1}_{[0, \bar{T}-t]}(\tau)$, since arrivals stop after calendar time $\bar{T}$, where $\mathbf{1}_{[0, \bar{T}-t]} = 0$.

Let $\mathcal{C}$ be the space of bounded continuous functions on $[0, \infty)^2$ with the sup norm. The option value W is a fixed point $W = \mathcal{T}W$ of the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ defined by

$$(5) \qquad (\mathcal{T}\tilde{W}_t)(a) \equiv E_t \left(B_t \max_{y \in [0, \min\{X, a\}]} \left[\left(P - \frac{k}{r} \right) y + \tilde{W}_{t+\tau}(a - y) \right] \right).$$

To wit, upon meeting a buyer with offer (p, x) before the deadline, the seller maximizes the present value $py + V_{t+\tau}(a - y) = (p - k/r)y + W_{t+\tau}(a - y) + ak/r$ by choosing sales y. The *supply function* $\mathcal{Y}_t(p, x, a)$ is the solution. We prove that the option value is uniquely defined.

LEMMA 1: $\mathcal{T}$ *is a contraction with a unique bounded and continuous fixed point W_t in $\mathcal{C}$.*

For starters, note that the option value function W_t admits an easy upper bound. Any expected discount factor $\beta_t \equiv E_t(e^{-r\tau}) < 1$ is at most $\bar{\beta} \equiv \sup_t \beta_t < 1$— hereby exploiting the assumed inequality $\Gamma_t(0) < 1$. With an infinite position $a = \infty$, one exploits all offers up to the purchase cap. Yet even here, one secures a finite present value $\bar{W} = \bar{\beta} E(PX)/(1 - \bar{\beta}) < \infty$.[13] We derive this upper bound in the Appendix by arguing that it is preserved by the operator $\mathcal{T}$.[14]

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[13] Earlier in Section IB, the analogous option value upper bound was $W_n \leq nE[P]/(1 - \beta) < \infty$, using (3).

[14] Since trading opportunities are bounded, the buyers' purchase caps bind more often with a larger position. The value function is concave even though the seller's asset dividends (the first term in (6)) rise linearly in his

Let $\mathcal{C}_1$ be the space of bounded and continuous functions on assets $a \in [0, \infty)$. We prove concavity using recursive methods and the convex duality theory of Fenchel.

THEOREM 1: *The option value $W_t(a)$ is a concave and increasing function of a.*

PROOF:
Since $\mathcal{T}$ is a contraction and concavity in a is a closed property in $\mathcal{C}_1$ with the sup norm, it suffices that $\mathcal{T}$ preserves concavity in a, for all $t \le \overline{T}$.[15] When meeting a buyer, the seller chooses the post trade asset position $z \equiv a - y$ from the convex constraint set $C(x) = \cup_a \{(z, a) \mid \max\{a - x, 0\} \le z \le a\}$. To use convexity theory, we eliminate the constraint, and instead use the characteristic function $\chi_{C(x)}(z, a) = 0$ if $(z, a) \in C(x)$ and $+\infty$ otherwise. As $C(x)$ is convex and $\chi_{C(x)}$ is convex, we rewrite the operator (5) as[16]

$$(6) \quad (\mathcal{T}\tilde{W}_t)(a) = E_t\left(B_t\left(a(P - k/r) - \min_{z \ge 0}[(P - k/r)z - \tilde{W}_{t+\tau}(z) + \chi_{C(X)}(z, a)] \right) \right).$$

For the recursion, assume that $\tilde{W}_t$ is concave in a for $t \le \overline{T}$. Then $(p - k/r)z - \tilde{W}_{t+\tau}(z) + \chi_{C(x)}(z, a)$ is convex in (z, a), and hence $\min_{z \ge 0}[(p - k/r)z - \tilde{W}_{t+\tau}(z) + \chi_{C(x)}(z, a)]$ is convex in assets a, for $t + \tau \le \overline{T}$, by Theorem 5.3 of Rockafellar (1970). As expectation preserves concavity, $\mathcal{T}\tilde{W}_t$ is concave for $t \le \overline{T}$, and therefore so too is its fixed point $\mathcal{T}W = W$. ∎

Theorem 1 implies that the value $V_t(a) = W_t(a) + ak/r$ is concave and increasing. Our concavity logic is unrelated to standard duality theory in economics, as it exploits the fact that *a minimization of a convex function over one argument is itself convex.*[17] Equivalently, our option value *maximization* (5) yields a *concave* value. Concavity ensues as we capture the constraint by subtracting the convex opportunity cost function $\chi(z, a)$ of holding z units.

A tractable special case is constant Poisson arrivals and no deadline. In this case, the expected discount factor is $E(B_t) = \int_0^\infty \rho e^{-(r+\rho)s} ds = \rho/(r + \rho) = \beta$, and the Bellman equation (5) for the stationary option value W is

$$(7) \qquad W(a) \equiv \beta E\left(\max_{y \in [0, \min\{X, a\}]} \left[\left(P - \frac{k}{r} \right) y + W(a - y) \right] \right)$$

since τ and (P, X) are independent. To fix ideas, consider two extreme cases. If the seller has no option to sell (so that $\rho = 0$), his value reduces to the linear discounted value of dividends $V_t(a) = ak/r$. Next, if the asset pays no dividends, so

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position. We are unaware of another dynamic programming problem in which strict concavity of the value is not inherited from the stage payoff function—as it is in growth theory.

[15] See Corollary 3.2.1 in Stokey and Lucas (1989).

[16] The seller chooses the new position $a' = a - y$, i.e., $a(p - k/r) - \min_{a' \in [a - \min\{x, a\}, a]}[(p - k/r)a' - W_{t+\tau}(a')]$ in (5), minimizing the opportunity cost $(p - k/r)a' - W_{t+\tau}(a')$ of holding assets.

[17] The convex profit function arises from an upper envelope maximization, and the concave cost function from a lower envelope minimization. Likewise, the convex bidder's profit function in a private value auction is an upper envelope maximization. But our concave value ensues from a maximization.

that $k = 0$, then $V = W$ is the present value of the pure option of meeting buyers with acceptable proposed terms of trade.

Wage search models are pure stopping exercises (panel A of Figure 4), where the worker's optimal behavior is fully summarized by a reservation wage. But here, as in Section IIA, we must derive a supply function, and this requires characterizing the marginal value V'. Since it is a concave function, V is almost everywhere differentiable. But we recursively prove more strongly that it is everywhere differentiable. For we find a unique fixed point of a *marginal value operator* $\mathcal{S}$ on $\mathcal{C}$. In the special case of Poisson arrivals and no deadline—recalling the dividend $\kappa = k/(r + \rho)$ and discount factor $\beta = \rho/(r + \rho)$—the marginal value obeys a simple recursion:[18]

$$(8) \quad V'(a) = \kappa + \beta\left(E(\max\{P, V'(a)\}) - \int_0^a \int_0^\infty \max\{p - V'(a - x), 0\}\, dF(p, x)\right).$$

THEOREM 2: *The marginal value $V'_t(a)$ exists on $[0, \infty)$, is continuous, and exceeds k/r.*

The proof in Appendix Section D argues that the marginal value operator $\mathcal{S}$ is a contraction on $\mathcal{C}$, with unique fixed point $V' \in \mathcal{C}$. The upper bound $k/r + \bar{\beta}E(P)/(1 - \bar{\beta})$ of $V'_t(a)$ is the marginal value of the unconstrained problem.[19] Theorem 1 also implies a lower bound for the marginal value: $V'_t(a) - k/r = W'_t(a) > 0$.

Equation (8) embeds some essential economics. Assets have a marginal expected dividend value κ until the next meeting, and then an expected continuation value $E(\max\{P, V'(a)\})$. Analogous to Ω_n in the indivisible units Section I, we call $\omega(z) \equiv E(\max\{P - V'(z), 0\}) > 0$ the *optionality* of the zth unit. The *holding cost* is analogously the subtracted integral in (8)—namely, the expected optionality gains $\beta\eta(a) \geq 0$ arising from all inframarginal units when the seller marginally decreases his position. In other words,[20]

$$(9) \quad \beta\eta(a) \equiv \beta\int_0^a \int_{V'(a-x)}^\infty (p - V'(a - x))g(p)\, dp\, h(x)\, dx \equiv \beta\int_0^a \omega(a - x)\, h(x)\, dx.$$

COROLLARY 2: *In the stationary Poisson arrival case, the holding costs $\eta(a)$ rise in the asset position a, and the marginal value function solves $(1 - \beta)V'(a) = \kappa + \beta(\omega(a) - \eta(a))$.*

As with indivisible units in Corollary 1, the time cost of the marginal value between buyers is the dividend plus the optionality of the marginal unit less its holding cost. Immediate optionality inflates the marginal value, and deferred optionality depresses it.

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[18] This is analogous to the reservation price equation (4) with indivisible units. As $\Omega_n + V_n = E(\max\{P, \Delta V_n\})$, it yields $\Delta V_n = \mathcal{R}_n = \kappa + \beta(E(\max\{P, \Delta V_n\}) - \sum_{j=1}^{n-1} \alpha_j E(\max\{P - \Delta V_{n-j}, 0\}))$.

[19] From (8), in the Poisson model, $V'(0+)$ solves McCall's (1970) wage search Bellman equation, i.e., $rV'(0+) = k + \rho E[\max\{P - V'(0+), 0\}]$, and thus $V'(0+) \leq k/r + \rho E[P]/r$. This is the same upper bound as for V_1 in the indivisible units model in Section IB. Indeed, $V_1 = \kappa/(1 - \beta) + \beta E(\max\{P - V_1, 0\})/(1 - \beta)$, by (3), and thus $V_1 \leq \kappa/(1 - \beta) + \beta E(P)/(1 - \beta)$. Finally, substitute $\beta = \rho/(r + \rho)$ and $\kappa = k/(r + \rho)$.

[20] This parallels the expressions $(1 - \beta)W_{n-1}$ in Section IA and $\beta\sum_{j=1}^{n-1} \alpha_j\Omega_{n-j}$ with indivisible units.

Holding costs rise in the asset position, by Theorem 1, because more optionality is delayed. Observe that the marginal value is increased by the optionality of the marginal unit $\omega(a)$ and decreased by the expected optionality of inframarginal units (i.e., the holding costs).

Without purchase caps, holding costs vanish, and the marginal value recursion (8) is the wage search Bellman equation $rV'(a) = k + \rho E(\max\{P - V'(a), 0\})$. The value function is linear $V(a) = aV'(0+)$, and the option to partially sell the position is worthless.

COROLLARY 3: *Assume $k > 0$. As the asset position a rises, search optionality is a falling fraction $W_t(a)/V_t(a)$ of value, vanishing as $a \to \infty$. In the limit, $\lim_{a\to\infty} V'_t(a) = k/r$.*

PROOF:

Since $V_t(0) = 0$ and V_t is increasing and concave in a, the secant slope $V_t(a)/a$ falls in a. Hence, $rW_t(a)/V_t(a) = r - ak/V_t(a)$ falls in a, recalling $V_t(a) = W_t(a) + ak/r$. Finally, $V'_t(a) \to k/r$ if and only if $V_t(a)/a \to k/r$ by l'Hôpital's Rule, which holds if and only if $W_t(a)/a \to 0$. This follows from the sandwich $0 \leq W_t(a) \leq \overline{W} < \infty$, where Lemma 1 gives $\overline{W} < \infty$. ∎

So the seller increasingly ignores search optionality as his asset position grows, and his value and marginal value eventually only reflects its dividend value to him—specifically, the marginal and average value $V'_t(a)$ and $V_t(a)/a$ both converge to k/r as $a \to \infty$.

We now use Theorem 2 to prove that V_t is strictly concave, enhancing Theorem 1.

THEOREM 3: *The value and option value functions $V_t(a)$ and $W_t(a)$ are each strictly concave.*

Toward a contradiction, assume that V_t is linear on some interval $[0, \overline{a}]$, so that W_t is too. For $a \in [0, \overline{a}]$, the optimal supply in (7) is then $y = \min\{x, a\}$ for prices $P \geq \overline{p}$, and zero otherwise. The first term in the maximand of (7) is then $E(P \min\{X, a\}|P \geq \overline{p})$. Since its derivative $E(\int_a^\infty Pf(P, x) \, dx|P \geq \overline{p})$ strictly falls in a, this is strictly concave, and thus so is the right side of (7). By the same logic, the value is linear on no subinterval.

Since the asset position confers valuable trade opportunities, its value exceeds the present value of dividends. But the seller cannot quickly exploit profitable sales due to the purchase caps. Intuitively, a larger position takes longer to unwind, as incremental assets are sold further in the future. We instead reason oppositely, thinking of the last unit as sold first. Since the last unit has a higher holding cost—i.e., it delays the sale of inframarginal units—the marginal value of assets falls, and thus the value is concave.

Waiting until a buyer arrives willing to buy everything is intuitively unwise. But how should the seller react to partial purchase orders? This yields a static sales exercise, in which the marginal value acts as the marginal (opportunity) cost, as in Section IIA.

COROLLARY 4: *For any asset position $a > 0$, bid price p, and cap $x > 0$, the optimal supply at time t is $\mathcal{Y}_t(p, x, a) = \min\{x, Y_t(p, a)\}$, where the uncapped supply $Y_t(p, a)$ is*

$$(10) \qquad Y_t(p, a) = \begin{cases} \max\{a - (V_t')^{-1}(p), 0\} & \text{for } p \leq V_t'(0+) \\ a & \text{for } p > V_t'(0+) \end{cases}.$$

PROOF:

In a meeting, the seller solves $\max_y [py + V_t(a - y)]$ subject to $0 \leq y \leq x$ and $y \leq a$. As V_t is strictly concave, and the constraints are linear, the FOC is necessary and sufficient for a maximum. The constraint qualification for the Kuhn-Tucker conditions is met, because all constraints are linear. If the multipliers are respectively $\lambda_1, \lambda_2, \lambda_3 \geq 0$, then the FOC is $p - V_t'(a - y) = -\lambda_1 + \lambda_2 + \lambda_3$. By complementary slackness, (i) if $y = x < a$, then $\lambda_1 = \lambda_3 = 0$ and $p - V_t'(a - x) = \lambda_2 \geq 0$, and (ii) if $y = a \not\leq x$, then $\lambda_1 = 0$ and $p - V_t'(0+) = \lambda_2 + \lambda_3 \not\geq 0$. (iii) If $y = 0$, then $\lambda_2 = \lambda_3 = 0$ and $p - V_t'(a) = -\lambda_1 \not\leq 0$, and (iv) if $0 < y < a$ and $y < x$, then all multipliers vanish, and $p = V_t'(a - y)$. The uncapped supply (10) applies when $y < x$, namely when either cases (ii), (iii), or (iv) hold. ∎

Let's flesh out the supply function plotted at the right of Figure 5. First, never trading is not optimal, as it pays $ak/r < V_t(a)$, by Theorem 2. Rather, the seller's supply (10) is the inverse marginal value function $(V_t')^{-1}$ until the purchase cap binds. Next, supply increases in the asset position and purchase cap. In a trading opportunity, supply (10) vanishes for positions $a < (V_t')^{-1}(p)$, and then rises with slope 1 in the asset position until $a = x$. Just as well, supply (10) vanishes at the cap $x = 0$, and increases dollar for dollar in the cap until hitting the *uncapped supply* $x = a - (V_t')^{-1}(p)$.

Next, Figure 5 plots the inverse supply function of the bid price, the uniform sell price for quantity y of the position a is $p = V_t'(a - y)$. Supply vanishes for prices below the *choke price* $V_t'(a)$, while any offer is fully acted upon at any position for prices above the *sell-all price* $V_t'(0+)$. For intermediate bid prices below the *ask price* $V_t'(a - x)$, offers are only partially acted upon, as $Y(p, a) < x$. For higher intermediate bid prices $p > V_t'(a - x)$, offers are fully acted upon, as the uncapped supply is $Y(p, a) > x$.

The seller increasingly ignores search optionality as his asset position explodes, and by Corollaries 3 and 4, the optimal supply becomes infinitely elastic near price k/r.

COROLLARY 5: *The choke and sell-all prices converge to the dividend present value k/r as $a \uparrow \infty$.*

The theory so far has simply relied on the finite mean of purchase caps. We now use the assumption that density $f(p, x)$ is weakly decreasing in x. Unlike with indivisible units in Section I, we find a sufficient condition for the convexity of the marginal value, which yields a supply curve increasing and convex in the bid price.
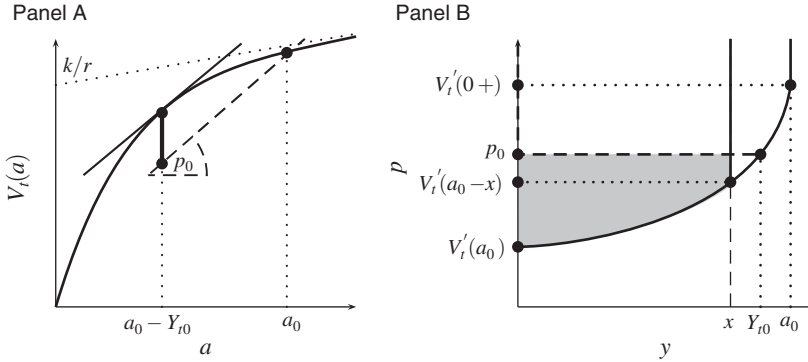
FIGURE 5. THE VALUE FUNCTION AND INVERSE UNCAPPED SUPPLY

Notes: In panel A, we schematically plot the increasing and concave value function $V_t(a)$, with slope $V_t'(a) \to k/r$ as $a \to \infty$. Given a bid price p_0 and asset position a_0 at time t, the trade surplus is the maximum vertical distance of $V_t(a)$ from the dashed line of slope p_0. In panel B, supply is the lesser of the purchase cap x and the uncapped supply Y_{t0}. The uncapped supply maximizes "producer surplus"; it equates price and marginal value $V_t'(a - y)$, soon called the ask price.

THEOREM 4: *The marginal value $V_t'(a)$ is decreasing and strictly convex in a. Moreover, its derivative $V_t''(a) < 0$ exists on $(0, \infty)$, is continuous, and is at least $\overline{V}'' > -\infty$.*

Notably, the derived value function of assets has the same properties typically assumed (for unrelated reasons) of money utility functions u: $u'' < 0 < u'$ with u' convex. So firms act as if risk averse and prudent when optimally selling an asset position.

To intuit Theorem 4, assume Poisson arrivals and independent limit orders $f(p, x) = g(p) h(x)$. Write the Corollary 2 formula as $V'(a) = \kappa + \beta(\omega(a) + V'(a) - \eta(a))$. If $V'(a)$ is convex, then so is the optionality $\omega(a) + V'(a) = E(\max\{P, V'(a)\})$. By recursive logic, the marginal value $V'(a)$ is convex if the holding costs $\eta(a)$ are concave. Loosely, holding costs arise from binding purchase caps, and so are concave when larger purchase caps are less likely, i.e., the cap density $h(x)$ is decreasing. More formally, differentiating (9) yields

$$(11) \qquad \beta \eta'(a) = \beta \omega(0) h(a) + \beta \int_0^\infty \omega'(z) h(\max(a - z, 0)) \, dz.$$

Now, an asset position increment da reduces the chance of full liquidation (sale of all units) by $H'(a) \, da = h(a) \, da$. Recalling the optionality of the 0th unit, this alone increases the holding cost by $\omega(0) h(a) \, da$. Next, since $\omega(z) \equiv E(\max\{P - V'(z), 0\}) > 0$, the optionality of each remaining unit $z = a - x \in (0, a]$ changes by $\omega'(z) \, dz$, which is positive by Theorem 3. The chance $1 - H(a - z)$ that one may sell the zth unit falls by $h(a - z) \, da$. This raises holding costs by $\beta \omega'(z) \, dz \, h(a - z) \, da$, explaining the integral term in (11). All told, a decreasing purchase cap density leads to marginal holding costs.

COROLLARY 6: *When positive, the uncapped supply $Y_t(p, a)$ is increasing and concave in p.*

For as V_t' is decreasing and convex by Theorem 4, its inverse is decreasing and convex in p, and uncapped supply $Y_t(p, a) = a - (V_t')^{-1}(p)$ in (10) is increasing and concave in p.

Altogether, with wage search, a trader stops when he secures a wage in excess of the reservation wage. But in our setting with divisibility and strict concavity, the average exceeds the marginal value, and the seller's trading strategy is governed by the marginal value. Trade may be choked off by the buyer or the seller.

The value function generally evolves with the passage of time. We now offer insights on a special case with a looming deadline $\overline{T}$, and time invariant waiting time distribution until that moment. Specifically, $\Gamma_t(\tau) = 1 - e^{-\rho \min\{\tau, \overline{T} - t\}}$ for all $t \leq \overline{T}$ and $\Gamma_t(\tau) = 0$ if $t > \overline{T}$. Then, for any asset position $a > 0$, we prove in Appendix Section E that the marginal value falls as the calendar time t advances; therefore, by Corollary 4, the supply curve $\mathcal{Y}_t(p, x, a)$ rises. Accordingly, the seller's ask price $V_t'(a - y)$ falls as the deadline approaches.[21]

Toward a more tractable analysis, we henceforth assume the time-stationary Poisson arrivals model for the comparative statics, supply, and bargaining analysis.

D. *Changing Search Frictions and Offer Distributions*

We now explore how the derived functions V, V', V'' change when the parameters k, ρ, r adjust. We next argue that while $V' > 0 > V''$ from Theorems 2 and 3, each inequality grows stricter as search frictions fall: the marginal value rises, but the second derivative falls. Our recursive proof exploits a lemma in Albrecht, Holmlund, and Lang (1991).

THEOREM 5: *For any position $a > 0$, the value $V(a)$ and marginal value $V'(a)$ fall in r, and rise in ρ and k, while $V''(a)$ rises in r and falls in ρ and k.*

The comparative statics of V and V' parallel those in the stationary single unit indivisible search model (Figure 6): as search frictions fall, the value and marginal value increase. But the marginal value falls faster at lower frictions. To wit, the value function flattens.

Next, we explore how shifts in the offer distributions affect the value. We consider changes in the price distribution P, conditional on a quantity, fixing the purchase cap marginal $h(x)$, and in the quantity distribution X, conditional on a price, fixing the price marginal $g(p)$. We call these *conditional* stochastic dominance changes.

THEOREM 6: *The value V rises with* (i) *conditional first-order stochastic dominance increases in P or X;* (ii) *conditional mean-preserving spreads in P; or* (iii) *conditional mean-preserving contractions in X. The marginal value V' rises with* (i).

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[21] Sweeting (2012) documents how ticket prices fall as the deadline approaches. He too assumes an exogenous and constant buyer arrival process, but unlike us, he explains it with a price posting model.
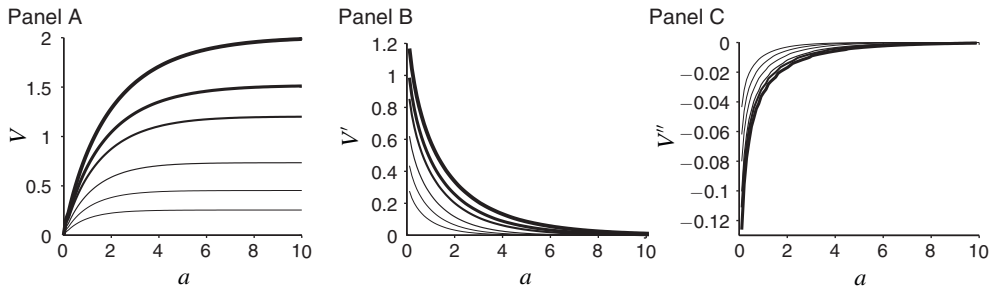
FIGURE 6. How Search Frictions Affect $V(a)$, $V'(a)$, $V''(a)$

Notes: With no dividends, the value function only depends on $\psi = r/\rho$. We posit $P \sim \Gamma(1,1)$ and $X \sim \Gamma(0.5,1)$ in numerical dynamic programming simulations. From thick to thin lines, we plot as search frictions ψ increase: 0.005, 0.1, 0.2, 0.5, 1, and 2. The marginal value falls dramatically; without purchase caps, the marginal value is constant at $V'(0+) \approx 0.4$ if $\psi = 1$.

As with single unit search theory, the seller profits from stochastically better or riskier prices. But quantity risk is different. For offers are truncated by his position, and he exploits only the lower tail of the purchase cap distribution. He profits from stochastically better purchase caps, as his ability to sell his position improves, whereupon values and marginal values both rise. For as the purchase caps stochastically improve, holding costs fall, and so the marginal value and ask price rise. To see this, write holding cost in (9) as $\beta\eta(a) = \beta(E(\omega(a - \min\{X, a\}) - (1 - H(a))\omega(0)))$. The function inside the expectation falls in X, as the optionality of inframarginal units $\omega(z)$ rises. To wit, as the purchase caps stochastically improve, and so the CDF $H(a)$ falls, holding costs fall.

E. *Trading Behavior and the Supply Curve*

If a buyer offers more generous terms of trade, the seller is willing to sell more (Figure 5). This reflects how the seller trades off sure money today and possible money tomorrow. There are a few measures of the willingness to sell. The endogenous arrival rate of acceptable offers equals $\rho\Phi(a) = \rho(1 - F(V'(a), \infty))$, where $\Phi(a)$ is the *trade chance*.

The expected time to trade is $\tau(a) = 1/(\rho\Phi(a))$ and its variance $\xi(a) = 1/(\rho\Phi(a))^2$. Since both fall in the trade chance, given a smaller asset position, higher dividends, or a lower interest rate, the seller is less eager to sell, and the mean and variance of trade times accordingly increase.[22] We argue that the seller finds it increasingly hard to trade as he unwinds his position, and it grows harder to predict the next trade time.[23]

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[22] When ρ rises, does the reservation price rise so much that search time rises? With a log-concave price density, we can show that the mean sales time and its variance fall, for low arrival rates $\rho > 0$.

[23] In this way, we shed light on Alan Greenspan's insight: "Super low interest rates can actually slow the process of liquidation, because the cost of carrying debt is so low" (Devin Leonard and Peter Coy, "Alan Greenspan on His Fed Legacy and the Economy," *Bloomberg Businessweek*, August 9, 2012).
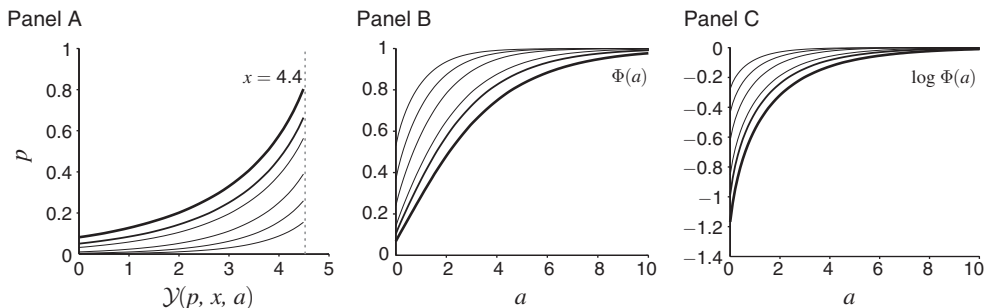
FIGURE 7. HOW SEARCH FRICTIONS AFFECT SUPPLY AND THE TRADE CHANCE

Notes: We plot numerical dynamic programming simulations for prices $P \sim \Gamma(1,1)$ and caps $X \sim \Gamma(0.5,1)$ and no dividend ($k = 0$). Panel A: the supply for $a = 5$ and $x = 4.4$. Panel B: the trade chance as a function of the position. Panel C: the log of the trade chance as a function of assets. In all cases, frictions $\psi = r/\rho$ increase from thick to thin lines, $\psi = 0.005, 0.1, 0.2, 0.5, 1$, and 2.

THEOREM 7: *The mean trade time $\tau(a)$ and its variance $\xi(a)$ fall in the asset position a and interest rate r, and rise in the dividend k. The expected trade price $E(P|P \geq V'(a))$ falls in a, and its variance $\sigma^2(P|P \geq V'(a))$ rises in a if the density $g(p)$ is log-concave.*

Next consider the supply curve. Since the mean trade price $E(P|P \geq V'(a))$ falls in the position a, by Theorem 1, the seller holds out for better prices as he sells his position. But as seen in Figure 8, with a log-concave price density $g(p)$, the variance of traded prices $\sigma^2(P|P \geq V'(a))$ falls in the position a (Heckman and Honoré 1990). Altogether, as the seller unwinds his position, his accepted terms of trade improve and grow more predictable, and the *expected mark-up $E[P - V'(a)|P \geq V'(a)]$* falls.

As the seller unwinds his position, he grows more picky, and trades less. Consider the *supply elasticity* for the uncapped supply (10), i.e., $\mathcal{E}_p(p,a) \equiv p(\partial Y(p,a)/\partial p)/Y$. This is the quotient of the secant and tangent slopes in panel B of Figure 5. For large positions, the secant and tangent coincide at price $\bar{p}(a) < V'(0+)$, where $\mathcal{E}_p(p,a) = 1$. So supply is elastic ($\mathcal{E}_p(p,a) > 1$) for low prices $p < \bar{p}(a)$, and otherwise inelastic ($\mathcal{E}_p(p,a) < 1$).

Next, consider the seller's transactional behavior. An instructive contrast is to Kyle's long-lived (1985) insider, say in possession of unfavorable information, and therefore wishing to sell. His equilibrium trading rule optimally trades off exploiting his informational edge and securing its fruits. Whereas Kyle's insider has a *falling supply curve*, because he depresses the price by selling more today, ours only sells more when offered a higher bid price. Kyle also finds that market depth—an inverse measure of the price impact of trades—is constant over time. In our model, depth is best captured by the slope (not elasticity) of the residual inverse supply $1/\Lambda(y,a) = -1/V''(a - y)$. And unlike in Kyle, it increases in the position and falls in the trade size, by Theorem 4.

Our search framework suggests a new measure of the seller's supply curve, namely, the *purchase premium $\pi(y,a) = V'(a - y) - V'(a)$* over his choke price for trades $y < a$. This is intuitively smaller when he has a smaller price impact, as the next result summarizes.
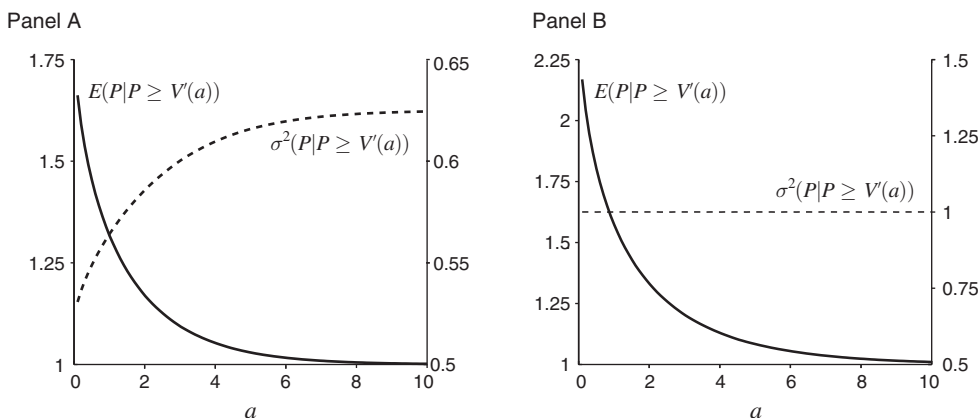
Panel A

Panel B

FIGURE 8. THE MEAN TRADE PRICE AND VARIANCE

Notes: The mean trade price increases and price variance falls as one sells off the position. In panel A, for prices $P \sim \Gamma(1.6, 1/1.6)$, we have $E(P) = 1$ and $\sigma^2(P) = 0.625$. In panel B, when $P \sim \Gamma(1,1)$, we have $E(P) = 1$ and $\sigma^2(P) = 1$. Both numerical dynamic programming simulations assume a cap distribution $X \sim \Gamma(0.5, 1)$.

THEOREM 8: (*i*) *Supply* $\mathcal{Y}(p, x, a)$ *is nondecreasing in* a, r, *and nonincreasing in* ρ, k. (*ii*) *The supply elasticity* $\mathcal{E}_p(p, a)$ *is decreasing and convex in* a, *and vanishes as* $a \to \infty$. *Depth* $1/\Lambda(y, a)$ *is increasing in* a *and decreasing in* y. *It falls in* ρ *and* k, *and rises in* r. (*iii*) *The purchase premium* $\pi(y, a)$ *is increasing in the trade size* $y < a$ *and decreasing in assets* a. *It falls in* r, *and rises in* ρ *and* k.

As the asset position falls, optionality figures more prominently in his optimization, by Lemma 3. Accordingly, the purchase premium rises, depth falls, and supply elasticity rises—the ask-price grows more responsive to the trades (Figure 9). As the position vanishes, the seller exploits asset divisibility less, and price converges to the sell-all price.

As the position a explodes, the optimal sales policy converges to a stationary rule—the seller avails himself fully of all limit offers with prices $p > k/r$, and otherwise abstains. Indeed, $V'(a) \to k/r$ as $a \to \infty$, by Lemma 3. But sales stochastically drift down as the seller's position unwinds. For the seller's own cap starts to bind more than the purchase caps, and he simultaneously grows more choosy due to value concavity—e.g., his choke price rises. A nearly stationary rule is once again optimal for small positions a, selling out for any price $p > V'(0+)$, and the purchase caps don't bind.

It might seem intuitive that trade worsens with greater search frictions. With a higher interest rate or a lower arrival rate, Theorem 8 asserts that the seller's chosen trade volume and depth rises, and the purchase premium falls. On the other hand, with a log-concave price marginal $g(p)$, trade prices fall and grow more volatile with more search frictions, and the expected markup rises[24]—for V' falls, by Theorem 5.

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[24] It might seem puzzling that the expected markup $E[P - V'(a) | P \geq V'(a)]$ rises and yet the purchase premium falls. But the seller's first order condition $p - V'(a - y) = 0$ only holds for interior solutions. Absent purchase caps, it would be an identity, and the expected markup equals the premium $\pi(y, a)$.
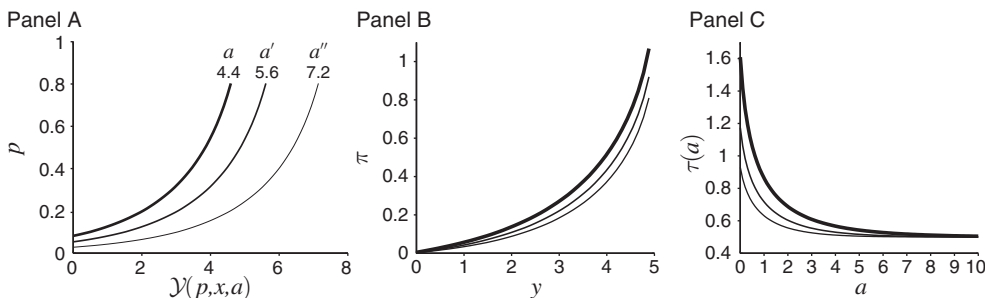
Panel A

Panel B

Panel C

FIGURE 9. EASE OF TRADE

Notes: Assume no dividends ($k = 0$). We plot the supply curve for changing asset positions (panel A), the purchase premium for $a = 5$ (panel B), and the time to trade (panel C), for search frictions $\psi = r/\rho$ ranging in $\{0.005, 0.1, 0.5\}$ from thick to thin lines.

III. Extension to Search and Nash Bargaining

Trade opportunities only arrive periodically, and so should be subject to negotiation. Nash bargaining over prices is commonly assumed in the monetary search literature. We argue below that we can easily modify our model to incorporate the Nash bargaining solution. We show that our results all naturally extend, but that the model is now richer, and affords further results about the negotiated trade price, quantity, and trade value.

For expositional purposes, consider the case of a land owner (the seller) liquidating his production stock in strawberries. Offers arrive randomly, when a buyer stops by driving his vehicle. Buyers vary in their willingness to pay w and in their carrying capacity x, since some drive bicycles, some small cars, some pickup trucks.

We now change the model, and assume bargaining weights $\delta \in [0, 1]$ and $1 - \delta$ on the seller's and buyer's surplus, respectively. The seller's trade surplus (over not trading) is $s(p, y, a) = py + \mathcal{V}(a - y) - \mathcal{V}(a)$, and the buyer's surplus is $(w - p)y$. The terms of trade dictated by the Nash solution entail a *negotiated price* $\mathcal{P}$ and *bargained supply* Υ functions:

$$(12) \qquad \big(\mathcal{P}(w,x,a), \Upsilon(w,x,a)\big) \equiv \underset{\{p,0 \leq y \leq \min\{x,a\}\}}{\arg \max} \; s(p,y,a)^{\delta}((w-p)y)^{1-\delta}.$$

We solve this maximization in stages. The FOC in p suffices by easily checked log-concavity in p. Given the reservation offer (w, x), the negotiated price is the weighted average of the traders' reservation prices:

$$(13) \qquad p = \delta w + (1 - \delta)(\mathcal{V}(a) - \mathcal{V}(a - y))/y.$$

The seller and buyer respectively secure fractions δ and $1 - \delta$ of the *total surplus* $\mathcal{S}(w, x, a)$. The bargained supply must maximize total surplus, namely,

$$\Upsilon(w,x,a) \equiv \underset{y \in [0,\min\{x,a\}]}{\arg \max} \; \mathcal{S}(w,x,a)$$

exactly as in our original model. We offer some insights.

(i) Observe that *the Nash bargaining model is formally equivalent to the original model with a lower arrival rate $\rho\delta$ of offers (w, x) drawn from the density f.* For since the seller is risk neutral, we can imagine that he secures price w with chance δ and otherwise gets his reservation (zero surplus) price. We recover our original model with $\delta = 1$. The case $\delta = 0$ erases all trade surplus, and the seller holds assets for their dividend stream, i.e., $\mathcal{V}(a) = ak/r$. Hence, *greater buyer bargaining power is formally equivalent to higher search frictions.* The seller's value, marginal value, and absolute second derivative are scaled lower with bargaining, since $\mathcal{V}(a|\rho, \delta) \equiv V(a\,|\,\delta\rho)$ and $\mathcal{V}'(a|\rho, \delta) \equiv V'(a|\delta\rho)$, and recalling Theorem 5. Given this logic, we now review how bargaining impacts our results.

(ii) We first observe that *the negotiated price and bargained supply rise in the cap.* The price is $p = \mathcal{P}(w, x, a)$ in (13) when evaluated at $y = \Upsilon(w, x, a)$. Since bargained supply $\Upsilon(w, x, a)$ rises in x, so too does $\mathcal{P}(w, x, a)$, by concavity of the value function.

   For example, assume two buyers with the same reservation price w. If one drives a large truck, and the other rides a bike, then we predict that the truck driver buys more, and yet pays a steeper price: there is no volume discount! The reason owes to the option value of asset. For the seller's marginal benefit is constant at w, while the seller's marginal cost is rising in the quantity sold, by the concavity of the value function.

(iii) We next note that *greater bargaining power for buyers raises the supply and lowers the negotiated price, the choke price, and the sell-all price:* the bargained supply $\Upsilon(w, x, a)$ is given by (10) but with meeting rate $\rho\delta$. By Theorem 8(i), it falls in the seller's bargaining power δ. Since the buyer secures a fraction $1 - \delta$ of total surplus $\mathcal{S}(w, x, a)$:

$$(14) \qquad [w - \mathcal{P}(w, x, a)]\,\Upsilon(w, x, a) = (1 - \delta)\,\mathcal{S}(w, x, a).$$

Notice that $\mathcal{S}(w, x, a) = \max_{y\in[0,\,\min\{x,a\}]}\int_0^y (p - \mathcal{V}'(a - z))\,dz$ falls in δ by Theorem 5. In the corner solution when $\Upsilon(w, x, a) = \min\{x, a\}$, the price $\mathcal{P}(w, x, a)$ rises in δ. We claim that this holds generally when $\Upsilon(w, x, a) < \min\{x, a\}$ and $w \equiv \mathcal{V}'(a - \Upsilon(w, x, a))$. To see this, define the trade surplus $s(y, a) \equiv \mathcal{V}(a - y) + \mathcal{V}'(a - y)y - \mathcal{V}(a)$, and rewrite (14) as $\mathcal{P}(w, x, a) = w - (1 - \delta)\,s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$. The Secant Claim in Appendix Section H verifies that $s(y, a)/y$ rises in y. Thus, $s(\Upsilon(w, x, a), a)/\Upsilon(w, x, a)$ falls in δ, since $\Upsilon(w, x, a)$ falls in δ. Finally, the two threshold prices fall, since the bargained marginal value is lower: $\mathcal{V}'(a) < V'(a)$.

   The logic of this point implies that with greater search frictions, not only does the bargained supply increase (as is true without bargaining), but the negotiated price falls.

(iv) Next, *the bargained supply rises in the position, and the negotiated price falls.* Supply rises just as in (10). Substitute the optimal supply $y = \Upsilon(w, x, a)$

into (12). This reduces to ($\P$): $\Upsilon(w,x,a)(p - c(\Upsilon(w,x,a),a))^{\delta}(w-p)^{1-\delta}$, where the secant slope of $\mathcal{V}$ is

$$(15) \qquad c(y,a) = [\mathcal{V}(a) - \mathcal{V}(a-y)]/y = \int_0^1 \mathcal{V}'(a-(1-z)y)\,dz.$$

First, by Topkis (1978), the price $\mathcal{P}(w,x,a)$ rises in a, because the earlier expression ($\P$) is log-supermodular in $(p,-a)$—since its middle factor $(p - c(\Upsilon(w,x,a),a))^{\delta}$ is log-supermodular in $(p,-a)$. The slope $\Upsilon_a(w,x,a) \leq 1$, as in (10). So substituting $y = \Upsilon(w,x,a)$ in (15), the argument $a - (1-z)y$ rises in a, i.e., $c(\Upsilon(w,x,a),a)$ falls in a, because $\mathcal{V}$ is concave.

(v) *The trade value is increasing and concave in the position a until the cap binds, and then is decreasing and convex.* Without bargaining, the trade value plot mirrors the supply (10), since the price is fixed—it is piecewise linear in the position a, rising with slope w until $V'(a - \min\{x,a\}) = w$, and then is constant. With bargaining, the trade value $\mathcal{P}(w,x,a)\,\Upsilon(w,x,a)$ initially vanishes, then is increasing and *strictly* concave in a until $\mathcal{V}'(a - \min\{x,a\}) = w$, and then decreasing and strictly convex.[25] For supply is fixed at x, but the price is decreasing and strictly convex in the asset position a.

(vi) *Bargaining lowers the trade value, except for low reservation prices and positions.* For reservation values w above the sell-all price $V'(0+) > \mathcal{V}'(0+)$, supply is unchanged and the price is lower, and so the trade value lower. Next, consider lower w. The bargained supply $\Upsilon(w,x,a)$ has unit slope in a, and surplus $\mathcal{S}(w,x,a)$ rises in a. So from (14), the trade value $\mathcal{P}(w,x,a)\,\Upsilon(w,x,a)$ has slope at most w in a. But the slope in a of the trade value $w\mathcal{Y}(w,x,a)$ without bargaining is w, recalling Theorem 4. Since the maximum trade revenue $\mathcal{P}(w,x,a)\,\Upsilon(w,x,a)$ lies below its no-bargaining counterpart wx, and falls after peaking, the two trade values cross (Figure 10).

(vii) *Greater bargaining power for buyers lowers the mean and variance of waiting times.* As in Theorem 7, this follows because the chance of a desirable trade $(1 - F(\mathcal{V}'(a), \infty))$ is now higher—because $\mathcal{V}'$ rises in ρ and thus in δ, by Theorem 5.

(viii) *Greater bargaining power for buyers raises depth and lowers the purchase premium.* For the FOC $w \equiv \mathcal{V}'(a-y)$ implies the *inverse uncapped supply curve* (13):

$$(16) \qquad p(y,a) = \delta\mathcal{V}'(a-y) + (1-\delta)\int_0^y \mathcal{V}'(a-z)\,dz/y.$$

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[25] By (13), the trade value is $\delta w\Upsilon(w,x,a) + (1-\delta)(\mathcal{V}(a) - \mathcal{V}(a-\Upsilon(w,x,a)))$, and $a - \Upsilon(w,x,a)$ is constant in a when the purchase cap does not bind, by (10). Finally, $\Upsilon(w,x,a)$ is piecewise linear, and $\mathcal{V}'(a)$ falls and is strictly convex.
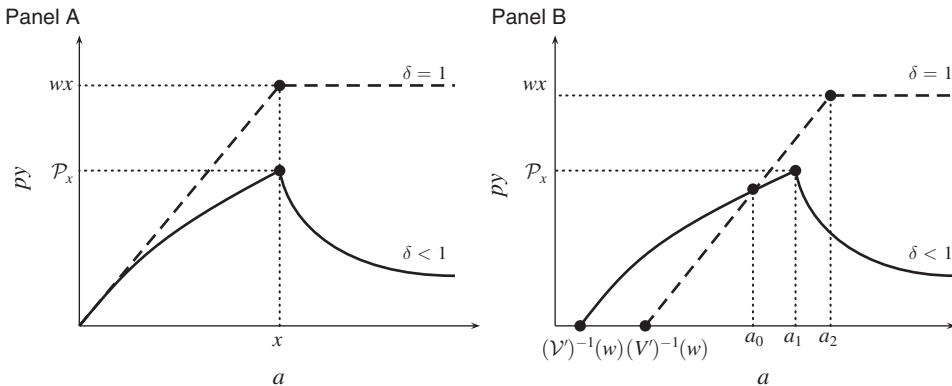
FIGURE 10. BARGAINING AND THE TRADE VALUE

Notes: We plot the trade value with and without bargaining (thick and dashed lines). In panel A, $w\mathcal{Y}(w,x,a) > \mathcal{P}(w,x,a)\,\Upsilon(w,x,a)$ when $w > V'(0+)$. In panel B, when $w \leq V'(0+)$, the trade value rises for low positions $a \leq a_0$. At positions a_1 and a_2, the purchase cap binds with and without bargaining, respectively.

Firstly, depth is the inverse slope $\Lambda(y,a) = (\partial p(y,a)/\partial y)^{-1}$, and the uncapped supply has slope $p_1(y,a) = -\delta\mathcal{V}''(a-y) + (1-\delta)c_1(y,a)$, where $c_1(y,a) = -y^{-2}\int_0^y \int_{a-y}^{a-z}\mathcal{V}''(u)\,du\,dz$, recalling (15). Next, using (16), rewrite the purchase premium $\Pi(y,a) = p(y,a) - \mathcal{V}'(a)$ as

$$\Pi(y,a)\,y = \delta\int_0^y \left(\mathcal{V}'(a-y) - \mathcal{V}'(a-z)\right)dz + \int_0^y \left(\mathcal{V}'(a-z) - \mathcal{V}'(a)\right)dz.$$

By our equivalence result, it suffices that $\mathcal{V}''$ fall in ρ and thus in δ (true by Theorem 5).

(ix) *The qualitative behavior of market depth, the purchase premium, and the sales elasticity claimed in Theorem* 8 *still hold with bargaining*, as verified in Appendix Section H.

IV. Conclusions

The large search literature in economics has assumed that individual optimizations either involve indivisible units, or only a single period, before access to an outside option.

We have first extended dynamic search theory to allow for multiple units. We first closely hewed to the standard wage or price search model, and assumed a seller owned many indivisible units. We first assume that buyers each only desire one unit. Inducting on the number of units, we prove that the reservation price strictly falls with more units, and has diminishing decrements. This is consistent with a Bellman value increasing at a decreasing rate, with a positive third difference. That is, the seller is choosier with a smaller position.

We then assume that arriving buyers may seek more than one unit, but that the seller can partially exercise the requests. Barring a high enough price, the seller only partially exercises the limit order. In this case, the reservation price strictly falls

with more units, but at an increasing or decreasing rate. Our formulation and result resolves a confusion from a 1993 operations research paper, and offers a definitive resolution of their problem.

A major idea that emerged here was how values and reservation prices reflect holding costs—namely, the expected cost of delayed search surplus from one's inframarginal units. As the asset position rises, so do holding costs, and reservation price falls.

We then revisited the notion of multi-unit search, by assuming a perfectly divisible asset, and generally supposed that buyers arrive with random limit order demands. We shift to recursive logic, exploiting contraction properties, and deduce stronger results than in our indivisible unit model. For instance, the value of assets is now concave, but the marginal value is convex whenever the purchase caps have a falling density. En route, we deduce a novel contraction proof of the differentiability of the value function, when existing methods cannot be employed.

Our model is rich and tractable, and allows a range of quick predictions for the change in values and reservation prices as the dividend changes, the seller grows more impatient, buyer arrivals increase, or the price or purchase cap distribution changes. Our model helps extend search insights to trade models with infrequent trading opportunities. Also, search models famously capture settings with temporal market power, where prices are bargained—as might aptly describe many financial settings. Our model predicts the agreed prices and trade sizes.

We hope that our model can be a key ingredient in future equilibrium models in which the buyers' behavior is derived and not exogenously specified. Our paper should allow, e.g., multiple periods of search before markets open in money papers. We are currently extending the analysis to a middleman managing his inventory, who both buys from periodically arriving sellers, and sells to periodically arriving buyers.

In our model, search intensity is not a choice variable. But since our holding costs are increasing, a referee has highlighted how the incentive to search increases in holdings. For instance, a seller holding a large position has an added incentive to advertise. Analyzing this is obviously beyond the scope of this paper, but is a natural open question.

Amongst the Pandora's box of model twists, a referee suggested a fixed flow search cost. This invalidates value concavity, and hence the simple search at the margin insights. In this case, the seller offers a quantity discount, for instance, when his position is small.

APPENDIX: PROOFS FOR DISCRETE TIME SEARCH WITH INDIVISIBLE UNITS

A. *Single Unit Sales: Proof of Proposition 1*

The first reservation price V_1 obeys the Bellman recursion $V_1 = \kappa + \beta E(\max\{P, V_1\})$. Let the seller's search *profit* be $\hat{p} = p - \kappa/(1-\beta)$, and $M(w) \equiv E(\max\{\hat{P}, w\})$ the expected maximum payoff. So $M(0) > 0$ and its right derivative is sandwiched $0 \leq M'(w) \leq 1$.[26] The *option value* $W_1 \equiv V_1 - \kappa/(1-\beta)$,

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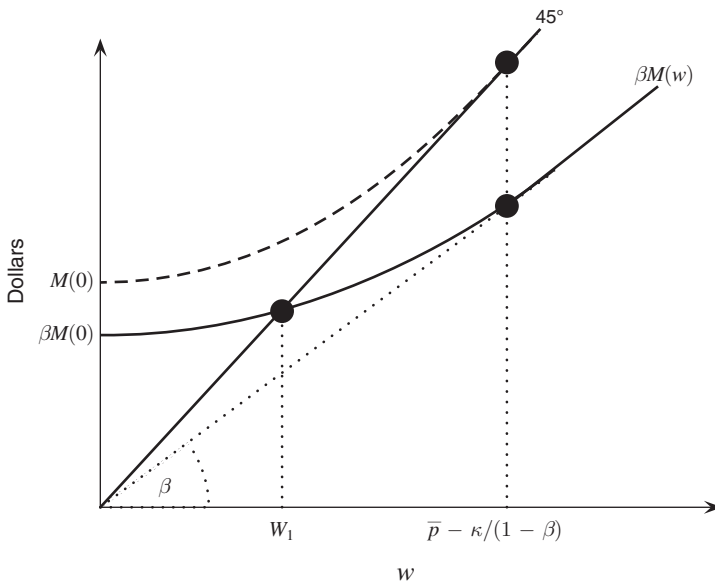[26] As a convex function, M has a right and left derivative everywhere.

FIGURE 11. THE EXPECTED MAXIMUM PAYOFF

Notes: The expected maximum payoff $M(w)$ (dashed curve) grows linearly for $w \geq \overline{p} - \kappa/(1 - \beta)$, as does $\beta M(w)$ (solid curve). The option value $W_1 \equiv \beta M(W_1)$ is sandwiched $0 < W_1 < \overline{p} - \kappa/(1 - \beta)$.

or expected surplus over rental income, uniquely solves $W_1 \equiv \beta M(W_1)$, and is sandwiched $0 < W_1 < \overline{p} - \kappa/(1 - \beta)$, as in Figure 11.

Result 1 (Option Values Rise in Holdings): The Bellman equation yields the recursion

$$(17) \qquad\qquad W_{n+1} = \beta[M(W_{n+1} - W_n) + W_n].$$

We argue inductively that $\Delta W_{n+1} \equiv W_{n+1} - W_n > 0$ for $n \geq 0$. First, $W_1 > 0 = W_0$. Assume $W_n > W_{n-1}$ and $W_n > 0$. Given the derived slope bound $M'(w) \leq 1$, we have $M(w - W_n) + W_n \geq M(w - W_{n-1}) + W_{n-1}$, and so the unique fixed point W_{n+1} of (17) is higher with index $n + 1$ than index n, namely, $W_{n+1} \geq W_n$. Then recursion (17) gives[27]

$$(18) \quad \Delta W_{n+1} = W_{n+1} - W_n = \beta M(W_{n+1} - W_n) - (1 - \beta) W_n < \beta M(\Delta W_{n+1}).$$

Since $W_1 = \beta M(W_1)$, Figure 11 implies $0 \leq \Delta W_{n+1} \leq W_1$. Hence, $\Delta W_{n+1} < \overline{p} - \kappa/(1 - \beta)$. The earlier inequality $M' \leq 1$ is strict inside the convex hull of the price support, as $M(w) < w$. As $M' < 1$ on $[0, \overline{p} - \kappa/(1 - \beta))$, we have $\Delta W_{n+1} > 0$.

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[27] Note that (17) admits a unique solution $W_n > 0$ since the map is a contraction, given $M'(w) \leq 1$ and $\beta < 1$. Also, $W_n > 0$ for all n since $M'(w) \leq 1$ and $W_{n-1} \geq 0$ implies $M(-W_{n-1}) + W_{n-1} \geq M(0) > 0$.

Result 2 (Reservation Prices Fall in Holdings):[28] The increments $\Delta \mathcal{R}_{n+1} = \Delta^2 W_{n+1} = \Delta W_{n+1} - \Delta W_n$ fall if and only if there is *diminishing option value increments*. Now, differencing (18) yields

$$(20) \qquad \Delta^2 W_{n+1} \equiv \beta(M(\Delta W_{n+1}) - M(\Delta W_n)) - (1 - \beta)\Delta W_n.$$

Then $\Delta^2 W_{n+1} < \beta(M(\Delta W_{n+1}) - M(\Delta W_n))$, as $\Delta W_n > 0$. Since $M' < 1$ in our domain, if $\Delta W_{n+1} \geq \Delta W_n$, then $0 \leq M(\Delta W_{n+1}) - M(\Delta W_n) < \Delta^2 W_{n+1}$. But then the last two inequalities for $\Delta^2 W_{n+1}$ contradict. So $\Delta^2 W_{n+1} < 0$ obtains for all $n = 1, 2, \ldots$, and the *reservation prices* $\mathcal{R}_n = V_n - V_{n-1} = W_n - W_{n-1} + \kappa/(1 - \beta)$ *fall in n*. Altogether, option values W_n rise at a falling rate, and thus reservation prices fall in holdings.

Result 3 (Reservation Prices Fall in Holdings at a Decreasing Rate): For the next step, we difference the marginal value expression (20) once more to get

$$\Delta^3 W_n \equiv \Delta^2 W_{n+1} - \Delta^2 W_n = \beta[M(\Delta W_{n+1}) + M(\Delta W_{n-1}) - 2M(\Delta W_n)] - (1 - \beta)\Delta^2 W_n.$$

Now $\Delta^2 W_n < 0$ implies $\Delta^2 W_{n+1} - \Delta^2 W_n > \beta[M(\Delta W_{n+1}) + M(\Delta W_{n-1}) - 2M(\Delta W_n)]$. But then if $\Delta^3 W_{n+1} = \Delta^2 W_{n+1} - \Delta^2 W_n \leq 0$, we get[29]

$$\Delta^3 W_{n+1} > \beta[\Delta W_{n+1} - 2\Delta W_n + \Delta W_{n-1}] = \beta\Delta^3 W_{n+1}$$

a contradiction, since $0 < \beta < 1$. Then $\Delta^3 W_{n+1} > 0$. Hence, *reservation prices $\mathcal{R}_n$ fall with diminishing absolute decrements*. In other words, option values vanish as $n \to \infty$, as values increase at diminishing rate and $V_n - n\kappa/(1 - \beta) < \beta E(P)/(1 - \beta) < \infty$. Eventually reservation prices reflect dividends. ∎

B. *Search with Multiple Unit Sales: Proofs of Results*

Let $\overline{W}_2$ be the value of two units if no one ever just wants to buy just one unit: $\alpha_1 = 0$. Then $\overline{W}_2 = 2W_1$, as the sales policy for one unit is feasible and so optimal. This yields an upper bound $W_2 \leq \overline{W}_2 = 2W_1$ across all α_1. The inequality is strict for $\alpha_1 > 0$: for (3) implies $F_2(W_1, 2W_1 | \alpha^2) = 2W_1 - \alpha_1(1 - \beta)W_1 < 2W_1$ but $F_2(W_1, W_2 | \alpha^2) = W_2$ and so $2W_1 > W_2$—by the logic captured in Figure 11. Then $\Delta W_1 > \Delta W_2$.

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[28] We now offer a more formal intuition in Section IA for why greater holdings increase the marginal gains of reservation price reductions. Fix the continuation value $\overline{V}_{n-1}$. Write the policy equation $\overline{V}_n(\mathcal{R}) = n\kappa + \beta[(1 - F(\mathcal{R}))(E(P|P \geq \mathcal{R}) + \overline{V}_{n-1}) + F(\mathcal{R})\overline{V}_n(\mathcal{R})]$ as

$$(19) \qquad \overline{V}_n(\mathcal{R}) - \frac{n\kappa}{1 - \beta} = \left(\frac{\beta(1 - F(\mathcal{R}))}{1 - \beta F(\mathcal{R})}\right)[E(P|P \geq \mathcal{R}) + \overline{V}_{n-1} - (n\kappa/(1 - \beta))].$$

The $\mathcal{R}$ derivative of the incremental value $\Delta\overline{V}_n(\mathcal{R})$ is negative—as the lead factor falls in $\mathcal{R}$ and the surplus $\overline{W}_n = \overline{V}_{n-1} - (n\kappa/(1 - \beta))$ rises in n, by Step 1. Thus, the optimal reservation price $\mathcal{R}$ falls in n.

[29] If $x > x' > x''$ and $c \in \mathbb{R}$, then $\max\{x'', c\} + \max\{x, c\} - 2\max\{x', c\} \geq \min\{x'' + x - 2x', 0\}$, as can be verified by checking cases. Finally, the inequality holds taking expectations over $c = P - \kappa/(1 - \beta)$.

We next argue inductively. Assume ($\ddagger$): $\Delta W_1 > \cdots > \Delta W_n > 0$ for $n \geq 2$. We want $\Delta W_n > \Delta W_{n+1}$. Consider the *upper envelope* of $\ell + 1$ linear ex post payoff functions:

$$(21) \qquad \mathcal{U}_n(\hat{p}, \ell) \equiv \max_{0 \leq i \leq \ell} (\hat{p}i + W_{n-i} - W_n).$$

We claim that this kinks upward in $\hat{p}$ at ΔW_{n+1-i} for $i = 1, \ldots, \ell$, or in p at ΔV_{n+1-i}, as in Figure 3 (panel B). For selling i units is best if and only if $\hat{p}$ exceeds the average (opportunity) cost of selling $m' \leq i$ fewer units, and is at most the average cost of selling m more units:

$$(22) \qquad (W_{n-i+m'} - W_{n-i})/m' \leq \hat{p} \leq (W_{n-i} - W_{n-i-m})/m$$

for

$$1 \leq m' \leq i, \qquad 1 \leq m \leq \ell - i.$$

By induction assumption ($\ddagger$), the discrete SOC globally obtains for the inframarginal units. So optimality reduces to the discrete FOC with $m = m' = 1$. Selling i units is optimal if and only if $\Delta W_{n-i+1} \leq \hat{p} \leq \Delta W_{n-i}$. As $\Omega_j = E(\max\{\hat{P} - \Delta W_j, 0\})$, the expected upper envelope is $E[\mathcal{U}_n(\hat{P}, \ell)] = \sum_{j=n+1-\ell}^{n} \Omega_j$, as seen earlier in Figure 3.

Using summation by parts (with the formula in Figure 3), rewrite (19) as

$$(23) \quad (1 - \beta)W_n = \beta \sum_{\ell=1}^{n} \alpha_\ell^n E[\mathcal{U}_n(\hat{P}, \ell)] = \beta \sum_{\ell=1}^{n} \alpha_\ell^n \sum_{j=n+1-\ell}^{n} \Omega_j = \beta \sum_{j=1}^{n} (1 - A_{n-j}) \Omega_j.$$

Next, twice differencing (23), using $A_0 = 0$ and $A_n - A_{n+1} = -\alpha_{n+1}$, yields[30]

$$(24) \qquad (1 - \beta)(\Delta W_{n+1} - \Delta W_n) = -\beta \sum_{j=1}^{n-1} \alpha_{n-j}(\Omega_{j+1} - \Omega_j)$$

$$- \beta \alpha_n \Omega_1 + \beta(\Omega_{n+1} - \Omega_n).$$

Since $(1 - \beta)W_1 = \beta E(\max\{\hat{P} - W_1, 0\})$, we have $\Delta W_1 \equiv W_1 < E(\max\{\hat{P}, 0\})$. Given price variance and $\hat{P} > 0$ sometimes, $\hat{P} > E(\max\{\hat{P}, 0\}) \geq \Delta W_1$ with positive chance. Since $\Omega_1 > 0$, and $\Omega_j = E(\max\{\hat{P} - \Delta W_j, 0\})$ by ($\clubsuit$), ($\heartsuit$), and ($\spadesuit$)—still valid in Section IB—and induction assumption ($\ddagger$), we have ($\dagger$): $\Omega_1 < \cdots < \Omega_n$. For a contradiction, assume $\Delta W_{n+1} \geq \Delta W_n$. Together, (24) and ($\dagger$) imply

$$\beta(\Omega_{n+1} - \Omega_n) > (1 - \beta)(\Delta W_{n+1} - \Delta W_n) \geq 0.$$

But $\Delta W_{n+1} \geq \Delta W_n$ implies $\Omega_{n+1} = E(\max\{\hat{P} - \Delta W_{n+1}, 0\}) \leq \Omega_n$. Contradiction.

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[30]For example, differentiating a smooth function $w(x) = \int_1^x [1 - F(x - t)]H(t)\,dt$ analogously yields $w'(x) = H(x) - \int_1^x f(x - t)H(t)\,dt$ and then $w''(x) = -\int_1^x F(x - t)H'(t)\,dt - F(x - 1)H(1) + H'(x)$ if $F(0) = 0$.

For the remaining part of Corollary 1, observe that for a sale of i units, (22) implies that reservation prices optimally depend on final holdings $n - i$ and not the sales quantity or original holdings, and that $\mathcal{R}_{n-i+1} = \Delta V_{n-i+1}$. When $i = 1$, (23) and $\mathcal{R}_n = \Delta W_n + \kappa/(1 - \beta)$ yield $(1 - \beta)\mathcal{R}_n = \kappa + \beta(\Omega_n - \sum_{j=1}^{n-1} \alpha_{n-j}\Omega_j)$. ∎

<div align="center">PROOFS FOR DIVISIBLE UNITS AND CONTINUOUS TIME</div>

<div align="center">C. The Option Value Function</div>

PROOF OF LEMMA 1 (The Surplus Recursion):

We first show that $\mathcal{T}$ maps $\mathcal{C} \to \mathcal{C}$ with bound $\overline{W} = \overline{\beta}E(PX)/(1 - \overline{\beta})$. Relaxing the asset constraint in (5), if we assume $\tilde{W}_t \le \overline{W}$, then $\mathcal{T}$ preserves the upper bound:

$$(\mathcal{T}\tilde{W}_t)(a) \le E_t(e^{-r\tau} \max_{y \in [0, X]} [Py + \overline{W}]) \le \overline{\beta}(E(PX) + \overline{W}) = \overline{W}.$$

Since the maximum of a continuous function $(p - k/r)y + W_{t+\tau}(a - y)$ is continuous in a (theorem of the maximum), $f \in \mathcal{C}$, and Γ_t is continuous in t, we have $\mathcal{T} : \mathcal{C} \to \mathcal{C}$. We check the Blackwell sufficient conditions for a contraction. By inspection, $\mathcal{T}$ is monotone. Likewise, $\mathcal{T}(W_t + b)(a) \le (\mathcal{T}W_t)(a) + \overline{\beta}b$, where $\overline{\beta} < 1$. As $\mathcal{C}$ is a complete metric space with the sup norm, by the contraction mapping theorem, $\mathcal{T}W_t = W_t \in \mathcal{C}$ is unique. ∎

PROOF OF THEOREM 1 (Monotonicity of Recursion Operator):

As $\mathcal{T}$ is a contraction, and monotonicity in a is a closed property in $\mathcal{C}_1$ with the sup norm, it suffices that monotonicity is preserved by $\mathcal{T}$ for all t (Corollary 3.2.1 in Stokey and Lucas 1989). From (5), since the choice set and the objective function increase in a (by assumption), if $a' > a$ then $(\mathcal{T}W_t)(a) \le (\mathcal{T}W_t)(a')$. So the fixed point increases in a, for all t. Next, $W_t(a)$ strictly rises in a, for one sales strategy available at $b > a$ is to act as if one's position is a, and for unexploited offers $\overline{F}(0, a) > 0$, sell at any $p > k/r$. So surplus is at least $E[\max\{P - k/r, 0\} \min\{b - a, X - \min\{X, a\}\}] > 0$. ∎

<div align="center">D. Value Function Differentiability: Proof of Theorem 2</div>

Let the discounted average dividend be $D_t = (k/r)(1 - E_t(B_t))$. The marginal value operator $\mathcal{S}$ is

(25) $(\mathcal{S}V_t')(a)$

$$= D_t + E_t\left(B_t \max\{\tilde{V}_{t+\tau}'(a), \min\{P, \tilde{V}_{t+\tau}'(a - \min\{X, a\})(1 + \chi_{[0, a]}(X))\}\}\right).$$

We show that there exists a unique bounded continuous function $U_t = \mathcal{S}U_t$, recalling (25). We also show that any such fixed point U is the derivative in a of $V_t(a) = W_t(a) + ak/r$, where $W_t = \mathcal{T}W_t$; in other words, $U_t = V_t'$. We attack these tasks in reverse order.

Step 1: We know from Lemma 1 that V_t satisfies the Bellman equation:

$$(26) \qquad V_t(a) = a D_t + E_t \left(B_t \max_{y \in [0, \min\{X, a\}]} [Py + V_{t+\tau}(a - y)] \right).$$

If V_t is continuously differentiable in a on $[0, \infty)$, then Corollary 4 is valid, as its proof only exploits concavity in a of V_t. So Corollary 5 in Milgrom and Segal (2002) applies, and the derivative of the second term, the expectation, in (26) is

$$(27) \qquad E_t(B_t[V'_{t+\tau}(a - \mathcal{Y}_{t+\tau}(P, X, a)) + \lambda_2(\mathcal{Y}_{t+\tau}(P, X, a))]).$$

Since $\lambda_2 = 0$ if $0 \le \mathcal{Y}_{t+\tau}(p, x, a) < \min\{x, a\}$, when $p \le V'_{t+\tau}(a - \min\{x, a\})$, then $V'_{t+\tau}(a - \mathcal{Y}_{t+\tau}(p, x, a)) + \lambda_2(\mathcal{Y}_{t+\tau}(p, x, a)) = \max\{p, V'_{t+\tau}(a)\}$. Otherwise, $\lambda_2 = 0$ if $x \le a$, so $V'_{t+\tau}(a - \mathcal{Y}_{t+\tau}(p, x, a)) + \lambda_2(\mathcal{Y}_{t+\tau}(p, x, a)) = \min\{p, V'_{t+\tau}(a - \min\{x, a\})(1 + \chi_{[0, a]}(x))\}$. Hence, (27) is $E_t(B_t \max\{V'_{t+\tau}(a), \min\{P, V'_{t+\tau}(a - \min\{X, a\})(1 + \chi_{[0, a]}(X))\}\})$. Add D_t to get (25). So, if (26) has a differentiable solution, then its derivative satisfies (25).

Step 2: We show that $\mathcal{S}$ is a contracting operator, and so has a unique bounded and continuous fixed point $U_t = \mathcal{S} U_t$, with bound $\overline{U} = k/r + \overline{\beta} E(P)/(1 - \overline{\beta})$. Assume $U \in \mathcal{C}$ with $0 \le U_t(a) \le \overline{U}$. Since $0 \le (\mathcal{S} U_t)(a) \le \overline{U}$ by (25), the bound $\overline{U}$ is preserved. The continuity in t follows from continuity of Γ_t in t. To see that $\mathcal{S}$ preserves continuity in a, let $a \ge 0$, and consider any sequence $a_n \to a$, for a fixed $t \le \overline{T}$. To understand (25), let

$$\mu_{t, n}(p, x) =: \max\{U_t(a_n), \min\{p, U_t(a_n - \min\{x, a_n\})(1 + \chi_{[0, a_n]}(x))\}\}.$$

Since $0 \le \mu_{t, n}(p, x) \le U_t(a_n) + p \le \overline{U} + p < \infty$, the dominated convergence theorem yields $\lim_{n \to \infty} E(B_t \mu_{t+\tau, n}(P, X)) = E_t(B_t \lim_{n \to \infty} \mu_{t+\tau, n}(P, X))$. So $\lim_{a_n \to \infty} \mathcal{S} U_t(a_n) = \mathcal{S} U_t(a)$, whence $\mathcal{S} : \mathcal{C} \to \mathcal{C}$. Next, $\mathcal{S}$ satisfies Blackwell's two sufficient conditions for a contraction: it is monotone, and obeys $\mathcal{S}(U_t + b)(a) \le (\mathcal{S} U_t)(a) + \overline{\beta} b$, where $\overline{\beta} < 1$. Hence, $\mathcal{S} U_t = U_t \in \mathcal{C}$ is unique. Finally $V'_t(a) - k/r = W'_t(a) > 0$, by Theorem 1. ∎

E. *The Marginal Value Function*

PROOF OF THEOREM 3 (Strict Value Concavity):

The marginal value exists, by Theorem 2. Write the max in (25) as $\max(u_i, \min(p, \gamma_i l_i))$ for $i = 1, 2$. The value is concave, by Theorem 1. So if $a_2 > a_1$, then $u_2 \le u_1 \equiv V'_{t+\tau}(a_1)$ and $\gamma_2 \le \gamma_1 \equiv V'_{t+\tau}(a_1 - \min\{x, a_1\})$, while $l_2 \le l_1$ as $\chi_{[0, a]}(x)$ is monotone in a. Now,

$$\max(u_2, \min(p, \gamma_2 l_2)) - \max(u_1, \min(p, \gamma_1 l_1))$$

$$\le \max(u_1, \min(p, \gamma_1 l_2)) - \max(u_1, \min(p, \gamma_1 l_1)).$$

The right side is nonpositive, and vanishes if the max is the first argument $(p < u_1)$. Otherwise, it is $\min\{p, \gamma_1 l_2\} - \min\{p, \gamma_1 l_1\}$, and so vanishes if $l_1 = l_2$. The only nonzero terms arise when $l_1 = \infty$ and $l_2 = 1$ (i.e., $a_1 < x < a_2$) and $p > V'_{t+\tau}(0+) > \gamma_1$, in which case: $\min\{p, \gamma_1 l_2\} - \min\{p, \gamma_1 l_1\} = \min\{p, \gamma_1\} - p = \gamma_1 - p < 0$.

Taking expectation yields $V'_t(a_2) - V'_t(a_1) \leq E_t\big(B_t \int \int_R [V'_{t+\tau}(0+) - p] F(dp, dx)\big)$ < 0 on the rectangle $R = [V'_{t+\tau}(0+), \infty) \times [a_1, a_2]$, since purchase caps have a convex support. Then $\mathcal{S}$ maps from the set of decreasing functions of a to the set of strictly decreasing functions of a. Hence, the marginal value fixed point V'_t of $\mathcal{S}$ strictly falls in a for all $t \leq \overline{T}$, by Corollary 3.2.1 in Stokey and Lucas (1989). ∎

PROOF OF THEOREM 4(a) (Marginal Value Is Falling and Strictly Convex):

Theorem 3 proves monotonicity. Next, write the continuation marginal value in (25) as

$$(28) \quad E_t(B_t \max\{P, \tilde{V}'_{t+\tau}(a)\}) - E_t\big(B_t \int_0^a \int_0^\infty \max\{p - \tilde{V}'_{t+\tau}(a - x), 0\}\, dF\big).$$

Write (28) with the uncapped supply curve $Y_t(p, a) = a - (V'_t)^{-1}(\min\{p, V'_t(0+)\})$ derived in (10). Since $\int_0^u (u - x)\, dF(x) = \int_0^u F(x)\, dx$ and $\int_b^\infty (x - b)\, d\overline{F}(x)$ $= -\int_b^\infty \overline{F}(x)\, dx$, if we integrate (28) by parts, and exchange integrations, we can rewrite the $\mathcal{S}$ operator in (25):

$$(29) \quad (\mathcal{S}V'_t)(a) = D_t$$
$$+ E_t\Big(B_t\Big(E(P) + \int_0^{\tilde{V}'_{t+\tau}(a)} F(p, \infty)\, dp - \int_0^\infty \int_0^{Y_{t+\tau}(p, a)} \int_p^\infty f(s, x)\, ds\, dx\, dp\Big)\Big).$$

Assume that $V'_t(a)$ is convex in a, for all $t \leq \overline{T}$. Since $\int_0^u F(p, \infty)\, dp$ is increasing and convex in u, by Theorem 5.1 in Rockafellar (1970), $\int_0^{V'_{t+\tau}(a)} F(p, \infty)\, dp$ is convex in a for all $t \leq \overline{T}$. Since $\partial Y_t(p, a)/\partial a \equiv 1$ by Corollary 4, the subtracted second integral in (29) is also convex (in a) since its derivative $E_t\big(B_t \int_0^\infty \int_p^\infty f(s, Y_{t+\tau}(p, a))\, ds\, dp\big)$ is weakly decreasing since $f(s, y)$ is weakly falling in y. In summary, $\mathcal{S}$ preserves convexity in a for all $t \leq \overline{T}$, which is a closed property under the sup norm. So the fixed point $\mathcal{S}V'_t = V'_t$ is convex in a for all $t \leq \overline{T}$. More strongly, by Corollary 3.2.1 in Stokey and Lucas (1989), it is strictly convex since $\mathcal{S}V'_t$ is strictly convex whenever V'_t is convex. This holds as $E_t\big(B_t \int_0^\infty \int_p^\infty f(s, Y_{t+\tau}(p, a))\, ds\, dp\big)$ strictly falls in a, as $f(s, y)$ eventually vanishes as $y \to \infty$—for if not, the expected price would be infinite. ∎

PROOF OF THEOREM 4(b) (Differentiability of the Marginal Value):

If V'_t is differentiable, then $\mathcal{H}V''_t = V''_t$, where $\mathcal{H}$ differentiates $\mathcal{S}$, where

$$(30) \quad (\mathcal{H}U_t)(a)$$
$$= E_t\Big(B_t\Big(\int_{V'_{t+\tau}(0+)}^\infty (V'_{t+\tau}(0+) - p)f(p, a)\, dp + F(V'_{t+\tau}(a), \infty)U_{t+\tau}(a)\Big)\Big)$$
$$+ E_t\Big(B_t \int_0^a \int_{V'_{t+\tau}(a-x)}^\infty U_{t+\tau}(a - x)\, dF\Big).$$

Conversely, if $\mathcal{H}$ is a contraction, its unique fixed point is $U_t = V_t''$, and V_t' is differentiable.

Recall the assumed upper bounds $\bar{h}$ for $h(x)$ and $\bar{p}$ for $E(P|x)$. Take $U_t \in \mathcal{C}$ with $0 \leq U_t(a) \leq -\bar{p}\bar{h}\bar{\beta}/(1 - \bar{\beta}) \equiv \underline{V}''$. As $V_t'(0+) > 0$, the first integral in (30) is negative and exceeds $-\bar{p}\bar{h}$. The last two terms in $(\mathcal{H}U_t)(a)$ are negative with sum at least $\underline{V}''$, by the assumed uniform bound on U_t. Adding terms,

$$0 > (\mathcal{H}U_t)(a) \geq E_t(B_t[-\bar{p}\bar{h} + \underline{V}'']) \geq \bar{\beta}[-\bar{p}\bar{h} + \underline{V}''] = \underline{V}''.$$

As V_t' and Γ_t are both continuous, and $f \in \mathcal{C}$, we have $\mathcal{H} : \mathcal{C} \to \mathcal{C}$.

We now argue that $\mathcal{H}$ is a contraction mapping, with unique fixed point $U_t \in \mathcal{C}$. Next, $\mathcal{H}$ obeys Blackwell's sufficient conditions for a contraction: Easily, $\mathcal{H}$ is monotone. To see that $\mathcal{H}$ obeys discounting, note that $V_{t+\tau}'(a) \leq V_{t+\tau}'(a - x)$ implies $F(V_{t+\tau}'(a), \infty) + \int_0^a \int_{V_{t+\tau}'(a-x)}^{\infty} dF \leq 1$. Consequently, $(\mathcal{H}U_t + b)(a) \leq (\mathcal{H}U_t)(a) + \bar{\beta}b$, where $\bar{\beta} < 1$. So the fixed point $\mathcal{H}U_t = U_t \in \mathcal{C}$ is unique, and V_t' is differentiable, with $V_t'' \geq \underline{V}''$. ∎

PROOF (Value Falls as Time t Advances):

As $\mathcal{S}$ is a contraction on $\mathcal{C}_2$, the bounded and continuous functions of $t \in [0, \infty)$, and t-monotonicity is a closed property in $\mathcal{C}_2$ in the sup norm, it suffices that $\mathcal{S}$ preserves monotonicity in a (Corollary 3.2.1 in Stokey and Lucas 1989). Write (1) as

$$(\mathcal{S}V_t')(a)$$

$$= k/r + E_t\left(B_t[\max\{\tilde{V}_{t+\tau}'(a), \min\{P, \tilde{V}_{t+\tau}'(a - \min\{X, a\})(1 + \chi_{[0, a]}(X))\}\} - k/r]\right).$$

For the recursion, assume that $\tilde{V}_t'$ rises in $\overline{T} - t$ for all a. As B_t rises in $\overline{T} - t$ and Γ_t is invariant, V_t' also rises as time t advances, and so does the fixed point $\mathcal{S}V_t' = V_t'$. ∎

F. *Changing Search Frictions and Offer Distributions*

PROOF OF THEOREM 5 (Increasing Search Frictions):

Poisson arrivals are the arrival distribution $\Gamma_t(\tau) = 1 - e^{-\rho\tau}$, and thus (1) becomes

(31) $(\mathcal{S}V_t')(a)$

$$= \frac{k}{r + \rho} + \frac{\rho}{r + \rho}E\left(\max\{\tilde{V}'(a), \min\{P, \tilde{V}'(a - \min\{X, a\})(1 + \chi_{[0, a]}(X))\}\}\right).$$

As $\mathcal{T}$ and $\mathcal{S}$ are monotone operators, changes that raise the operator increase its fixed point. Since $\mathcal{S}$ falls in r and rises in k, so do the marginal value and value, V' and V. But V' falls in r and rises in k if and only if V is strictly submodular in (a, r) and supermodular in (a, k). As $E_t(B_t) = \rho/(r + \rho)$ in (5), since $\mathcal{T}$ rises in ρ, the option value W and value V rise too. If V' solves $V'(a) = \mathcal{S}(V', \theta)(a)$, recalling (31), we claim that V' increases in $\theta = \rho, k$ and falls in r. For by Albrecht, Holmlund, and Lang (1991), V' is differentiable in parameters θ, and its derivative unique solves

$V'_\theta(a) = \mathcal{S}_\theta(V', \theta)(a) + \mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$, where $\mathcal{S}_{V'} = \partial \mathcal{S}/\partial V'$. So define operators for the partial derivatives in ρ, r, and k of V':

$$(32) \qquad (\mathcal{Q}V'_\rho)(a) = r(V'(a) - k/r)/[\rho(r + \rho)] + \mathcal{S}_{V'}(V'_\rho, V', \rho)(a),$$

$$(33) \qquad (\mathcal{J}V'_r)(a) = -V'(a)/(r + \rho) - \mathcal{S}_{V'}(V'_r, V', r)(a),$$

$$(34) \qquad (\mathcal{K}V'_k)(a) = 1/(r + \rho) + \mathcal{S}_{V'}(V'_k, V', k)(a).$$

Let $\theta = \rho$, and focus on the $\mathcal{Q}$ recursion. If $V'_\rho \geq 0$ then $\mathcal{S}_{V'}(V'_\rho, V', \rho)(a) \geq 0$, and so the second term in (32) is nonnegative. Since $V'(a) > k/r$ by Theorem 2, we have $(\mathcal{Q}V'_\rho)(a) > 0$. By the second conclusion of Corollary 3.2.1 in Stokey and Lucas (1989), the fixed point obeys $\mathcal{Q}V'_\rho = V'_\rho > 0$, i.e., V is strictly supermodular in (a, ρ).

By strict concavity of V, the first term in (32) falls in a, and the first term in (33) rises in a. It follows that $V'_\rho(a)$ and $V'_k(a)$ fall in a, and $V'_r(a)$ rises in r if $\mathcal{S}_{V'}(V'_\theta, V', \theta)(a)$ falls in a. To see this, differentiate the second term of (32) in θ, and change the order of integration using the new variable $z = a - x$:

$$(35) \quad \mathcal{S}_{V'}(\theta, V'; V'_\theta)(a)$$
$$= \beta\left(V'_\theta(a) F(V'(a), \infty) + \int_0^\infty \int_{a - \mathcal{Y}(p, a, a)}^\infty V'_\theta(z) f(p, a - z)\, dz\, dp\right).$$

The first term in (35) falls in a by the concavity of V, and the second since, by Theorem 4, $a - \mathcal{Y}(p, a, a) = \min\{a, (V')^{-1}(\min\{p, V'(0+)\})\}$ rises in a, and $f(p, a - z)$ falls in a. ∎

PROOF OF THEOREM 6 (Changing Offer Distribution):

Rewrite the *trade payoff* in (5) as $\max_y[(p - k/r)y + W(a - y) - \chi_{[0, \min\{x, a\}]}(y)]$. First, consider price changes. The max increases in p. Since it is the conjugate function of $(k/r)y - W(a - y) + \chi_{[0, \min\{x, a\}]}(y)$, it is convex in p, by Theorem 12.2 of Rockafellar (1970). If $\tilde{P}$ dominates P by first order stochastic dominance for all x, or if $\tilde{P}$ is a mean preserving spread of P, then by the first stochastic dominance ranking theorem:

$$E\left(\max_{y \in [0, \min\{X, a\}]} \left(\left(\tilde{P} - \tfrac{k}{r}\right)y + W(a - y)\right)\right)$$
$$\geq E\left(\max_{y \in [0, \min\{X, a\}]} \left(\left(P - \tfrac{k}{r}\right)y + W(a - y)\right)\right).$$

Let the left-hand side define a Bellman operator $\tilde{\mathcal{T}}W$ for $(\tilde{P}, X)$. As the x-marginals coincide, $\tilde{h}(x) = h(x)$, in the X expectation, we have $(\tilde{\mathcal{T}}W)(a) \geq (\mathcal{T}W)(a)$, i.e., the fixed points obey $\tilde{W} \geq W$. We use a similar logic and consider cap changes. Since the constraint set $0 \leq y \leq \min\{x, a\}$ is convex in (y, x), so is the characteristic function $\chi_{[0, \min\{x, a\}]}(y)$. As a result, *the trade payoff is concave in x*, as in the proof of

Theorem 1, by concavity of W. So by ranking theorems, the fixed point W rises in FSD shifts and MPS in X.

Lastly, $\max\{V'(a), \min\{p, V'(a - \min\{x, a\})(1 + \chi_{[0, a]}(x))\}\}$ in (31) rises in p and x. So its expectation V' rises with first order stochastic dominance increases in P, X. ∎

G. *Trading Behavior and the Supply Curve*

PROOF OF THEOREM 7 (Waiting Times):
It suffices that $\Phi(a)$ rises in r, falls in k, and is increasing and log-concave in a. Now, $V'(a)$ falls in a by Theorem 3, and falls in r and increases in k by Theorem 5. So the trade chance $\Phi(a) = 1 - F(V'(a), \infty)$ increases in a and r, but falls in k. The expected trade price $E(P|P \geq V'(a))$ falls in a by Theorem 3, and its variance $\sigma^2(P|P \geq V'(a))$ rises by Heckman and Honoré (1990), when $g(p)$ is log-concave. ∎

PROOF OF THEOREM 8 (Trading Behavior):
Since the seller's objective function $py + V(a - y)$ in (7) is supermodular in (y, θ), for $\theta = r, -\rho, -k$ by Theorem 5, $\mathcal{Y}(p, x, a)$ rises in θ (Theorem 6.1 in Topkis 1978). By the same logic, $py + V(a - y) - \chi_{[0, a](y)}$ is supermodular in (y, a), so $\mathcal{Y}(p, x, a)$ rises in a.

Since $V'(a)$ is decreasing and convex by Theorem 4, $\pi(y, a)$ is increasing in y, and decreasing in a. Likewise, $\pi(y, a)$ falls in $\theta = r, -\rho, -k$, since $V'(a)$ is supermodular in each pair (a, θ), by Theorem 5.

If $p \geq V'(0+)$, then supply is constant in p, and equal to $\min\{x, a\}$, and supply vanishes if $p \leq V'(a)$. For intermediate p, the supply is $Y(p, a) = a - (V')^{-1}(p)$. In this case, the elasticity is $\mathcal{E}_p(p, a) = Y_1(p, a)p/Y(p, a) \propto 1/(a - (V')^{-1}(p))$, namely, decreasing and convex in a, and vanishing as $a \to \infty$.

Finally, $1/\Lambda(y, a) = -1/V''(a - y)$ is decreasing in a and increasing in y by Theorem 4. It rises in θ, since $V''(a)$ rises in θ, by Theorem 5. ∎

H. *Trading Behavior with Nash Bargaining*

Recalling (16), $\mathcal{P}(w, x, a) \equiv p(\Upsilon(w, a), a)$, the bargained elasticity $\mathcal{H}(\mathcal{P}(w, x, a), a)$ solves

$$(36) \quad \frac{1}{\mathcal{H}(\mathcal{P}(w, x, a), a)} = \frac{1}{\mathcal{P}(w, x, a)}\left(\frac{\delta w}{\mathcal{E}_w(\Upsilon(w, a))} + (1 - \delta)(w - c(\Upsilon(w, a), a))\right).$$

We have shown (in Section III, point (iv)) that $c(\Upsilon(w, a), a)$ and $\mathcal{P}(w, x, a)$ fall in a, and Theorem 8 proves that the elasticity $\mathcal{E}_w(\Upsilon(w, x, a))$ falls in a. The right side of (36) rises in a. So the bargained supply elasticity $\mathcal{H}(\mathcal{P}(w, x, a), a)$ falls in the position a.

Next, since the two other transactional liquidity measures—depth and purchase premium—are expressed in terms of the inverse uncapped supply (16), it suffices to understand the marginal value $\mathcal{V}'$ and secant slope $c(y, a)$.

THE SECANT CLAIM: *The secant slope $c(y, a)$ is increasing in y, ρ, k, and δ, falling in a and r, supermodular in $(y, -a)$, $(y, -r)$, (y, ρ), (y, δ), and (y, k), and convex in y.*

PROOF:

By the earlier equivalence result for the bargaining model, all monotonicity claims in Theorems 4 and 5 are inherited by $\mathcal{V}'$, and thus by the integral $\int_0^y \mathcal{V}'(a - z)\,dz/y$. By the same logic, supermodularity claims about $c(y, a)$ follow from monotonicity of $c_1(y, a) = -y^{-2} \int_0^y \int_{a-y}^{a-z} \mathcal{V}''(u)\,du\,dz$. For the convexity in y, change the order of integration, and change variables, to get $c_1(y, a) = -\int_0^1 \mathcal{V}''(a - zy)\,z\,dz$. This rises in y by Theorem 4. ∎

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