An Economic Theory Masterclass

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March 4, 2018

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[†]I am grateful to the initial typesetting by my Wisconsin Econ 713 TA Tono Carrasco, and subsequent improvements by TAs and students, as well as Axel Anderson and Andrea Wilson. The sorting proof for imperfect TU is the brainchild of Hector Chade. I welcome improvements or mistake discovery. The bug finding fee is my praise and will double with each new bug found. Updated: March 4, 2018. Eventually, these notes will be embellished with the best of the exam questions with solutions that I have co-created over the years, with the TAs.

1 Matching Foundations of Markets

Economics explores how societies allocate scarce resources. Most allocation problems are solved by a price system. Sometimes a price faces legal or ethical objections (e.g. organ donors, public-school places to children, etc). Moreover, even if the price system exists the perfect competition assumption is hardly satisfied. How do we solve the allocation problem in this kind of markets? We explore a theory for the optimal allocation of resources when there is no price system.[1] We then see how prices and money change everything. We explore this model, eventually arriving at standard double auctions.

1.1 Pairwise Matching with Nontransferable Utility

Assume individuals are of several observable types. We develop a model of a two-sided market with complete information, or just a two sided market if the information structure is understood, e.g. buyer-seller, worker-firm, worker-task, etc. We will use the label men and women to refer to the general case, but this is just metaphorical. Their different types will determine the payoff each agent in one side of the market receives from being matched with each player of the other side of the market. Ultimately we want to answer the question about the optimal way men and women should match.

A *finite* number of women $x \in \mathbb{X}$ and men $y \in \mathbb{Y}$ can pair off in a "matching market" — inspired by the genetics XX and XY. We will think of $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}_+$, for simplicity. Everyone can match with at most one partner (polygamy is infeasible). Write $m(x, y) = 1$ if woman x is matched to man y, and $m(x, y) = 0$ otherwise. A *feasible matching* obeys symmetry $m(x, y) = m(y, x)$ for all x, y (so if x is matched with y then the opposite holds), and *no overmatching*: for every x, we have $m(x, y) = 1$ for at most one y. So woman x remains single if $m(x, y) = 0$ for all $y \in Y$. We let $\mathcal{M}$ denote the set of all feasible matchings, and if it is unique, $\mu(x)$ the partner of woman x, $\mu(y)$ the partner of man y.

Everyone only has rank order preferences over all matches as well as the single status. While we will represent these by numerical payoffs (in consumer, called utility), they cannot be traded and we thus describe this as *nontransferable utility* (NTU).

Economics is based on the notion of mutual consent, or the "double coincidence of wants".[2] To do so, we instead characterize matching allocations that ensure that no matches are severed. If no one prefers a different pairing (possibly with the empty set), we call such a matching allocation **stable**, and otherwise it is unstable. More formally: An assignment of men and women is **unstable**[3] if there is a **blocking pair**, namely, a woman x and man y not matched, even though x prefers y to his partner, and y prefers x to his

[1] The 2012 Nobel prize was awarded jointly to Alvin E. Roth and Lloyd S. Shapley "*for the theory of stable allocations and the practice of market design*".

[2] The Alfred Hitchcock film "Strangers on a Train" was premised on a murderer finding someone who also wants a murder done, and each doing the murder for the other. William Stanley Jevons (1875), *Money and the Mechanism of Exchange*, Chapter 1.

[3] This stability notion is in Gale and Shapley's (1962) "College Admissions and Stability of Marriage".

partner. Indeed, *double coincidence of wants* is at the heart of economics: x prefers y and y prefers to match with x.

EXAMPLE — RAPPERS & CLASSIC ROCK ARTISTS. Assume a contest in which a rapper that has to match with a classic rock artist to collaborate on a song. We represent preferences by the utilities below, and assume a zero utility for not matching.

	Pink Floyd	Bob Dylan	J.Hendrix
Jay-Z	**6,9**	12,12	18,15
50 C	4,16	**8,18**	12,20
Snoop	2,23	4,24	**6,25**

The bold match is stable if the unmatched payoff is zero, because Snoop and J. Hendrix get their most desirable partner when 50 C and Bob Dylan get their most desirable partner among the rest, and matching with Jay-Z or Pink Floyd beat remaining single.

The following algorithm arrives at this allocation: Rappers first propose, everyone asking their top rocker, J. Hendrix. Jimmy Hendrix accepts his most preferred option, Snoop. Then 50 C and Jay-Z ask their top remaining partner, Dylan, who in turn accepts his most preferred suitor 50 C. Finally, Jay-Z and Pink Floyd match.

This is an example of the *Gale-Shapley deferred acceptance algorithm* (1962).[4] The algorithm involves the following stages:

1. All men start unengaged and women start with no suitors.
2. Each unengaged man *proposes* to his most-preferred woman (if any) among those he has not yet proposed to, if matching with her beats remaining single;
3. Each woman gets *engaged* to the most preferred among all her suitors, including any engagements, if matching with him beats remaining single.
4. Repeat steps 2–3 until no more proposals are possible. Engagements become matches.

Proposition 1 *(a) The algorithm stops in finite time. With an equal number of men and women, if matching beats remaining single, then everybody matches. (b) The algorithm produces stable marriages. With strict preferences, the algorithm yields a unique matching.*

Proof: To see (*a*), notice that at each iteration, one man proposes to some woman to whom he has never proposed before. With n men and n women, this is n^2 possible events.[5] Let Alice and Bob be married, but not to each other. After the algorithm, both cannot prefer each other to their current partners. If Bob prefers Alice to his current partner, then he must have proposed to Alice before his current partner. If Alice accepted, yet is not married to him at the end, then she must have dumped him for someone she prefers;

[4] This is a stylized algorithm, and not intended to be realistic. It ignores waiting or search costs, and assumes obedience to the algorithm. For an attempted search foundation of sorting with NTU, see Smith (2008), The Marriage Model with Search Frictions, *Journal of Political Economy*. For the TU version, see Shimer and Smith (2000), *Assortative Matching and Search*, *Econometrica*.

[5] The maximum number of steps in the DAA is precisely $n^2 - 2n + 2$. See Appendix A.

so she doesn't prefer Bob to her current partner. If Alice rejected his proposal, then she was already engaged to someone she prefers to Bob. □

It is easy to see that if everyone prefers any match to being single, then everyone gets married in the DAA with an equal number of men and women.

If there are several stable matches, then a stable allocation is called *male (female) optimal* if all men (women) agree that they weakly prefer their match under this allocation than under any other stable allocation. Similarly an allocation is male (female) pessimal if they all agree on it being the worst allocation among the stable allocations.

Suppose there is more than one stable matching. In this case, we can compare them in a natural way. We call a stable matching m *male-optimal* if given any other stable matching m', each man y prefers his match with m to his match m'. Similarly, define m as *female-optimal*. Call m *female-pessimal* if given any other stable matching m', each woman x prefers her match with m' to her match m.

Proposition 2 *The DAA chooses a male-optimal and female-pessimal stable matching.*

Proof: If male optimality fails, then some man M is rejected by his most preferred woman in a stable matching μ, say $W = \mu(M)$, in favor of another man M' — for whom $M' \succ_W M$. Assume ($\bigstar$): this is the first period say k where this happens to any man.

Since M' proposed to W, and was not yet rejected by any woman partner by period k in any stable matching (assumption ($\bigstar$)), in particular by $W' \equiv \mu(M')$, he must have preferred $W \succ_{M'} W'$. But then

$$M' \succ_W M \quad \text{and} \quad W \succ_{M'} W'$$

In this case, (M', W) block the stable matching μ that pairs man M and woman W.

To see female-pessimal, suppose woman W strictly prefers the DAA matching μ to a stable matching μ'. Then W must match with some man $M = \mu(W)$. Since μ' is stable, man M prefers his partner $W' = \mu'(M)$ to W, contradicting μ male-optimal. □

Corollary 1 *The DAA produces the same matching, regardless of which side proposes, if and only if there is a unique stable matching.*

Proof: Since the DAA always produces a stable matching, if the stable matching is unique, then the DAA produces the same result regardless of which side proposes. Conversely, if the algorithm produces the same result regardless of which side proposes, it must be the unique outcome because it is both optimal and pessimal for both sides. □

We now give an example illustrating this result. The example in Figure 1 (with a zero outside option payoff) has several stable matchings.

If women propose to men, the DAA finishes in round one. W_1 propose to M_2, W_2 to M_3 and W_3 to M_1. All of them accept since the value of the outside option is 0. If men propose to women, the algorithm also finishes in one: M_1 propose to W_2, M_2 to W_3 and M_3 to W_1. Of course each woman accepts. Hence, the DAA gives two stable pairings:

$$(M_1, W_3), (M_2, W_1), (M_3, W_2) \text{ and } (M_1, W_2), (M_2, W_3), (M_3, W_1)$$

	M_1	M_2	M_3
W_1	5,5	6,2	2,6
W_2	2,6	5,5	6,2
W_3	6,2	2,6	5,5

Figure 1: **Three Stable Matchings, but two Outcomes from the DAA.** In the male-optimal and female pessimal matching, all men earn 6 and all women earn 2. In the female-optimal and male pessimal matching, all men earn 2 and all women earn 6. A third stable matching yields payoffs of 5 for everyone.

as in Proposition 1. If men propose, then each man gets a payoff of 6 and each woman 2. If women propose, then each man gets 2 and each woman 6. So, when men propose, the matches are male optimal, that is each man is better off.

To see that while the DAA always yields a stable outcome, the converse is not true. Here is a third stable matching that gives everyone his/her second choice:

$$(M_1, W_1), (M_2, W_2), (M_3, W_3)$$

For no two people want to divorce and rematch: at least one of them is made worse off.

In order to characterize the stable allocations, let us now index types by a scalar from shortest to tallest, poorest to richest, etc. *Positive assortative matching* (PAM) with respect to the index occurs if the relationship between matched types is monotone increasing — from low types matched with low types to high types with high types. *Negative assortative matching* (NAM) means a monotone decreasing relation — high types with low types and vice-versa. Whether a matching is PAM or NAM (or even assortative at all) crucially depends on how the agents are indexed. For example, a matching that is positively assortative on one attribute might be negatively assortative in another.

Let $f(y|x)$ the utility of man x when matched with woman y and $g(x|y)$ the payoff of woman y when matched with man x. Let us assume that the payoffs of an unmatched man x or woman man y is $f(0|x)$ or $g(0|y)$, writing 0 for matching with no partner. These are *strictly comonotone* if both are increasing, or both decreasing:

$$[f(y_2|x) - f(y_1|x)] \cdot [g(x_2|y) - g(x_1|y)] > \ 0 \ \ \forall x, y \text{ and } \forall y_2 > y_1, \ x_2 > x_1 \tag{1}$$

With differentiable functions, these notions are easily expressed by the partial derivative signs. It is *reverse comonotone* if f is increasing while g is decreasing, or vice versa.

Proposition 3 *PAM is the unique stable matching if f and g are strictly comonotone. The unique stable matching is NAM if f and g are reverse comonotone.*

Proof for PAM: We assume f and g are comonotone, since the reverse comonotone case is similar. First, PAM is stable since if (x', y') and (x, y) are matched, where $x' > x$ and $y' > y$, and if perchance y' prefers x to x', then $g(x|y') > g(x'|y')$; therefore, by comonotonicity, we have $f(y|x) > f(y'|x)$, and so x prefers y to y'.

	Pink Floyd	Bob Dylan	Jimi Hendrix
Jay-Z	**6,9**	12,12	18,15
50 C	4,16	**8,18**	12,20
Snoop	2,23	4,24	**6,25**

$\rightsquigarrow$

	PF	BD	JH
JZ	15	24	**33**
50C	20	**26**	32
S	**25**	30	31

Figure 2: **Rappers and Rockers.** At left are the musicians' contest payoffs, and at right are the total match payoffs. These are respectively relevant to NTU and TU matching.

Next, assume that PAM is not the unique stable matching. Then there exists a stable matching with matched pairs (x, y') and (x', y), where $x' > x$ and $y' > y$. In other words, if $f(y|x) \geq f(y'|x)$ then $g(x'|y) \geq g(x|y)$. In other words, if x prefers y to y' then this preference is not reciprocated. This violates comonotonicity, since it says it requires that f and g both be increasing or both decreasing. □

Example — Rappers & Rockstars Sorting. To generate payoffs here, label rapper Jay-Z as $x = 3$, 50 C as $x = 2$ and Snoop as $x = 1$, and rock stars Pink Floyd as $y = 1$, Bob Dylan as $y = 2$ and Jimi Hendrix as $y = 3$. $f(y|x) = 2xy$ for rappers, and $g(x|y) = x(y - 8) + 30$ for rock stars. Observe that

$$\frac{\partial f(y|x)}{\partial y} = 2x > 0 \quad \text{and} \quad \frac{\partial g(x|y)}{\partial x} = y - 8 < 0$$

Proposition 3 predicts that NAM is the stable outcome.

Note that in every case we have considered so far the number of people on each side have been equal. In general, however, one side may have more (less) people than the other. We will refer to this side as the *long (short)* side of the market. In this case, a stable matching will necessarily have unmatched individuals.

Additionally, we have only thus far considered cases with discrete agents. We could additionally consider a model with a continuum of agents. While the DAA will not finish in finite time, there is an intuitive prediction nonetheless with co-monotone preferences. In this case, instead of a number of agents, we will have a *mass* of agents. In the continuum case, an allocation must have the property that any subset of matched men must have the same mass as the subset of women to whom they are matched.

1.2 Pairwise Matching with Transferable Utility

In nontransferable utility (NTU) world, the match partner determines your payoff, and so the theoretical predictions are the same with ordinal utility. Assume these payoffs are actually money, and thus transferable across agents (which utility normally is not). A richer matching problem then emerges, in which individuals decide on whom to match with and money transfers. In this world of *transferable utility (TU) matching*, agents can be compensated his partner for an inferior match quality, or might pay for a better match.

NTU presumes that match differences cannot be equalized by monetary transfers. The TU model formally introduces side payments which will later be called transfers or wages.

Example — Rappers & Rock stars Revisited: Let us start with the unique stable NTU matching we found, but then allow transfers (it might help to think of these as "bribes").[6] In this case, the matching unravels. For Jay-Z is willing to offer Jimi Hendrix up to $18 - 6 = 12$ to match with him and sever his match with Pink Floyd. This strictly exceeds Jimi's loss of $25 - 15 = 10$ from doing this rematch. Thus, any payment to Jay-Z between 10 and 12 leaves both musicians willing abandon their partners for this new match. The NTU stable matches are bolded at left, and the different matching that will arise with TU is bolded at right.

It is natural to ask then which matchings are immune to side payments, namely, stable even in the presence of transfers. We claim that it is:

(Jay-Z, Jimmy Hendrix); (50 C, Bob Dylan); (Snoop, Pink Floyd)

Intuitive only the total match payoffs matter with side payments. Let $h(x,y) = f(y|x) + g(x|y)$ be the payoff of the match between woman x and man y, as given in the table. In the unisex model, the two sides of the market enter symmetrically, and thus $h(x,y) \equiv h(y,x)$); this best captures a partnership model.

A matching m is **pairwise efficient with TU** if for all matched pairs (x_1, y_1) and (x_2, y_2):

$$h(x_1, y_1) + h(x_2, y_2) - h(x_1, y_2) - h(x_2, y_1) \geq 0 \quad (2)$$

If this condition failed, then a simple rematching to (x_1, y_2) and (x_2, y_1) and suitable side payments would undo the original matching. Otherwise, one can split the gains from rematching among all four agents via side payments and make everyone happy about the divorce and repairing. With NTU, we did not need to worry about the losses of dumped partners, but this condition essentially incorporates their payoffs.

But one might imagine that a richer rematching involving three, four, or more couples might be needed to secure a payoff improvement. One in general could imagine rematching all agents, and securing a higher total payoff across all matches. An **efficient** matching $m \in \mathcal{M}$ maximizes the sum of all match outputs, and thereby precludes any such possible profitable rematching. Naturally, pairwise efficiency is necessary for efficiency.

Lemma 1 *Any efficient matching $m \in \mathcal{M}$ is pairwise efficient.*

The converse of this lemma is false (Figure 3, by Andrea Wilson at Georgetown).

Assume a matching market for men and women. the monetary payoffs are the *wage* $w(y)$ to man y and $v(x)$ to woman x. Imagine that match makers can compete to offer wages $v(x)$ and $w(y)$ to attract men and women, and earn *profits* $h(x,y) - v(x) - w(y)$. With free entry of match makers (or free creation of matches), all such profits are nonpositive for *any* match, or some other match maker would enter, and tease away both agents by offers higher wages, and yet still earn strictly positive profits. On the other hand, there is free exit of match makers, so that if a match does indeed form, then it must be possible

[6] This analysis sheds light on why we sometimes see bribes emerge in matching environments when transfers are disallowed. For instance, the NCAA bans payments to athletes, since it has monopoly power (!). Not surprisingly, every year some teams engage in bribery.

	Y_1	Y_2	Y_3
X_1	**3**	**3**	0
X_2	0	**3**	**3**
X_3	**2**	0	**3**

Figure 3: **Pairwise Efficiency versus Efficiency.** The pairwise efficient green matching has a lower total payoff than the pairwise efficient cyan matching, and is inefficient.

to pay wages out of match output. Equivalently, the match maker earns nonnegative profits. A **competitive equilibrium** is a matching and wage profile (m, w, v) satisfying:

(a) FEASIBILITY: $m(x,y) \geq 0$ and $\sum_x m(x,y) \leq 1\ \forall y \in \mathbb{Y}$, and $\sum_y \hat{m}(x,y) \leq 1\ \forall x \in \mathbb{X}$

(b) FREE ENTRY: $v(x) + w(y) \geq h(x,y)$ for any $(x,y) \in \mathbb{X} \times \mathbb{Y}$

(c) FREE EXIT: $v(x) + w(y) \leq h(x,y)$ for any matched $(x,y) \in \mathbb{X} \times \mathbb{Y}$, i.e. $m(x,y) > 0$

In other words, since there is free entry of agents into matches, it cannot be that some potential match exists that yields positive profits to the agents (or to a matchmaker). Likewise, since there is free exit from matches, it cannot be that some actual match yields negative profits. The word "competitive" reflects the free entry and exit conditions. Just as well, agents *take wages as given* when they match, as they are set by match makers competing with each other. Conditions (b) and (c) imply $v(x) + w(y) = h(x,y)$ for all matched agents x, y. We might more simply think of this as a feasibility constraint.

In other words, wages of matched agents precisely exhaust output and no agents could match and earn positive residual profits. So the matching survives free entry and free exit — specifically, a new match cannot form that earns positive profits (no profitable entry) and an existing match can dissolve and save money.

We now introduce a strong endorsement of the free market.

Proposition 4 (First Welfare Theorem of Matching) *Any competitive equilibrium (m, v, w) yields an efficient matching m.*

Proof: We argue by contradiction. Suppose there exists a competitive equilibrium (m, v, w) that is not efficient. Then, there exists a feasible matching $\hat{m} \in \mathcal{M}$ with a strictly higher payoff across all matches. This is the middle inequality below:

$$\sum_x v(x) + \sum_y w(y) \geq \sum_y \sum_x h(x,y)\hat{m}(x,y) > \sum_y \sum_x h(x,y)m(x,y) = \sum_x v(x) + \sum_y w(y)$$

The first inequality reflects free entry: For $v(x) + w(y) \geq h(x,y)$ for all $(x,y) \in \mathbb{X} \times \mathbb{Y}$, as well as feasibility: $1 \geq \sum_x \hat{m}(x,y)$ for all $y \in \mathbb{Y}$, and $1 \geq \sum_y \hat{m}(x,y)$ for all $x \in \mathbb{X}$. □

COMPUTING WAGES EXAMPLE. Consider a simple example of a divergence between NTU and TU in Figure 4. Then the best stable outcome for the X's matches each X_i with Y_i, but the best stable outcome for the Y's matches (X_1, Y_2) and (X_2, Y_1). Only the

	Y_1	Y_2
X_1	2,0	**0,7**
X_2	**0,7**	2,0

	Y_1	Y_2
X_1	2	**7**
X_2	**7**	2

Figure 4: **NTU versus TU matching, with wages.** Here we see how the inability to make transfers can worsen social welfare.

latter matching is efficient. And now we claim that it alone is a competitive equilibrium. For if outside options are zero, wages obey $v_1, v_2, w_1, w_2 \geq 0$ and:

$$\begin{array}{ll} v_1 + w_1 \geq 2 & v_1 + w_2 = 7 \\ v_2 + w_1 = 7 & v_2 + w_2 \geq 2 \end{array}$$

Many wages solve this system. Given the two degrees of freedom, say v_1, v_2, feasibility gives $w_1 = 7 - v_2$, $w_2 = 7 - v_1$. Then free entry gives $2 \leq v_1 + w_1 = v_1 + 7 - v_2$, and thus $v_2 - v_1 \leq 5$. Likewise, $2 \leq v_2 + w_2 = v_2 + 7 - v_1$ implies $v_1 - v_2 \leq 5$. Hence, $-5 \leq v_1 - v_2 \leq 5$, which is nonempty. One set of wages is $v_1 = 5, v_2 = 0, w_1 = 7, w_2 = 2$.

But it would have been impossible to support the black allocation in Figure 4. For then

$$\begin{array}{ll} v_1 + w_1 = 2 & v_1 + w_2 \geq 7 \\ v_2 + w_1 \geq 7 & v_2 + w_2 = 2 \end{array}$$

But then $2+2 = (v_1+w_1)+(v_2+w_2) = (v_1+w_2)+(v_2+w_1) \geq 7+7$, which is impossible.

Does an efficient outcome exist? And does the converse of Proposition 4 obtain? Namely, is an efficient matching also a competitive equilibrium? To answer this, we turn to another example of transferable utility matching: the **assignment problem**, namely, the efficient allocation of goods to traders that maximizes total value. Koopmans and Beckmann (1957) posed this problem for the optimal assignment of plants to locations.[7] Assume that plants i in location j yields output $h_{ij} \geq 0$. Shapley and Shubik later in 1971 explored the formally identical problem in a trading context that yields the welfare theorem we seek.[8] Assume every trader either supplies or demands exactly one indivisible unit, like a house or a car. The units need not be alike, and different buyers may have different values for the same unit. The model is a special case of our TU matching model, and all proofs extend to any TU matching model.[9]

As in any TU matching model, the double coincidence of wants vanishes, since sellers have no preferences over buyers, but simply wish to sell at the highest positive price. Assume $I \geq 1$ sellers (homeowners) and $J \geq 1$ prospective buyers. The i-th seller values his house at (opportunity cost) $c_i > 0$ dollars, while the j-th buyer values the same house at $\xi_{ij} > 0$ dollars. If $\xi_{ij} > c_j$, and seller i to sell his house to buyer j for some price p_i dollars, then i's payoff is exactly $p_i - c_i$ and j's payoff is exactly $\xi_{ij} - p_i$ (quasilinear

[7]This was the primary paper for Koopmans's 1975 Nobel Prize in Economics.

[8]This was a key cited paper for Shapley's 2012 Nobel Prize in Economics.

[9]Shapley and Shubik (1971) gave no algorithm for reaching stable allocations with transferable utility, but Crawford and Knoer (1981) showed that a generalized Gale-Shapley algorithm does the trick.

House	Seller Costs	Buyer Valuations		
i	c_i	ξ_{i1}	ξ_{i2}	ξ_{i3}
1	18	23	26	20
2	15	22	24	21
3	19	21	22	17

Figure 5: **The Housing Example.** The numbers are in units of \$10,000.

utility). Since seller i need not sell his house to buyer j, their match payoff is

$$h_{ij} = \max\{0, \xi_{ij} - c_i\}$$

This specifies how to divide the gain h_{ij} between players i and j. The choice variable is $m_{ij} = 1$ or 0 depending on whether j buys i's house.

Just as in Kantorovich's solution of the transportation problem (see Appendix B), we no longer build our theory on matchings $m \in \mathcal{M}$ but instead convexify the choice set by allowing fractional purchases of houses. Let seller i sell fraction $m_{ij} \geq 0$ of house i to buyer j. Think of this as buying and selling "time shares" on condominiums. We end up with six constraints on the sums of $m_{ij} \geq 0$, three stating that no house can be oversold, and three stating that no buyer can buy more than one house. These constraints incorporate the possibility that some seller i might not sell a house (if $m_{ij} = 0$ for all j) or some buyer j might not purchase a house (if $m_{ij} = 0$ for all i):

$$\max_{(m_{ij})} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij} m_{ij} \text{ s.t. } \sum_{j=1}^{J} m_{ij} \leq 1 \ \forall i \in \{1, \ldots, I\} \text{ and } \sum_{i=1}^{I} m_{ij} \leq 1 \ \forall j \in \{1, \ldots, J\} \quad (3)$$

Proposition 5 *An efficient matching $m \in \mathcal{M}$ exists.*

As (3) is a maximization of a continuous function on a compact set, a solution exists.

Figure 5 gives an example with $J = 3$ buyers (also called $1, 2, 3$) and $I = 3$ sellers (called $1, 2, 3$). Assume the following costs and valuations for houses (in \$100K units): As the objective function is continuous in $m \in \mathbb{R}^9$, on a compact domain, the problem has a solution. More subtly, *the maximum includes a "vertex" of this six-dimensional polytope, with $m_{ij} = 0$ or 1*, so that time sharing is never required. This is geometrically intuitive.[10]

We now apply the LP duality Lemma 5 in §C to our optimization (3). Here, the fraction of the good sold is the primal variable $z = m$, and the wages $u = (v, w)$ are the dual variable. In other words, v_i is seller i's shadow value, or his equilibrium producer surplus. Meanwhile, w_j is buyer j's shadow value, or his equilibrium consumer surplus.

Lemma 2 *The dual problem to the output maximization* (3) *is the cost minimization:*

$$\min_{v_i, w_j} \sum_{i=1}^{I} v_i + \sum_{j=1}^{J} w_j \quad s.t. \quad v_i + w_j \geq h_{ij} \ \forall i, j \quad and \quad v_i, w_j \geq 0 \ \forall i, j \quad (4)$$

[10] For a convincing intuition, imagine dropping a polyhedron on the floor, and think of "down" as the direction of maximization. While it may by chance land on an edge or even perchance a face of the polyhedron, it is always the case that a vertex lands first, possibly in a tie. See Appendix C.

Proof: We prove this by applying Lemma 5, for suitable A, q, p, z, and y. For simplicity, we assume two buyers and two sellers. Let $m' = (m_{11}, m_{12}, m_{21}, m_{22})$, where m_{ij} is the share bought from seller i by buyer j. Let $h' = (h_{11}, h_{12}, h_{21}, h_{22})$, where h_{ij} is the (i, j) match trade surplus. Let $q' = (1, 1, 1, 1)$. Finally, define

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Write the primal problem (3) as

$$\sum_i \sum_j h_{ij} m_{ij} = \max_{m \geq 0} \; h'm \quad \text{s.t.} \quad Am \leq q$$

The first two constraint rows correspond to sellers' no-overselling constraints, and the second two rows to the buyers' no-overbuying constraints.

Next, define the shadow values for sellers' and buyers' no-overselling and no-overbuying constraints $(v, w)' = (v_1, v_2, w_1, w_2)$. These are the respective payoffs to the buyers and sellers. Then the dual problem in Lemma 5 can be written as (4) since

$$\min_{w,v \geq 0} \{v_1 + v_2 + w_1 + w_2\} = \min_{v,w \geq 0} \; (v, w) \cdot q \quad \text{s.t.} \; (v, w) \cdot A \geq h \qquad \square$$

Accounting for the complementary slackness conditions, the dual trading problem (4) is a cost minimization over shadow values $v, w \in \mathbb{R}^3_+$ such that

$$v_i + w_j \begin{cases} \geq h_{ij} & \text{for all } i, j \\ = h_{ij} & \text{if buyer } x_i \text{ and seller } y_j \text{ trade } (m_{ij} > 0) \end{cases} \tag{5}$$

In words, the sum of the shadow value of any buyer and seller is at least the trade surplus, and equal if they trade. These inequalities ensure that a buyer trades with a seller only when the seller and buyer cannot afford to pay their surpluse shadow values from the house sale. So the cheapest way to afford all match output subject to entry and free exit constraints (5) of a competitive equilibrium occurs at the efficient matching. Further, these two ways of measuring output — corresponding to gross national product and gross national income — coincide at the optimum.

Proposition 6 (Second Welfare Theorem of Matching) *An efficient matching m is a competitive equilibrium (m, v, w) for some prices (v, w).*

The welfare theorems still hold for a continuum type model that we assume in §1.3.[11,12]

[11] They are formally established in "The nonatomic assignment model" by Gretsky, Ostroy, and Zame (1992). But the math is naturally far more involved than simple linear programming.

[12] A major part of Becker's paper was linking the efficient outcomes and competitive equilibrium, but he was apparently unaware of the prior work by Shapley and Shubik (1971).

	y_1	y_2	y_3	Seller payoffs v_i
House 1	5	**8**	2	$v_1 = 4$
House 2	7	9	**6**	$v_2 = 5.5$
House 3	**2**	3	0	$v_3 = 0$
Buyer payoffs w_j	$w_1 = 2$	$w_2 = 4$	$w_3 = 0.5$	

Figure 6: **Example of Wages Solving the Housing Example.**

The welfare theorem is natural in light of the dual formulations. For the primal maximum equals the dual minimum at the optimum. The shadow value interpretation of the multipliers (v, w) is consistent with the welfare theorems — for the wages of individuals coincide with the marginal values of the planner. In other words, individual decisions do not diverge from the socially optimal decisions.

The proof argued that in any supposed efficient matching with a higher total output than an existing competitive equilibrium, total wages exceed the value of match output.

Lemma 1 and Propositions 4 and 6 can be summarized as:

$$\text{competitive equilibrium} \iff \text{Efficient outcome} \implies \text{Pairwise efficient} \tag{6}$$

The welfare theorems arise more readily from the equivalence of the dual and primal problems.[13] The dual problem is the decentralized outcome to the primal social planner's problem, since every trader best responds to prices, independent of other traders.

The last implication in (6) asserts that the competitive equilibrium (m, w, v) is graft-free: matches can be undone by side payments: no buyer and seller that are not trading with each other, that would like to do so. This is a nice stability property of free market outcomes, and is an important reason why economists prefer the free market.

The unique optimal assignment in the housing example is shown in Figure 6. While the allocation is generally unique (generically, more precisely), *the shadow values are far from unique!* For the incentive constraints define a convex set six-dimensional subset of $\mathbb{R}^6$ — i.e., not only are there infinitely many equilibrium prices, but the set is of full measure. To wit, price multiplicity is not a "zero chance" event. It must be treated seriously.

$$\begin{array}{lll} v_1 + w_1 \geq 5 & v_1 + w_2 = 8 & v_1 + w_3 \geq 2 \\ v_2 + w_1 \geq 7 & v_2 + w_2 \geq 9 & v_2 + w_3 = 6 \\ v_3 + w_1 = 2 & v_3 + w_2 \geq 3 & v_3 + w_3 \geq 0 \end{array}$$

The matrix indicates one solution. The total shadow value measure of gains is $(2 + 4 + 0.5) + (4 + 5.5 + 0) = 16$ thousand dollars. Using these surpluses, and the initial matrix of costs c_i and valuations v_j, we find that the prices supporting this equilibrium are $p_1 = c_1 + v_1 = 18 + 4 = 22$, $p_2 = c_2 + v_2 = 15 + 5.5 = 20.5$ and $p_3 = c_3 + v_3 = 19$. Equally well, we could have computed prices $p_1 = h_{12} - w_2 = 26 - 4 = 22$, $p_2 = h_{23} - w_3 = 21 - 0.5 = 20.5$ and $p_3 = h_{31} - w_1 = 21 - 2 = 19$. Given these prices, the first buyer clearly prefers the

[13] Since efficiency is stronger than Pareto efficiency later in the course, the first equivalence is a stronger version of the *First Welfare Theorem* and *Second Welfare Theorem* later on.

third house, as his gain is maximum and equal to \$2,000. Similarly, the second buyer prefers the \$4,000 gain at house 1 and the third buyer only considers the second house.

From a computational perspective, in this example the dual problem is harder to solve than the primal. One need only check the outputs of the $3! = 6$ matchings in the primal problem. But assume as the number $I = J = n$ of matched agent types increases, the number of possible matchings and the number of primal variables m_{ij} rises as $n!$, and grows exponentially fast.[14] But in the dual problem, the number $2n$ of prices rises linearly. So the decentralized matching market tackles a computationally much easier problem.

1.3 Assortative Matching in the Marriage Model

We have so far described equivalent centralized and decentralized approaches to finding the best allocation. To say something about who matches with whom, we assume a continuum of female and male types $x, y \in [0, 1]$. Specifically, let the mass cdf of women be Φ and the mass cdf of men be Γ. So a measure $\Phi(x)$ of women lies below type x and a measure $\Gamma(y)$ of men lies below type y. If $\Phi(1) > \Gamma(1)$ then there are more women than men, and so women are on the long side of the market; oppositely, when $\Phi(1) < \Gamma(1)$, men are on the short side of the market. But not everyone on the short side of the market need will choose to match, if the option of remaining unmatched is sufficiently enticing.

We explore the "Marriage Model" by Gary Becker (1973). Assume supermodular match output $h(x, y)$, and let us usually assume a zero unmatched payoff. Our benchmark matching is *positive assortative matching* (PAM), i.e. the partner of woman x is man $y(x)$, where $\Gamma(1) - \Gamma(y(x)) = \Phi(1) - \Phi(x)$, so that higher type women are paired to higher type men. The function $h(x, y)$ is *supermodular* (*strictly supermodular*) if

$$h(x', y') + h(x, y) \geq (>)\ h(x', y) + h(x, y') \tag{7}$$

for any pair of women $x' \geq x$ and men $y' \geq y$. If match output is supermodular, then PAM is pairwise efficient, by (2). If inequality (7) is strict whenever $x' > x$ and $y' > y$, then h is *strictly supermodular*, and PAM is uniquely pairwise efficient. *Submodularity* corresponds to the reverse inequality $h(x', y') + h(x, y) \leq h(x', y) + h(x, y')$ always, and *modularity* is $h(x', y') + h(x, y) = h(x', y) + h(x, y')$ always. Consequently,

Proposition 7 *If production is supermodular, then PAM is an efficient matching. If it is strictly supermodular, then PAM is the unique efficient matching. If production is modular for a set of agents that match, then any re-matching among them is also efficient.*

If production is strictly supermodular, then PAM is pairwise efficient. For if there is at least one ordinal mismatch, with marriages (x, y') and (x', y) for women $x' > x$ and men $y' > y$, then (7) fails. More strongly, PAM is uniquely pairwise efficient. Since an efficient matching exists, PAM is also uniquely efficient, by Lemma 1. Next, add εxy to any supermodular production, where $\varepsilon > 0$, and get a strictly supermodular production; thus PAM is the limit of efficient matchings (as $\varepsilon \downarrow 0$), and therefore is efficient.

[14] By Stirling's factorial approximation, this grows exponentially as $n! \sim \sqrt{2n\pi}(n/e)^n$.

On the other hand, purely additive (modular) payoffs $h(x,y) = a(x) + b(y)$ are also supermodular, as (7) holds with equality, and in this case, any matching is efficient. Less extreme, in any intermediate case where (7) is sometimes strict and sometimes weak, PAM is a possible efficient allocation, but there may be others. For any other allocation, total output weakly rises with each pairwise swap that undoes sorting mismatch.[15] In other words, the very definition of supermodularity coincides with the claim that pairwise devisions from PAM are unprofitable.[16]

Corollary 2 *If production is submodular, then* negative assortative matching *(NAM) is efficient. If production is strictly submodular, then the unique efficient matching is NAM.*

Prices with a finite number of agents are not unique, because the competitive forces on sellers (respectively buyers) only come from a finite number of other sellers (respectively sellers). But with a continuum of types there can be arbitrarily nearby types that offer competitive forces. That holds here, and we shall find that a unique competitive wage emerges. Let $v(x)$ be the wage of woman x and $w(y)$ the wage of man y. In a competitive equilibrium, the matchmaker *profits* namely, output minus wages paid:

$$\pi(x,y) = h(x,y) - v(x) - w(y) \tag{8}$$

of a potential match of x and y are nonpositive, and vanish for all matches that form.[17]

Since match profits (8) attain their maximum of zero at the optimal partner for an agent, by free entry and exit, namely at $y(x)$ for every x, and this is an interior optimum of a smooth function, the FOC holds:

$$\frac{\partial \pi(x,y)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \pi(x,y)}{\partial y} = 0 \tag{9}$$

The partner y that maximizes profits $\pi(x,y)$ increases in the type x when $\pi(x,y)$ is supermodular, and thus when $h(x,y)$ is supermodular (by Topkis' Theorem). By considering this maximization, we can deduce what the wage schedule is. We see that the FOC pins down the slope of the wage profile, and thus in any competitive equilibrium, the wage is uniquely pinned down if the production function is smooth.

Consider the unisex model where $h(x,y) \equiv h(y,x)$. Suppose $h_{12} > 0$ so that PAM is uniquely efficient, and is a competitive equilibrium, and wages should be $v(x) = h(x,x)/2$, namely equal sharing of the output. Also, if there is an outside option payoff of the single status that pays v^0, then every type x with $h(x,x) > 2v^0$ matches according to PAM.

Next, relax the unisex assumption. Consider the payoff function $h(x,y) = x^2 y$ of woman x with man y, both uniformly distributed. Assume single option payoffs v_O and w_O. Since $h_{xy} = 2x > 0$, PAM is the unique efficient outcome. Profits $\pi(x,y) = x^2 y - v(x) - w(y)$ achieve a maximum of 0 at $y = x$ under PAM. The FOC are:

$$2xy - v'(x) = 0|_{y=x} \Rightarrow v'(x) = 2x^2 \qquad \text{and} \qquad x^2 - w'(y) = 0|_{x=y} \Rightarrow w'(y) = y^2$$

[15] Think of the bubble sort algorithm in computer science for putting words in alphabetical order.

[16] Becker kindly attributes this proof or observation to William Brock.

[17] We can deduce that the competitive equilibrium matching exhibits PAM. For by basic monotone comparative statics, the profit-maximizing partner $y(x)$ maximizing $\pi(x,y)$ increases in x if $\pi_{xy} = h_{xy} > 0$.

Integrating yields:

$$v(x) = \tfrac{2}{3}x^3 + b \qquad \text{and} \qquad w(y) = \tfrac{1}{3}y^3 + d$$

Given PAM and $\Phi(x) \equiv x$ and $\Gamma(y) \equiv y$, the least man and woman who match have a shared type x_0. Also, by arbitrage, each earns the same matched and unmatched. Thus, $\frac{2}{3}x_0^3 + b = v_O$, and $\frac{1}{3}x_0^3 + d = w_O$. Since match output of (x, x) is $x^3 = \frac{1}{3}x^3 + \frac{2}{3}x^3$, the transfers b and d across the two sides of the market balance: $b = -d$. Altogether,

$$b = v_O - \tfrac{2}{3}x_0^3 = -[w_O - \tfrac{1}{3}x_0^3] \quad \Rightarrow \quad x_O = (v_0 + w_O)^{1/3} \quad \text{and} \quad b = v_O - \tfrac{2}{3}(v_0 + w_O)^{1/3}$$

In the matching model, women compete against other women, and men compete against other men. But there is a degree of freedom in pinning down how women vs. how men do, collectively. In general, one side of the market can make transfers to the other side. A *dowry* $d > 0 > b$ (transfer paid from women to men) arises if the outside option of women is zero, namely $v^O = 0$. In another special case, if the outside option of men is zero, namely $w^O = 0$, then there is a *bride price* $b > 0 > d$.

1.4 Double Auctions

Let us now relax the double coincidence of wants in a pairwise matching model. We simplify the housing assignment model, and explore the double auction with homogeneous goods. We let ξ_j denote buyer j's value of *any* seller i's good. This is a special case of the assignment model, now with $\xi_{ij} = \xi_j$ for all j.

Let $h(\xi, c) \equiv \max\{0, \xi - c\}$ be the gains from trade for a buyer with value ξ and a seller with cost c. The efficient allocation maximizes the sum of the realized match surpluses $\sum_i \sum_j x_{ij} h(\xi_j, c_i)$, where $x_{ij} = 1$ if seller i trades with buyer j, and $x_{ij} = 0$ otherwise.

Lemma 3 (Trade Surplus Function) *The trade surplus function h is submodular: If $\xi_1 \le \xi_2$ and $c_1 \le c_2$, then $h(\xi_2, c_2) + h(\xi_1, c_1) \le h(\xi_2, c_1) + h(\xi_1, c_2)$.*

Proof: If $\xi_1 \ge c_2$, then two trades should occur, and so the total trade surplus is the same regardless of who trades: $h(\xi_2, c_2) + h(\xi_1, c_1) = h(\xi_1, c_2) + h(\xi_2, c_1) = \xi_2 + \xi_1 - c_1 - c_2$. If $\xi_2 \le c_1$, then no trades should occur, and so $h(\xi_2, c_2) + h(\xi_1, c_1) = h(\xi_1, c_2) + h(\xi_2, c_1) = 0$. Finally, consider cases where exactly one trade should happen. If $\xi_2 \ge c_2 \ge c_1 \ge \xi_1$, then $h(\xi_2, c_2) + h(\xi_1, c_1) = \xi_2 - c_2 \le \xi_2 - c_1 = h(\xi_1, c_2) + h(\xi_2, c_1)$. And if $c_2 \ge \xi_2 \ge \xi_1 \ge c_1$, then $h(\xi_2, c_2) + h(\xi_1, c_1) = \xi_1 - c_1 \le \xi_2 - c_1 = h(\xi_1, c_2) + h(\xi_2, c_1)$. In each case, inequality is strict if $c_1 < c_2$ and $\xi_1 < \xi_2$, since trade surplus falls when the wrong good is traded. □

Given Lemma 3, Corollary 2 implies NAM, i.e., the highest value buyers matched with the lowest cost sellers. Corollary 2 also asserts that NAM is uniquely optimal when the submodular inequality is globally strict. But equality holds in cases where both trades should occur, or neither. In other words, swapping among them entails no loss. The inequality is only strict when exactly one of the agents should trade. So *there is NAM — highest value buyers trade with lowest cost sellers, but who trades with whom is irrelevant.*

Since all permutations of goods in the final assignment are equivalent, the **Law of One Price** holds: $p_i = p$ for all i. To see this, observe that the Shapley-Shubik dual solution now requires a shadow value $v_j = \xi_j - p_i$, and thus a constant price $p_j = p$.

Let's interpret the cost c_i for seller i as the cost of producing the good. The dual conditions mean that in the competitive equilibrium, all traders maximize, taking prices as given — namely, every buyer j with a **seller surplus** $\xi_j - p > 0$ buys, every seller i with a **buyer surplus** $p - c_i > 0$ sells, and it is optional whether a buyer j with a value $\xi_j = p$ buys, or a seller i with a cost $c_i = p$ sells. So NAM means that the highest value buyers buy and the lowest cost sellers sell.

Re-order buyers from high to low valuations, and the sellers from low to high costs:

$$\xi_1 > \cdots > \xi_k > \xi_{k+1} > \cdots > \xi_N \quad \text{and} \quad c_M > \cdots > c_{k+1} > c_k > \cdots > c_1$$

For this homogenous good world, assume that a competitive equilibrium is achieved via a fictitious *Walrasian auctioneer*. He must choose a price p to **clear the market**, namely, so that supply and demand balance because $c_k \leq p \leq \xi_k$ and $\xi_{k+1} \leq p \leq c_{k+1}$. In other words, k units are traded, and the next unit sold by a seller or bought by a buyer would yield nonpositive surplus. Unlike a normal auctioneer who merely raises the price when there is excess demand, this auctioneer also lowers the price when there is excess supply.

Proposition 8 *An efficient allocation exists, with trades by the highest k value buyers and lowest k cost sellers, with positive trade if $v_1 \geq c_1$. This allocation is a competitive equilibrium for any price $p \in [\max(c_k, \xi_{k+1}), \min(c_{k+1}, \xi_k)]$. Any competitive equilibrium is efficient, and thus maximizes the sum of gains from trade. It is immune to side bribes.*

Non-economists are suspicious of markets. The very word scalping, e.g., is suggestive of a nefarious activity. Likewise, "price gouging" is often illegal. But by Proposition 8, no such retrade is profitable unless we did not start with a competitive allocation, and any attempts to stifle retrade harms social welfare.[18] Attempts years ago by Jay Leno to stamp out retrade of his Tonight Show tickets could only serve to hurt the ticket owners that he sought to help, had they succeeded; fortunately, they did not. Stubhub improves our welfare, facilitating reallocation of tickets to those who prize them more.

For some historical background, consider the military draft, that used a lottery. A draftee might receive a salary of \$15,000. The equilibrium salary needed to attract workers to an all-volunteer military might be \$25,000. But some soldiers who were drafted could have received a salary of \$35,000 a year. This is the opportunity cost of drafting such individuals. But under an all-volunteer military, individuals who volunteer for the military do so if the value of their time in the military is higher than in alternative employments.[19]

In the spirit of this anti-market solution is an ingenious efficiency attack on gift-giving. In his provocatively-labelled "The Deadweight Loss of Christmas", Waldfogel (1993) highlighted how giving gifts rather than cash invariably results in costs that swamp benefits that in totality can amount to around ten billion dollars per holiday season.

[18] Adam Smith's (1776) *Wealth of Nation'* describes how free markets acts *as if* guided by an "invisible hand" toward the best social outcome.

[19] Milton Friedman was instrumental in helping make the economic case to end the draft. Opposed to an all-volunteer military, Gen. Westmoreland asserted that he did not want to command "an army of mercenaries." Milton Friedman, on President's Commission on an All-Volunteer Force, immediately shot back: "General, would you rather command an army of slaves?"

Example. Consider 20 traders indexed 1 to 20. Assume traders $j = 2, 4, 6, \ldots, 20$ are buyers and traders $i = 1, 3, 5, \ldots, 19$ are sellers. Buyer valuations are $\xi_i = 2i$ and sellers costs are $c_j = 3j$. After re-ordering, values and costs are:

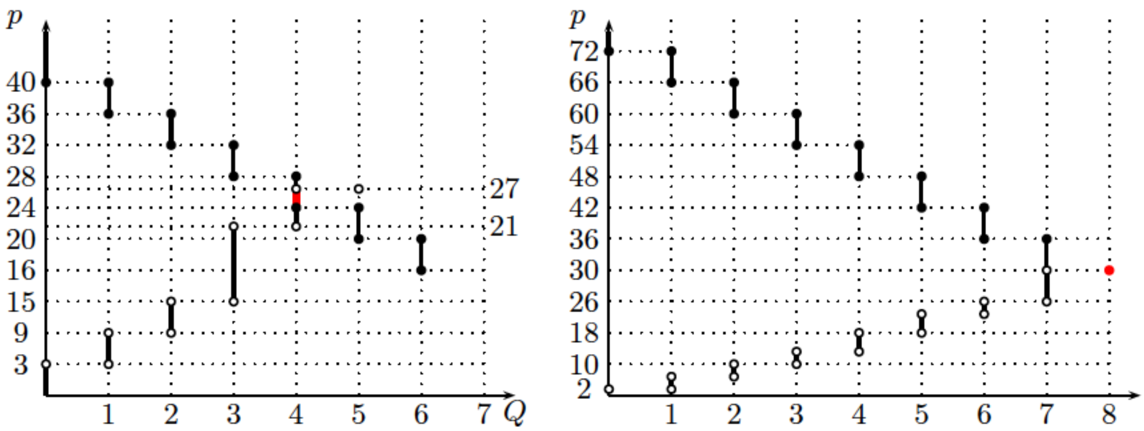
$$\xi \in \{40, 36, 32, 28, 24, 20, 16, 12, 8, 4\} \quad \text{and} \quad c \in \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57\}$$

It is efficient for sellers $j = 1, 3, 5, 7$ to sell their good to buyers $i = 20, 18, 16, 14$. Also, any price in the interval $[24, 27]$ clears the market, since $24 \leq p \leq 28$ and $21 \leq p \leq 27$.

Suppose now agents indexed 1-25, with even traders buyers and odd traders sellers. Assume instead that $\xi_i = 3i$ and cost is $c_j = 2j$. After re-ordering, values and costs are:

$$\xi \in \{72, 66, \ldots, 12, 6\} \ \& \ c \in \{2, 6, \ldots, 46, 50\}$$

In this case the unique equilibrium price is $p = 30$ and the equilibrium quantity is eight. When the price is not uniquely pin down, the quantity is uniquely pinned down. When the quantity is not uniquely pin down, the price is uniquely pinned down.



Notably, heterogeneity is the source of all gains from trade. If everyone had identical valuations, then no consumer secures consumer surplus at the market clearing price; the more heterogeneous are consumers or producers, the larger the total gains from trade. Heterogeneity is good.

A Maximum Number of Stages in the DA Algorithm

We draw on a claim by Gale and Shapley, later proved in Itoga (1978).[20]

Lemma 4 *The maximum number of steps in the DAA is $n^2 - 2n + 2$.*

[20] Stephen Y. Itoga (1978), "The Upper Bound for the Stable Marriage Problem", *The Journal of the Operational Research Society.* I am grateful to Han Wang for finding this article, and some insights above.

Itoga's Proof: First, at most one man M ends up with his worst possible partner W. For M must have proposed to all $n-1$ of preferred women. Since he was eventually rejected, each woman preferred someone else. So when M proposes to W this must end the DAA, and he must be the only proposer at this stage.

Next, there are only n^2 possible proposals initially, in any event — namely, every man proposing to every woman. In fact, the bound is less. For suppose n initial proposals by the n men; this will occupy one step. At this point, each man has at most $n-1$ proposals left. But we noted that at most one man can actually make n total proposals, and the others must be limited to $n-1$. Hence the number of proposals after the first stage is at most $(n-1)+(n-1)(n-2)$. Since each subsequent stage requires at least one proposal, the total number of stages cannot exceed $1+(n-1)+(n-1)(n-2) = n^2-2n+2$.

Next, the paper shows that can we attain this bound in an example. To maximize the number of steps in the DAA, we minimize the number of men rejected each step, holding it to one. So below, with $n = 5$ men and women, there are 17 DAA rounds. In this example, each of $n-1$ women rejects $n-1$ men, and thus end up with their favorite man. Only the last woman might not end up with her best partner. Also, the men's and women's preferences are negative correlated, so the men initially ask the women most inclined to dump them. The last woman's preferences do not matter, so we omit them.

M_1	1	2	3	4	5
M_2	4	1	2	3	5
M_3	3	4	1	2	5
M_4	2	3	4	1	5
M_5	1	2	3	4	5

W_1	4	3	2	1	5
W_2	3	2	1	5	4
W_3	2	1	5	4	3
W_4	1	5	4	3	2

	W_1	W_2	W_3	W_4	W_5
round 1	M_1	M_4	M_3	M_2	$\varnothing$
round 2	M_1	M_5	M_3	M_2	$\varnothing$
round 3	M_1	M_5	M_4	M_2	$\varnothing$
round 4	M_1	M_5	M_4	M_3	$\varnothing$
round 5	M_2	M_5	M_4	M_3	$\varnothing$
round 6	M_2	M_1	M_4	M_3	$\varnothing$
round 7	M_2	M_1	M_5	M_3	$\varnothing$
round 8	M_2	M_1	M_5	M_4	$\varnothing$
round 9	M_3	M_1	M_5	M_4	$\varnothing$
round 10	M_3	M_2	M_5	M_4	$\varnothing$
round 11	M_3	M_2	M_1	M_4	$\varnothing$
round 12	M_3	M_2	M_1	M_5	$\varnothing$
round 13	M_4	M_2	M_1	M_5	$\varnothing$
round 14	M_4	M_3	M_1	M_5	$\varnothing$
round 15	M_4	M_3	M_2	M_5	$\varnothing$
round 16	M_4	M_3	M_2	M_1	$\varnothing$
round 17	M_4	M_3	M_2	M_1	M_5

Essentially, once any man gets rejected $n-1$ times, then DAA must end in the next step. For then each one of the $n-1$ women whom he proposed to also receives a proposal from another preferred man.

B Historical Note on the Transportation Problem

In 1781, Gaspard Monge proposed what is now known as **the transportation problem**: How do you most cheaply move sand from n source piles into n destination holes? Let $\mathcal{S}$ be the set of all maps σ from equal size sources i to equal size destinations j, namely, $j = \sigma(i)$. Knowing the cost $c_{ij} \geq 0$ per ton of sand transported from a place i to hole j, the problem is to decide how to allocate piles to holes so as to minimize the total transportation costs:

$$\min_{\sigma \in \mathcal{S}} \sum_i c_{i\sigma(i)}$$

If we write output $h(i,j) = -c_{ij}$ then this is equivalent to the assignment problem.

Much later, in WWII in 1942, Leonid Kantorovich returned to the transportation problem. He transformed it into a linear problem on a convex set: the bistochastic $n \times n$ matrices $\Pi = \pi_{ij}$ (i.e., $\sum_{i=1}^n \pi_{ij} = \sum_{j=1}^n \pi_{ij} = 1$). In the discrete case, his problem reduces to minimizing:

$$\min_{\pi \in \Pi} \sum \pi_{ij} c_{ij}$$

Impressively, Kantorovich essentially invented dynamic programming to solve this. A few years later, Dantzig (1946) derived linear programming duality that we we used earlier.

One optimal transport plan in Kantorovich's problem solves Monge's problem, since we argued that some linear programming solution is at a vertex.[21] The dual formulation is:

$$\min_{\Pi} \sum \pi_{ij} c_{ij} = \max_{v,v} \left(\sum_i v_i + \sum_j w_j \right) \quad \text{s.t.} \quad v_i + w_j \leq c_{ij}$$

Intuitively, to transport soil from piles i to holes j, you can do it yourself, pay c_{ij} to transport from place i to place j; this is the primal. Alternatively, you can hire someone else to do the job. He could set a two-part price, charging one price v_i for loading at pile i, and another price w_j for unloading at hole j. Then the shipping prices obey $v_i + w_j \leq c_{ij}$.

C Linear Programming and Duality

We digress into optimization theory. For a given *primal linear programming* (LP) problem:

$$\max\{pz | Az \leq q, z \geq 0\} \tag{10}$$

there is a related *dual problem*

$$\min\{uq | uA \geq p, u \geq 0\} \tag{11}$$

These two problems have the same values, a fact known as *Linear Programming Duality.*

Lemma 5 (Duality) *If problems* (10) *and* (11) *are finite, then the solutions coincide.*

Roughly speaking, this theorem asserts that *the dual of the dual is the primal.*

[21]Not only did Shapley get the 2012 Nobel Prize in part for his work on thus dual matching problem, but Kantorovich won the 1975 Nobel prize (with Tjalling C. Koopmans) "*for their contributions to the theory of optimum allocation of resources*".

Proof: Primal feasibility[22] ensures $Az \leq q$, while dual feasibility implies $uA \geq p$. So $pz \leq uAz \leq uq$ for all $u, z \geq 0$: *the value of the primal is at most the value of the dual.* This easy claim is known as *weak duality.* The reverse (strong) direction is harder to show.

Next, we look for a **saddle point** (z, u) of $\mathcal{L}(z, u) = pz + uq - uAz$, simultaneously maximal in $z \geq 0$ given $u \geq 0$, and minimal in $u \geq 0$ given $z \geq 0$. Such a saddle point exists by the Minmax Theorem in game theory. In particular, this means that:

$$\max_{z \geq 0} \min_{u \geq 0} [pz + uq - uAz] = \min_{u \geq 0} \max_{z \geq 0} [pz + uq - uAz] \tag{12}$$

Since $\mathcal{L}(z, u) = uq + (p - uA)z = pz + u(q - Az)$, the saddle point is not finite unless $p - uA \leq 0 \leq q - Az$, with $z_\ell = 0$ when $p_\ell - (uA)_\ell < 0$, and $u_k = 0$ when $q_k - (Az)_k > 0$. Given these complementary slackness conditions, the value of (10) is the left side of (12), and the value of (11) is the right side of (12). In other words, both primal and dual programs are feasible, and have the same optimum. □

The multipliers have economic meaning — namely, they are the *shadow value* of the corresponding constraint. Specifically, u_k measures the marginal value of additional slack in the constraint $q_k - (Az)_k > 0$. To see this, note that $(\partial/\partial q_k)\mathcal{L}(z, u) = u_k$, so that the objective $\mathcal{L}(z, u)$ rises by $u_k(dq_k)$ given the increment dq_k. Likewise, the marginal value of more slack in the constraint $(uA)_\ell - p_\ell > 0$ is z_ℓ because $(\partial/\partial p_\ell)\mathcal{L}(z, u) = z_\ell$.

Here is the earlier promised graphical argument in Koopmans and Beckmann (1957) for why the maximum is attained at a vertex:

It is intuitively obvious, and not hard to prove,[4] that a linear function defined on a convex polyhedron will reach its maximum in a vertex. If this maximum is not reached in any other vertex, then it is not reached in any other point of the polyhedron either. If it is reached in more than one vertex, then it is also reached in all points of a "face" of the poly-

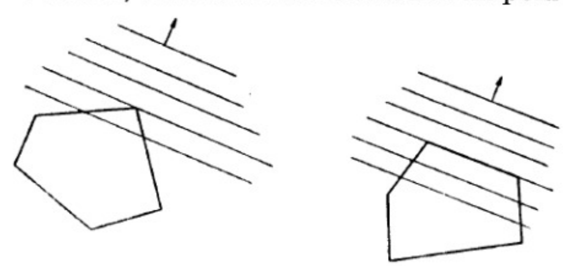
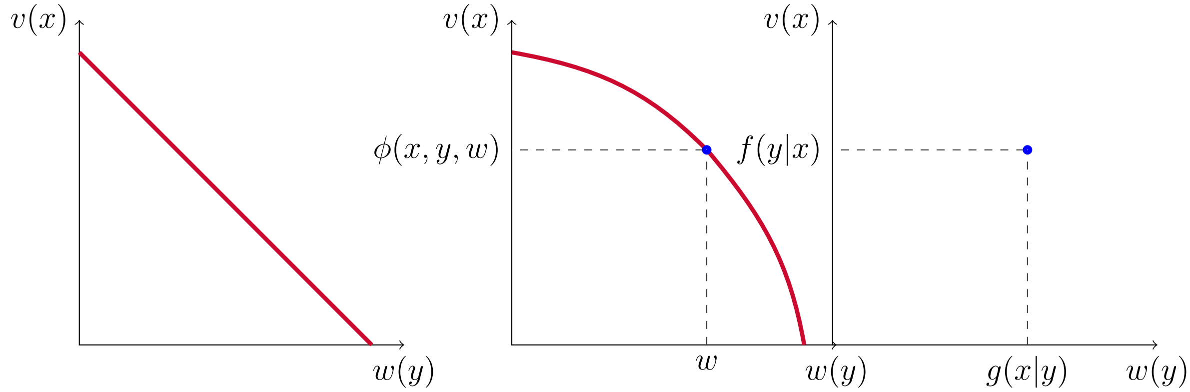


Figure 1.—Maximum of a Linear Function on a Polyhedron.

hedron, that is, a polyhedron contained in the boundary[5] of the original polyhedron, having as its vertices all those vertices of the original polyhedron where the function rea its maximum. The two cases are illustrated in Figure 1.

[22] The maximum of any function on an empty set is defined to be $-\infty$, and the minimum of any function on an empty set is defined to be ∞.



(a) TU (b) Imperfect TU (c) NTU

Figure 7: We depict examples of the payoff frontiers respectively for transferable utility, imperfectly transferable utility, and strict nontransferable utility.

D Extension: Imperfectly Transferable Utility

We have explored assortative matching in both the TU and the NTU cases. The frontier of payoffs achievable by a pair of matched agents is linear in the TU case, and collapses to a point in the strict NTU case. In the arguably typical intermediate case — called **imperfectly transferable utility** — where agents can transfer utility but not at a constant rate, the frontier is decreasing but neither linear nor a single point, as in Figure 7.

To describe the payoff frontier for matched agents, let $\phi(x, y, w)$ be the maximum utility that woman x generates when matched with man y who earns payoff w. For simplicity, assume that ϕ is twice differentiable. Also, $\phi(x, y, w)$ is decreasing in w: $\phi_w < 0$.

We assume what is known as a strict *Spence Mirrlees* condition, that

$$\phi_{xy} - \phi_{wx}[\phi_y/\phi_w] > 0 \quad \Leftrightarrow \quad -[\phi_y/\phi_w]_x > 0 \tag{13}$$

Namely, the marginal rate of substitution $-\phi_y/\phi_w$ between the man's type y and his payoff w increases in one's type x. So higher woman types x are willing to pay more payoff per increment in the men's types. Becker's TU model automatically meets this condition, for supermodularity asserts $\phi_{xy} > 0$ while transferable utility implies that $\phi_{wx} = 0$.

In equilibrium,[23] given the wage function $w(y)$, woman x solves $V(x) = \max_y V(x|y)$, where $V(x|y) \equiv \phi(x, y, w(y))$ is the payoff to man x. Firstly, note that $w'(y) > 0$ — for by way of contradiction, if $w'(y) \leq 0$ on an interval $[y_0, y_1]$, then any woman x earns more profits in a match with y_1 than y_0 (strictly higher output for a weakly lower payment).

Next, we use monotone comparative statics to deduce PAM. By the single crossing property, the optimal man y for woman x is nondecreasing in x if $V_y(x|y) \geq 0$ implies $V_x(x'|y) \geq 0$ for $x' > x$. To verify this implication, let's compute:

$$V_y(x|y) \equiv \phi_y(x, y, w(y)) + \phi_w(x, y, w(y))w'(y) \tag{14}$$

[23]Legros and Newman (2007) generalizes Becker's supermodularity condition for PAM in this framework, and Chade, Eeckhout and Smith (2016) find the differential version of this.

Since $\phi_w < 0$, this holds if and only if $-(\phi_y(x)/\phi_w(x)) \geq w'(y)$. But if $x' > x$ then $-(\phi_y(x')/\phi_w(x')) \geq w'(y)$, by (13). Then $V_y(x', y) \geq 0$, verifying the SCP. This proves:

Proposition 9 *If the woman's maximum utility $\phi(x, y, w)$ obeys $\phi_{xy} \geq 0$ and the strict Spence-Mirrlees condition* (13)*, then PAM is the unique efficient matching $m \in \mathcal{M}$.*