

# Cross-Sectional Dynamics in a Two-Sided Matching Model\*

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WORK IN PROGRESS!!  
Conjectures, theorems, and proofs welcome!

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## Abstract

We explore a two-sided matching model with a continuum of agents indexed by efficiency parameters in  $(0,1)$ , and assume that the flow output of the match  $(x,y)$  is  $2xy$ . To incorporate nominal rigidities, we consider the nonstandard assumption of equal-output sharing, and show that the multiplicative production function engenders a natural segmentation of  $(0,1)$  into equivalence classes of agents willing to match with one another. This produces a discontinuous wage profile. We also analyze the non-steady state dynamics of the model, where the no-discounting case proves a fruitful benchmark. We find that quits endogenously arise as a non-steady state phenomenon. In the appendix, we describe the steady-state of the model under the standard Nash equal-surplus division rule.

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\*The model of this paper was inspired by a conversation with Doug Galbi. Computer simulations done by Parag Gupta under the UROP have also been very helpful.

## 1. INTRODUCTION

(to be completed)

- Under the equal output division rule (analogous to a minimum wage), an intertemporal competition for rents arises, producing a wage distribution riddled with discontinuities: Some agents greatly profit at the expense of others.
- The efficient and actual decision matching rules in steady-state of a model with finitely-many ability levels and an additive “match-specific” production component are analyzed in Lockwood (1986).
- The effect of coordination on production is studied by Jovanovic (1991).

## 2. SOME DISCRETE-TIME EXAMPLES

Suppose there is a continuum of individuals with efficiency parameters distributed uniformly in  $(0,1)$ . If individuals  $x$  and  $y$  are matched, they produce a joint product (literally) of  $2xy$  in each period in which they remain matched. Future payoffs are discounted at the common rate  $\alpha < 1$ . What is the matching that maximizes total output here?<sup>1</sup> Rather than appeal to optimal control theory, we consider the following heuristic argument. An interchange argument due to Hardy asserts that for any two positive increasing sequences  $\{a_1, \dots, a_n\}$  and  $\{b_1, \dots, b_n\}$ , the maximum of the expression  $\sum_{i=1}^n a_i b_{\pi(i)}$  over permutations  $\pi$  of  $\{1, 2, \dots, n\}$  is the identity permutation: Pair the highest  $a$ 's with the highest  $b$ 's. For our scenario, this suggests that each  $x \in (0, 1)$  should be paired with another  $x$ . This would yield an average output of  $\int_0^1 2x^2 dx = 2/3$ . How closely can optimal selection by individuals in a two-sided matching model approximate this socially optimal outcome? Much of the insight is found in two and three period models, to which we now turn.

### 2.1 A Two Period Example

Let there be two periods. In the first period, each individual is randomly paired with another potential partner, and either individual to a proposed match can veto it. If this occurs, each receives zero payoff that period. Otherwise, they split the output produced. Assume for now an equal output division. In the second period, all individuals remaining unmatched — and those wishing to terminate their current match and pair anew — are randomly paired, once for all. No vetoes are permitted.<sup>2</sup> We also assume that search is a time-consuming process, and that if one is matched in a given period, one cannot simply quit that match for another the very next period; rather, a one period sabbatical is required to search; therefore, no individual will ever opt to terminate a match.

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<sup>1</sup>This simple question has the flavour of more general optimal pairing questions being studied by Kremer and Maskin (private communication, 1992).

<sup>2</sup>Alternatively, vetoes are allowed, but would never be used since the alternative payoff equals zero.

Note that the first period decision criterion of all individuals is identical: Given the multiplicatively separable production function, one's own parameter has no effect on preferences. Each individual simply seeks to maximize the discounted expected parameter of her partner. Thus, in period one, all individuals will accept any parameter above some threshold  $\theta$  equal to their discounted expected period two partner's parameter. Given that everyone uses the same threshold,

$$\theta = \alpha \frac{\theta(\theta/2) + (1 - \theta)\theta(1 + \theta)/2}{\theta + (1 - \theta)\theta}$$

so that  $(2 - \alpha)\theta^2 - (4 - \alpha)\theta + \alpha = 0$  or  $\theta = 0$ . There is also an equilibrium with  $\theta = 0$  that is possible: All individuals who meet in a given period agree to match, knowing that no one will be around next period. This *trivial* outcome is excluded from consideration. Thus,

$$\theta = \frac{4 - \alpha - \sqrt{5\alpha^2 - 16\alpha + 16}}{2(2 - \alpha)},$$

and there is an equivalence class  $[\theta, 1)$  of individuals willing to pair with one another in the first period, and no one with an efficiency parameter below  $\theta$  will pair until period two.

It can be shown that the earnings profile is discontinuous at  $\theta$ . Individuals with indices above  $\theta$  earn rents, due to the equal output division rule. As a result some agents (those just below  $\theta$ ) may actually prefer increases in the interest rate (more heavy discounting of future payoffs) because  $\theta$  is increasing in  $\alpha$ .

Now consider a more standard output division, leaving no economic rent unexploited.

(Nash division: to be completed)

## 2.2 A Three Period Example

Now suppose that there are three periods, with those matches proposed in the final period necessarily consummated. We describe all (non-trivial) such equilibria using backward induction. Independent of non-trivial first period behaviour, the optimal second period strategy is to accept a proposed partner exactly when one's index is at least  $\theta_2 > 0$ , where  $\theta_2$  equals the discounted expected period three partner's parameter.<sup>3</sup> Thus, there is some equivalence class  $(\theta_2, 1)$  in period two of mutually desirable individuals. Moreover, by the rationale of the previous example, no individual will ever opt to terminate a match after this period.

Next consider the first period behaviour. Since  $\theta_2$  is the lower threshold for accepting any new matches in period two, it follows that anyone initially paired in

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<sup>3</sup>This expectation depends in an obvious fashion upon the common period two behaviour. Thus, whether such a  $\theta_2$  exists or is unique is at issue. We shall, however, shortly exhibit a solution which happens to be unique.

period one will quit her match in period two (and thus forego period two payoffs) if and only if her partner's parameter lies below  $\theta_2$ . Who in the first period then will agree to match? Clearly, an individual  $x$  declines a match exactly when her discounted expected payoff  $\alpha r_x$  from being *eligible* for next period's matching exceeds that of the proposed match. Since those with indices below  $\theta_2$  will always be turned down (or dumped, if already matched) in the second period, the option of being eligible for the next period is worthless ( $r_x = 0$ ). They will therefore accept any proposed match in the first period.

Consider now those with indices at least  $\theta_2$ . Since they are all in the same equivalence class in the second period, they all have the same first period threshold  $\theta_1$ . Note that  $\theta_1$  equals the discounted expected eventual partner's parameter, conditioned on the threshold decision rule of period two. We claim that  $\theta_1 > \theta_2$  necessarily obtains.

*Proof:*

Denote by  $P_t(a, b)$  the period  $t$  fraction of eligible individuals with efficiency parameters  $y$  in  $(a, b)$ , and by  $E_t$  the expectation operator with respect to the measure  $P_t$ . Then  $\theta_1 = P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)$ , and so

$$\begin{aligned}
(\theta_1 - \theta_2)/\alpha &= [P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)] - E_3(y) \\
&= [P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)] - \\
&\quad [P_3(0, \theta_2)E_3(y \mid y < \theta_2) + P_3(\theta_2, 1)E_3(y \mid y \geq \theta_2)] \\
&= P_2(0, \theta_2)\theta_2 + [P_2(\theta_2, 1) - P_3(\theta_2, 1)]E_2(y \mid y \geq \theta_2) - \\
&\quad P_3(0, \theta_2)E_3(y \mid y < \theta_2) \\
&> P_2(0, \theta_2)\theta_2 + [P_2(\theta_2, 1) - P_3(\theta_2, 1)]\theta_2 - P_3(0, \theta_2)\theta_2 \\
&= 0
\end{aligned}$$

because the second period matching equivalence class implies that

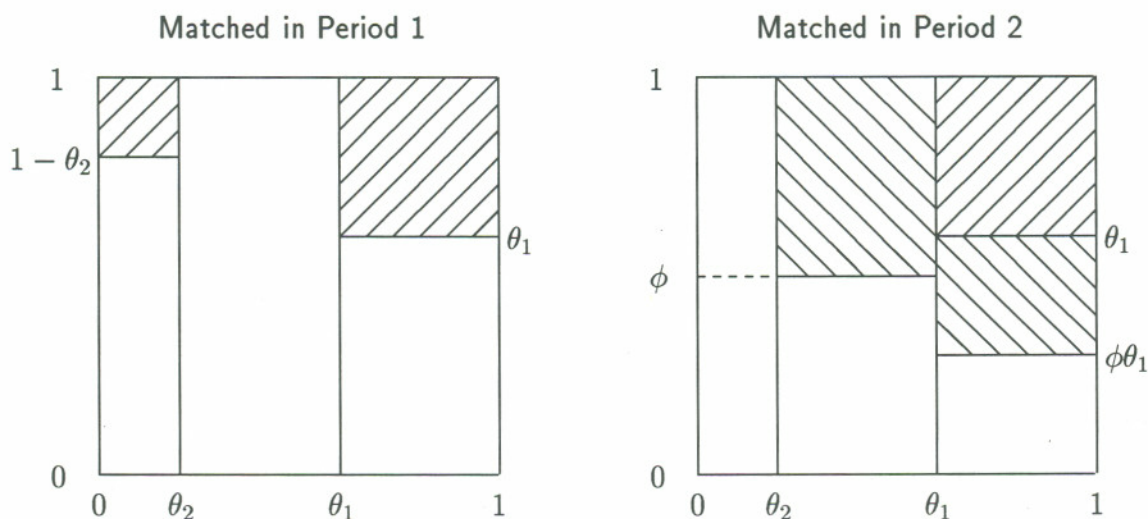
$$P_3(\theta_2, 1) = \frac{P_2(\theta_2, 1) - P_2(\theta_2, 1)^2}{1 - P_2(\theta_2, 1)^2} = \frac{P_2(\theta_2, 1)}{1 + P_2(\theta_2, 1)} < P_2(\theta_2, 1). \quad \text{QED}$$

Given  $\theta_1 > \theta_2$ , the histogram diagram of Figure 1 depicts the matches that occur. In period one, there are two equivalence classes of mutually agreeable (forward shaded) matches  $(0, \theta_2)$  and  $[\theta_1, 1)$ . It is common knowledge that matches consummated within  $(0, \theta_2)$  are only temporary, and that those individuals are not eligible for second period matches. A variety of *Groucho Marx result* manifests itself here: In the first period, no individual in  $(\theta_2, \theta_1)$  is willing to match with anyone who's willing to match with her — namely those in  $(0, \theta_2)$ . In period two, there is one such (backward shaded) equivalence class  $[\theta_2, 1)$ , and in the final period the equivalence class constitutes all of  $(0, 1)$ .<sup>4</sup>

<sup>4</sup>The previous definitions of  $\theta_1$  and  $\theta_2$  yield

$$\theta_1 = \alpha \frac{(1 - \theta_1)\theta_1(\theta_1 + 1)/2 + (\theta_1 - \theta_2)(\theta_1 + \theta_2)/2 + (1 - \theta_2)\theta_2^2}{(1 - \theta_1)\theta_1 + (\theta_1 - \theta_2) + (1 - \theta_2)\theta_2}$$

Figure 1: Three Period Matching Histogram



(Nash division: to be completed)

In the model that follows, we switch to continuous time.

### 3. THE CONTINUOUS-TIME MODEL

Consider a model with a continuum of individuals indexed by *efficiency parameters* (or *indices*) distributed uniformly in  $(0,1)$ . At any moment in time, each individual is either *matched* or *unmatched*. Only those unmatched engage in search for a new partner. When two individuals meet, either can veto the proposed match. There are two possible matching technologies involving Poisson arrival times. We first consider a *linear search technology*.<sup>5</sup> Here, an unmatched individual can expect to have a match proposed to her within time  $t > 0$  with probability  $1 - \exp(-\rho t)$ . Perhaps more intuitively, potential partners for unmatched individuals arrive with constant flow probability  $\rho > 0$ , independent of the Lebesgue measure of unmatched individuals (the *unemployment rate*). Call  $\rho$  the *rendezvous rate*. With this match-

and

$$\theta_2 = \alpha \frac{(1 - \theta_1)\phi\theta_1(\theta_1 + 1)/2 + (\theta_1 - \theta_2)\phi(\theta_1 + \theta_2)/2 + \theta_2^2/2}{(1 - \theta_1)\phi\theta_1 + (\theta_1 - \theta_2)\phi + \theta_2},$$

where

$$\phi = \frac{(1 - \theta_1)\theta_1}{(1 - \theta_1)\theta_1 + (\theta_1 - \theta_2) + (1 - \theta_2)\theta_2} = \frac{(1 - \theta_1)\theta_1}{2\theta_1 - \theta_1^2 - \theta_2^2}.$$

There is a unique solution of the form  $0 < \theta_2 < \theta_1 < 1$  to these equations.

<sup>5</sup>This terminology is taken from Diamond and Maskin (1979) and (1981).

ing technology, the flow of matchings is *linearly* proportional to the unemployment rate. We later consider a *quadratic search technology*, in which the arrival rate equals  $\rho > 0$  times the unemployment rate. Thus, the flow of matchings declines as the *square* of the unemployment rate.

If  $x$  is paired with  $y$  then the flow output of that relationship is  $2xy$ .<sup>6</sup> Individuals discount the flow of payoffs at the common *interest rate*  $\beta$ . How that payoff flow is divided is crucial. We again investigate two possibilities. Under the *nominal sharing rule*, both  $x$  and  $y$  receive a flow reward of  $xy$ . This equal output division, the nonstandard assumption made in the body of this paper, is intended as a rough proxy of nominal rigidities that might persist — or be mandated — such as the existence of a minimum wage, or internal pay equity laws.

Under the *Nash sharing rule*, the flow surplus from the match is equally divided. That is, if the option of remaining (or becoming) unmatched entails *flow rewards* of  $r_x$  and  $r_y$  for  $x$  and  $y$  respectively, where  $r_x + r_y > 2xy$ , then  $x$  receives a flow payoff equal to

$$r_x + (2xy - r_x - r_y)/2 = xy + (r_x - r_y)/2$$

and  $y$  receives a flow payoff of  $xy + (r_y - r_x)/2$ . In the appendix, we explore how radically matters change under this more standard surplus division rule.

A *strategy* for  $x \in (0, 1)$  in this set-up is a measurable correspondence  $\Gamma_x : R^+ \rightarrow (0, 1)$  specifying who  $x$  is willing to match with at each time  $t > 0$ .<sup>7</sup> We assume that all individuals of the same efficiency parameter  $x$  use the same  $\Gamma_x$ , so that every  $x$  is willing to match with any member in her *acceptance set*  $\Gamma_x(t)$  at time  $t > 0$ . If we further posit that an individual accepts a proposed match if indifferent, then  $\Gamma_x(t)$  is closed in  $(0, 1)$ .

An individual only matches if she expects to receive more from the match than from remaining unemployed. Thus, under the nominal sharing rule, all individuals' preferences are monotonic increasing in the partner's efficiency parameter. Consequently, acceptance sets are of the form  $\Gamma_x(t) \equiv [\theta_x(t), 1)$ , where  $\theta_x(t)$  is the *threshold* partner for individual  $x$  at time  $t$ . Notice that under the nominal sharing rule, *the current threshold  $\theta_x$  for any individual equals the expected normalized flow reward  $r_x/x$  of remaining unmatched*. Now consider two agents  $y < x$ . Anyone who is willing to match with  $y$  is clearly willing to match with  $x$ , by monotonicity. Thus,  $r_x/x \geq r_y/y$ . Hence, we discover the

**Nominal Matching Lemma** *If at any time,  $x$  is willing to match with  $y < x$ , then  $y$  is willing to match with  $x$ , i.e.  $\theta_x(t) < y$  and  $y < x \implies \theta_y(t) \leq \theta_x(t) < x$ .*

Under the Nash sharing rule, the decision whether to match is somewhat less

<sup>6</sup>Optimal strategies under the *nominal* sharing rule are unchanged with the joint production function  $x + y$ , reminiscent of Lockwood (1986). The *Nash* results require some modification.

<sup>7</sup>Note that the efficiency index of a potential partner is perfectly observable. Moreover, as individuals have the right to sever a match at will, they will exercise that right when their current partner is no longer an element of  $\Gamma_x$ .

involved: Two individuals agree to a match exactly when there is a positive flow surplus from it. Consequently, the matching lemma is symmetric:

**Nash Matching Lemma** *At any time,  $x$  is willing to match with  $y$  if and only if  $y$  is willing to match with  $x$ , i.e.  $y \in \Gamma_x(t)$  if and only if  $x \in \Gamma_y(t)$ .*

Given the multiplicative separability of the output function, and the equal output sharing rule, everyone seeks to maximize the expected present value of all her future partners' indices. But two individuals may differ in their future opportunity sets if both aren't equally acceptable to all others. This fact, together with the Matching Lemma, establish the following:

**Nominal Sorting Lemma** *If an unmatched  $x$  is willing to match with  $y < x$  for all times  $s \geq t$ , then  $x$  and  $y$  have the same lower threshold. That is,  $\theta_x(s) \leq y$  for all  $s \geq t$  and  $y < x \implies \theta_y(s) = \theta_x(s)$  for all  $s \geq t$ .*

## PART I: LINEAR SEARCH TECHNOLOGY

### 4. Steady-State Analysis

In steady-state, each individual  $x$  employs a time-invariant strategy  $\Gamma_x(t) \equiv \Gamma_x$  for all  $t \geq 0$ . Consequently, *no one will ever quit*. To see this, just realize that because of time-invariance, if  $y$  is acceptable to  $x$  at some moment, then  $y$  is always acceptable to  $x$ . We can therefore simply restrict attention to the lower parameter threshold for each individual.

So as to maintain the steady-state, we also assume that matches dissolve with constant flow probability  $\delta > 0$ , the *dissolution rate*.<sup>8</sup> This implies that a matched individual can expect to be separated from her current partner within  $t > 0$  time units with probability  $1 - \exp(-\delta t)$ . Normalizing by own indices, let  $V_x(y)$  be the *present value* to  $x$  of matching with  $y$ , and  $V_x$  the present value of being unmatched. Finally, since thresholds are constant, say  $1 = \theta_0 > \theta_1 > \theta_2 > \dots > 0$ , the Nominal Sorting Lemma allows us to refer to the values by their equivalence class number:  $k$  for  $[\theta_k, \theta_{k-1})$ . Thus,  $V_k(y)$  depends upon the dissolution rate and  $V_k$  in the following fashion:

$$V_k(y) = \int_0^\infty [(1 - e^{-\beta t})y/\beta + e^{-\beta t}V_k] \delta e^{-\delta t} dt$$

which simplifies to  $V_k(y) = (y + \delta V_k)/(\delta + \beta)$ .

Let  $u_k$  be the *unemployment rate* of  $[\theta_k, \theta_{k-1})$ , and let  $u$  be the overall unemployment rate. Then the probability that a given match lies in  $[\theta_k, \theta_{k-1})$  is  $q_k = u_k(\theta_{k-1} - \theta_k)/u$ . The value to being unmatched depends upon  $q_k$  and the rendezvous rate  $\rho$ .

$$V_k = \int_0^\infty (q_k E[V_k(y) | \theta_k \leq y < \theta_{k-1}] + (1 - q_k)V_k) \rho e^{-(\rho+\beta)t} dt$$

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<sup>8</sup>Here, everyone is infinite-lived. The analysis changes somewhat when  $\delta$  is interpreted as the death rate (equal to the birth rate).

which, upon substitution of the expressions for  $V_k(y)$  and  $q_k$ , and appeal to the simple properties of the uniform distribution, yields

$$(\beta + \delta)\beta V_k = \rho[(\theta_{k-1} + \theta_k)/2 - \beta V_k]u_k(\theta_{k-1} - \theta_k)/u$$

This equation reduces further with the simple observation that  $\theta_k = \beta V_k$ . We may now rewrite the previous equations as

$$u(\beta + \delta)\theta_k = \rho u_k[(\theta_{k-1} + \theta_k)/2 - \theta_k](\theta_{k-1} - \theta_k)$$

Completing the square, this simplifies to the following

$$\text{SS-1 [Optimality Equation]} \quad \theta_0 = 1 \quad \text{and} \quad 2u(\beta + \delta)\theta_k = \rho u_k(\theta_{k-1} - \theta_k)^2$$

Also true in steady-state is the fact that the unemployment rate in each equivalence class is constant. In other words, the flow of matches severed  $\delta(1 - u_k)(\theta_{k-1} - \theta_k)$  equals the flow of matches created  $q_k u_k(\theta_{k-1} - \theta_k)$ . Or,

$$\text{SS-2 [Constant Regional Unemployment]} \quad \delta u(1 - u_k) = \rho u_k^2(\theta_{k-1} - \theta_k)$$

All that remains for a characterization of equilibrium is to define  $u$ , the *natural rate of unemployment*:

$$\text{SS-3 [Labour Market Equilibrium]} \quad u = \sum_1^\infty u_k(\theta_{k-1} - \theta_k)$$

The analysis proceeds more smoothly if we write  $\Delta_k = (\theta_{k-1} - \theta_k)$ .

(to be completed)

A corollary of (SS-1) is that  $\theta_k > 0$  for all  $k$ . Hence, *there is an infinite number of equivalence classes*.

Questions: How many equivalence classes are there with a labour market  $(c, 1)$ ? Who profits and who loses from the rent sharing?

## 5. Non Steady-State Analysis

We now forego any exogenous source of match dissolutions (i.e. set  $\delta = 0$ ), and consider the evolution of the model beginning with all individuals unmatched.

The results of the last section only obtain when  $\beta > 0$ , for otherwise no equivalence class would have positive measure and no matching would ever occur. The same is not true when we consider non steady-state analysis, where falling flow values provides an implicit discounting.

### 5.1 No Discounting Case

When individuals do not discount future payoffs, we must first worry about their objective function. Suppose simply that individuals seek to maximize their average payoff (or the liminf of truncated average payoffs, if the former fails to exist). Then the non-steady-state analysis admits a simple formulation. Equivalence class segmentation will arise at each moment in time, for the same reasons as before;



however, there are dynamics. In light of the interpretation of thresholds as flow rewards, the threshold for an equivalence class cannot increase without inducing quits. That is, if  $\theta(t_2) > \theta(t_1)$  for  $t_2 > t_1$ , then any individual  $x \geq \theta(t_2)$  paired with  $y \in [\theta(t_1), \theta(t_2))$  will quit her match. Thus, given the average payoff objective function, we can assume WLOG that these matches never occur. Hence, we have the

**Threshold Monotonicity Lemma** *If  $\beta = 0$  and  $\theta(t)$  is the threshold of an equivalence class of individuals at time  $t$ , then  $\theta'(t) \leq 0$ .*

Another useful fact true only without discounting is that if an individual  $x \in (0, 1)$  will at some future time  $t_0 < \infty$  be subsumed within an equivalence class from above, then she will never consider matching until time  $t_0$ . This follows from the fact no individual will ever “match down” given that her expected value *will* equal  $x$  at time  $t_0$  (and hence *now* equals  $x$ , given  $\beta = 0$ ). From this observation and from the previous lemma, we can conclude the following:

**Proposition [Non Steady-State Equivalence Class Characterization]**

*Under the equal output sharing rule, at each moment in time, only the following two possibilities can arise:*

1. *One equivalence class  $[\theta(t), 1)$  obtains at each time  $t \geq 0$ , with  $\theta(0) \in (0, 1)$  and  $\theta'(t) \leq 0$ , and  $\theta(t) \downarrow 0$ .*
2. *There is a finite or countable partitioning  $(\theta_1, 1) \cup \{(\theta_k, \theta_{k-1})\}_2^\infty$  of  $(0, 1)$  such that for all  $k = 1, 2, \dots$ , one equivalence class  $(\theta_k(t), \theta_{k-1}]$  obtains at each time  $t \geq 0$ , with  $\theta_k(0) \in (\theta_k, \theta_{k-1})$ ,  $\theta'_k(t) \leq 0$ , and  $\theta_k(t) \downarrow \theta_k$ .*

We now show that possibility (1) above can in fact arise, and precisely describe the dynamics. Let  $u(t)$  be the fraction of unmatched individuals at time  $t$ ,  $\theta(t)$  the time- $t$  threshold of the one equivalence class, and  $\mu(t)$  the average efficiency parameter among unmmatched individuals above  $\theta(t)$ . These three parameters *can* act as state variables for the dynamical system; however, it turns out to be analytically more tractable to consider  $\pi(t) \equiv \mu(t)[u(t) - \theta(t)]$  instead of  $\mu(t)$ .

Then  $\theta(t)$  is an optimal threshold for individuals above it if the alternative of waiting provides the same expected eventual partner. Now, an agreeable match consummated at time  $t$  provides an expected partner’s index of  $\mu(t)[1 - \theta(t)/u(t)] + \theta(t)[\theta(t)/u(t)]$ . Hence,

$$\theta(t) = \int_t^\infty \rho e^{-\rho(s-t)} [\pi(s) + \theta(s)^2] / u(s) ds$$

so that differentiation yields the following *optimality equation*:

$$\theta'(t) = \rho\theta(t) - \rho[\pi(t) + \theta(t)^2] / u(t) \tag{1}$$

This equation implies that  $\theta'(0) = -\rho[1 - \theta(0)]^2/2$ , since  $u(0) = 1$  is assumed.

Next notice that the fraction of mutually agreeable meetings equals  $[1 - \theta(t)/u(t)]^2$ . Hence, the quadratic matching technology implies that

$$u'(t) = -\rho[1 - \theta(t)/u(t)][u(t) - \theta(t)] = -\rho[u(t) - \theta(t)]^2/u(t) \quad (2)$$

Finally, we discover a differential equation for  $\pi(t)$ . To this end, first define the “point” unemployment rate  $u(x, t)$  of index  $x$  at time  $t$ . Thus,  $u(t) = \int_{\theta(t)}^1 u(x, t) dx$  and  $\pi(t) = \int_{\theta(t)}^1 x u(x, t) dx$ . Then notice that

$$u_t(x, t) = -\rho[1 - \theta(t)/u(t)]u(x, t)$$

Hence, because individual  $x$  accepts no match until his time  $\theta^{-1}(x)$ , we have  $u(\theta(t), t) \equiv 1$ , so that

$$\begin{aligned} \pi'(t) &= -\theta'(t)\theta(t) + \int_{\theta(t)}^1 x u_t(x, t) dx \\ &= -\theta'(t)\theta(t) - \int_{\theta(t)}^1 x \rho[1 - \theta(t)/u(t)]u(x, t) dx \\ &= -\theta'(t)\theta(t) - \rho\pi(t)[u(t) - \theta(t)]/u(t) \end{aligned}$$

It is possible to eliminate  $\pi(t)$  from this system, and discover that  $u'(t)\theta'(t) = -\theta''(t)u(t)$ . Using the initial conditions, this implies that

$$u(t) = \frac{-\rho[1 - \theta(0)]^2}{2\theta'(t)}$$

Thus, *there is a continuum of equilibria, indexed by  $\theta(0) \in (0, 1)$ , all of which close the market in finite time.*

## 5.2 The Discounting Case

(to be completed)

# PART II: QUADRATIC SEARCH TECHNOLOGY

## 6. Steady-State Analysis

(to be completed)

Big Question: Does the quadratic matching technology engender a Diamond-style multiplicity of equilibria? i.e. Is there a unique natural rate of unemployment?

## 7. Non Steady-State Analysis

(to be completed)

Basic insight: There is no substantive difference between  $\beta = 0$  and  $\beta > 0$ . Let  $\beta > 0$ . Then individuals whom the top cohort is not planning to match with (for a while) impose no externality on the top cohort by matching early due to the

quadratic technology. If the threshold of the top cohort ever reaches a particular individual, then she will part with any current partner, because the flow rewards of being unemployed exceed those of the current match. Thus the problem that the top cohort solves is totally independent of any matching decisions made by those not in the top cohort. Hence, if the threshold tends to zero when  $\beta = 0$  then it also does so when  $\beta > 0$ .

Possible Application: What is the optimal transformation from a socialist regime (with random matching) to a capitalist one? How much unemployment should be created?

## APPENDIX: DYNAMICS WITH THE NASH SURPLUS SPLIT

(to be completed)

Question: Does this surplus division rule induce the maximum possible average output — i.e. would a social planner (constrained by the matching technology) make the same matching decisions?

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