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RUSHES IN LARGE TIMING GAMES

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RUSHES IN LARGE TIMING GAMES

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We develop a continuum player timing game that subsumes standard wars of attrition and pre-emption games, and introduces a new rushes phenomenon. Payoffs are continuous and single-peaked functions of the stopping time and stopping quantile. We show that if payoffs are hump-shaped in the quantile, then a sudden "rush" of players stops in any Nash or subgame perfect equilibrium.

Fear relaxes the first mover advantage in pre-emption games, asking that the least quantile beat the average; *greed* relaxes the last mover advantage in wars of attrition, asking just that the last quantile payoff exceed the average. With greed, play is inefficiently late: an accelerating war of attrition starting at optimal time, followed by a rush. With fear, play is inefficiently early: a slowing pre-emption game, ending at the optimal time, preceded by a rush. The theory predicts the length, duration, and intensity of stopping, and the size and timing of rushes, and offers insights for many common timing games.

KEYWORDS: Timing games, war of attrition, pre-emption game, tipping, bubbles, sorority rushes, bank runs, population games.

1. INTRODUCTION

"Natura non facit saltus."—Leibniz, Linnaeus, Darwin, and Marshall

MASS RUSHES PERIODICALLY GRIP MANY ECONOMIC LANDSCAPES—such as fraternity rush week; the "unraveling" rushes of young doctors seeking hospital internships; the bubble-bursting sales rushes ending asset price run-ups; land rushes for newly-opening territory; bank runs by fearful depositors; and flight from an at-risk neighborhood. These important topics are so far removed from one another that they are studied in wholly disparate fields of economics. Yet by stepping back from their specific details, we capture them in a simple unified model of timing games.

We venture a continuum of players, whose payoffs solely reflect the stopping time and the fraction (*quantile*) of players who have already stopped. The fundamental nonstrategic portion of the payoff is a deterministic function of time—when to cut the metaphorical tree. A strategy is a stopping time distribution function on the positive reals. We show that rushes arise in equilibrium—that is, positive measures of agents simultaneously stopping—whenever preferences over the stopping quantile are hump-shaped. We explain the players' rate of stopping, and the size and timing of rushes. Our theory also agrees with many known results for these games.

Timing games have usually assumed a small number of identified players, as befits settings like industrial organization. This paper introduces a tractable class of population timing games (so-called "large games") for anonymous environments like the motivational examples. We thus assume a continuum of homogeneous players, ensuring that no single individual has any impact, and dispensing with strategic uncertainty. A large

[1]This follows on the manuscript "Greed, Fear, and Rushes" (2010) by Andreas and Lones, growing out of joint work in Andreas' 2004 Ph.D. thesis. It was presented at the 2008 Econometric Society Summer Meetings at Northwestern and the 2009 ASSA meetings. While including some preliminary analysis from this earlier work, the current paper reflects the joint work of Axel and Lones since 2012. We have profited from seminar comments at Wisconsin, Western Ontario, Melbourne, Columbia, UCLA, Penn State, Boston College, Simon Fraser University, the 2016 Miami Theory Conference, and Brown. We thank Faruk Gul for a helpful observation about subgame perfection. This version reflects suggestions of the co-editor and three referees.

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population of players facilitates equilibrium coordination, since everyone can honestly communicate intentions to surveys, etc. and learn from them—for the actions of no finite set of individuals matters. We characterize the subgame perfect equilibria of the timing game—thereby also ignoring any learning about exogenous uncertainty.

When the stopping c.d.f. is continuous in time, there is *gradual play*, as players slowly stop; a *rush* occurs when a positive mass suddenly stops, and the c.d.f. jumps. We assume no discontinuities, with payoffs smooth and hump-shaped in time, and smooth and single-peaked in quantile. This simultaneously ensures a unique optimal *harvest time* for any quantile, when fundamentals peak, and a unique optimal *peak quantile* for any time, when stopping is strategically optimal.

Two opposing flavors of timing games have long been studied. A *war of attrition* entails gradual play in which the passage of time is fundamentally harmful and strategically beneficial. The reverse holds in a *pre-emption game*—the strategic and exogenous delay incentives oppose, balancing the marginal costs and benefits of the passage of time. Consequently, standard timing games assume a monotone increasing or decreasing quantile response, so that the first or last mover is advantaged over all other quantiles. But in our class of games, the peak quantile may be interior. The game exhibits *greed* if the very last mover's payoff exceeds that of the *average* quantile, and *fear* if the very first mover's does. So the standard war of attrition is the extreme case of greed, with later quantiles more attractive than earlier ones. Likewise, the standard pre-emption game captures extreme fear. Since payoffs are single-peaked in quantile, greed and fear are mutually exclusive—in other words, a game either exhibits greed or fear or neither.

Gradual play requires constant equilibrium payoffs, balancing the fundamental and quantile considerations. Nash equilibrium usually entails gradual play for all quantiles in the well-studied case with a monotone quantile response (Proposition 1); however, an initial rush happens here when the immediate stopping gains dominate fundamental payoff growth. But when preferences are hump-shaped in quantile, a rush always occurs (Proposition 2A). For if no mass of players ever stopped in a Nash equilibrium, then the players' indifference in gradual play would require that quantiles below the quantile peak stop after the harvest time, and quantiles above the quantile peak stop before the harvest time—an impossibility. Apropos our lead quotation, despite a continuously evolving model with an interior optimal stopping time and quantile, aggregate behavior must jump. Notably, this jump is not driven by higher order belief subtleties, nor even Bayesian updating. Rather, it simply reflects elementary best response forces in Nash equilibrium.

Ruling out faster than exponential growth, we assume that payoffs are log-concave in time. This affords a sharp characterization of gradual play: Any gradual pre-emption game ebbs to zero after the early rush, whereas any gradual war of attrition accelerates from zero towards its rush crescendo (Proposition 3). This means that even inclusive of the stopping rush, stopping rates wax after the harvest time, and wane before the harvest time. To wit, wars of attrition intensify and pre-emption games taper off. We can thus identify timing games from the stopping rate data.

Not only do rushes occur, but there are two cases: Absent fear, a war of attrition starts at the harvest time, and is followed by a rush. Meanwhile, absent greed, a rush is followed by a gradual pre-emption game that ends at the harvest time. So rushes occur inefficiently early with fear, and inefficiently late with greed. Both types of equilibria arise with neither greed nor fear. This yields a useful big picture insight for our examples: the rush occurs before fundamentals peak in a pre-emption equilibrium, and after fundamentals peak in a war of attrition equilibrium.

Rushes create timing coordination problems, and thereby a multiplicity of equilibria. We focus on *safe equilibria*, a Nash refinement robust to slight timing mistakes. When the stopping payoff is monotone in the quantile, there exists a unique Nash equilibrium (Proposition 1). But with hump-shaped quantile preferences, safety strictly refines Nash equilibrium. Using a graphical apparatus, we depict all safe equilibria simply by intersecting two curves: one locus equates the rush payoff and the adjacent quantile payoff, and another locus imposes constant payoffs in gradual play. Proposition 4 then implies that exactly one or two safe equilibria exist: absent fear, a war of attrition starting at the harvest time, immediately followed by a rush; and absent greed, a rush immediately followed by a gradual pre-emption game ending at the harvest time.

We deduce in Section 7 comparative statics for safe equilibria using our graphical framework. Any changes in fundamentals affect the timing, duration, and stopping rates in gradual play, and rush size and timing. Proposition 5 considers a monotone ratio shift in fundamentals postponing the harvest time, like a faster growing stock market bubble. With payoff stakes so magnified, stopping rates in any war of attrition phase monotonically attenuate before the swelling terminal rush. Less intuitively, stopping rates in any pre-emption game intensify monotonically, but the initial rush shrinks. All told, an inverse relation between stopping rates in gradual play and the rush size emerges—stopping rates intensify as the rush size shrinks, despite the heightened payoff stakes.

Proposition 6 likewise considers an increase in greed. With log-supermodular payoff shifts towards later quantiles, extreme fear eventually transitions into extreme greed. As greed rises in the war of attrition equilibrium, or oppositely, as fear rises in the pre-emption equilibrium, the gradual play phase lengthens; stopping rates fall and the rush shrinks. So perhaps surprisingly, the rush is smaller and farther from the harvest time the greater is the greed or the fear (Figure 7).

To prove how robust our results are, we revisit the equilibrium concept in Section 8. Proposition 7 asserts that any Nash equilibrium can only differ from a safe equilibrium by the size of the rush, and the length of the inaction phase that separates it from gradual play. We depict this finding graphically, by adding a third locus to our earlier apparatus. Reversing gears in Appendix A, we refine equilibrium instead. For Nash equilibrium often cannot capture dynamics; however, Proposition A.1 argues that any Nash equilibrium is automatically subgame perfect if we suitably specify off-path play. The proof that a Nash equilibrium obtains in each subgame exploits the players' indifference over stopping times. To address a different aspect of economic realism, we next dispense with the assumption of homogeneous players. We argue in Proposition A.2 that with slightly heterogeneous players, the resulting pure strategy subgame perfect equilibria closely approximate our equilibria. When players are ranked by a single crossing condition, we can exploit the revelation principle to construct equilibria in strictly monotone pure strategies.

The paper concludes in Section 9 with our motivational applications, showing that our model is easily specialized to capture the essentials of each context. In each case, we argue that our comparative statics findings agree with established ones in richer models, but also supply many new insights.

First consider the popular "tipping point" rushes. In Schelling's 1969 Nobel-cited model, one population group "A" responds myopically to thresholds on the number of neighbors from group "B". Our timing game avoids spatial thresholds. Rather, a tipping rush occurs even though all A's enjoy smooth single-peaked preferences over the mass of type B's in the neighborhood. Moreover, we predict that this rush occurs early, before fundamentals dictate, due to the fear.[2]

[2]Meanwhile, tipping models owing to the "threshold" preferences of Granovette (1978) penalize early quantiles, and so exhibit greed. Their rushes are late, as our theory predicts.

We next turn to a famous and well-documented timing game that arises in matching contexts. We create a reduced form model incorporating economic considerations found in Roth and Xing (1994), who assumed a stigma of early matching. All told, fear rises when hiring firms face a thinner market, while greed rises in stigma. Firms also value learning about the caliber of the applicants. We find that matching rushes occur inefficiently early if the early matching stigma is not too great. By assuming that stigma reflects recent matching outcomes, our model delivers the matching unraveling without appeal to any tatonnement process (Niederle and Roth (2004)).

Next, consider two common market forces behind the sales rushes ending asset bubbles: a desire for liquidity fosters fear, whereas a demand for superior relative performance engenders greed. Since Abreu and Brunnermeier (2003)—also a large timing game—ignores relative performance, a rush does not precede their pre-emption game. Their bubble bursts when rational sales surpass a threshold. Like them, we, too, deduce a larger and later bubble burst with lower interest rates. Our model also speaks to a different puzzle: By conventional wisdom, the NASDAQ bubble burst in March 2000 after fundamentals peaked. For because our game no longer exhibits fear with enough relative compensation, a sales rush after the harvest time is an equilibrium.

Our last application is bank runs. As in Diamond and Dybvig's two-period model in 1983, a run occurs in our simple continuous time setting when too many depositors inefficiently withdraw before the harvest time. Here, payoffs monotonically fall in the quantile, and the threat of a bank run is an example of alarm or panic. By Proposition 1, either a slow pre-emption game arises or a rush occurs immediately. We predict that while a reserve ratio increase shrinks the bank run and delays the withdrawals, it surprisingly raises the withdrawal rate during any pre-emption phase.

Literature Review

Applications aside, there is a large theory literature on timing games. Maynard Smith (1974) formulated the war of attrition as a model of animal conflicts. Its biggest impact in economics may be the all-pay auction literature (e.g., Krishna and Morgan (1997)). We think the economic study of pre-emption games dates to Fudenberg, Gilbert, Stiglitz, and Tirole (1983) and Fudenberg and Tirole (1985). Brunnermeier and Morgan (2010) and Anderson, Friedman, and Oprea (2010) have experimentally tested it. Park and Smith (2008) explored a finite player game with rushes and wars of attrition; however, slow pre-emption games were impossible. Ours may be the first timing game with all three flavors of timing game equilibria.

2. MODEL

There is a continuum of identical risk-neutral players i indexed on the unit interval $[0, 1]$. The measure $Q(t)$ of players *stopping* at time $\tau \leq t$ is the *quantile function*. At time zero, each player decides independently on when to stop. To ensure pure strategies, we assume that each player $i \in [0, 1]$ stops at time $T(i) = \inf\{t \in \mathbb{R}_+ | Q(t) \geq i\} \in [0, \infty)$, the generalized inverse distribution function of Q. It suffices to refer to players by their quantile.

Payoffs are a function of stopping time t, called the *fundamental*, and the stopping quantile q—capturing the anonymous strategic interaction. By Lebesgue's decomposition theorem, the c.d.f. Q is the sum of continuous portions, called *gradual play*, and *atoms*, where Q jumps. The common *stopping payoff* is $u(t, Q(t))$ given gradual play at t, and is the average quantile payoff $\int_q^p u(t,x)\,dx/(p-q)$ given a rush at t by quantiles $[q,p]$, for the quantiles $p=Q(t)>Q(t-)=q$.

A *Nash equilibrium* is a c.d.f. Q whose support contains only times that maximize stopping payoffs. Two salient deviations address the vagueness of the phrases "immediately before" or "immediately after" in the continuum: A player *gains from pre-empting* a time-$t>0$ rush of quantiles $[q,p]$ if $u(t,q)>\int_q^p u(t,x)\,dx/(p-q)$ and *gains from post-empting* a time-t rush of quantiles $[q,p]$ if $u(t,p)>\int_q^p u(t,x)\,dx/(p-q)$. A time-0 rush cannot be pre-empted.

While we explore Nash equilibria played by homogeneous players, we argue in Section A that any such equilibrium is also subgame perfect; therefore, we need not assume that stopping choices are irrevocably made at time 0. We show, a fortiori, in Section A.2 that each Nash equilibrium arises as a limit of purified strict subgame perfect equilibria with slightly heterogeneous players. In other words, our analysis of "silent" timing games fully captures the essence of dynamic games.

We assume that $u(t,q)$ is C^2, and for fixed q, is quasi-concave in t with a unique argmax, the *harvest time* $t^*(q)$. The payoff $u(t,q)$ is either strictly monotone or log-concave in q, with unique *peak quantile* $q^*(t)$. We embed strategic interactions by assuming that $u(t,q)$ is log-submodular—for example, $u(t,q)=\pi(t)v(q)$.[3] Given log-submodular payoffs, the proportionate gains $u(t,q_2)/u(t,q_1)$ to postponing until a later quantile $q_2>q_1$ are weakly larger at earlier stopping times t. Relatedly, by Theorem 2.8.1 in Topkis (1998), *the harvest time $t^*(q)$ is weakly decreasing in the quantile q, while the peak quantile $q^*(t)$ is weakly decreasing in t.* To ensure a finite harvest time, waiting forever for the peak quantile payoff is dominated by stopping in finite time:[4]

$$\lim_{s\to\infty} u\big(s,q^*(s)\big)<u(t,q)\quad \forall(t,q). \tag{1}$$

3. MONOTONE PAYOFFS IN QUANTILE

In this section, we focus on standard timing games whose payoffs are monotone in the quantile. Since players earn the same Nash payoff, indifference $u(t,Q(t))=\bar{w}$ prevails for times t in any gradual play interval. Because the stopping quantile function is increasing in time, the slope signs u_q and u_t are mismatched during any gradual play *phase*. So two such phases are possible:

(a) *Pre-emption phase*: a connected open time interval of gradual play on which $u_t>0>u_q$, so that the passage of time is fundamentally beneficial but strategically costly.

(b) *War of attrition phase*: a connected open time interval of gradual play on which $u_t<0<u_q$, so that the passage of time is fundamentally harmful but strategically beneficial.

When $u_q>0$ always, each quantile has an absolute advantage over all earlier ones: $u(t,q')>u(t,q)$ for all $q'>q$ and $t\ge 0$. In this case, rushes cannot occur, since players gain from post-empting any rush. So the only possibility is a *pure war of attrition*, namely, gradual play for all quantiles of the form (b). In fact, gradual play must begin at the harvest time $t^*(0)$ for quantile 0. For if gradual play starts later, then quantile 0 could profitably deviate to $t^*(0)$, while if it starts earlier, payoff indifference is impossible, since

[3]Almost all of our results only require the weaker complementary condition that $u(t,q)$ be *quasi-submodular*, so that $u(t_L,q_L)\ge(>)u(t_H,q_L)$ implies $u(t_L,q_H)\ge(>)u(t_H,q_H)$, for all $t_H\ge t_L$ and $q_H\ge q_L$.

[4]Indeed, $t^*(0)$ is finite since $\lim_{s\to\infty}u(s,0)\le\lim_{s\to\infty}u(s,q^*(s))<u(t^*(0),0)$, where the strict inequality is (1) evaluated at $(t,q)=(t^*(0),0)$. But then, $t^*(q)$ is finite for all q, since $t^*(q)$ is weakly decreasing.

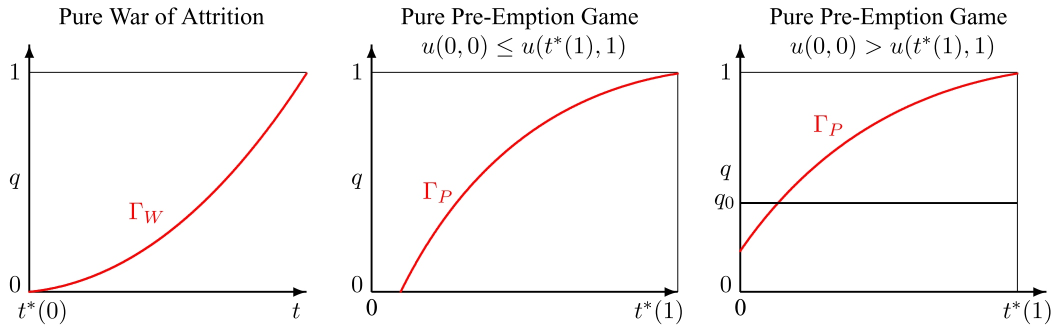

FIGURE 1.—*Monotone Timing Games*. At left, $u_q > 0$ and so gradual play follows the war of attrition locus Γ_W. When $u_q < 0$, there are types of two pre-emption games. If $u(0,0) \leq u(t^*(1),1)$, then play is wholly gradual, as stopping entirely follows Γ_P (middle). If $u(0,0) > u(t^*(1),1)$, as with alarm and panic, then entirely gradual play cannot arise as the indifference curve Γ_P crosses the q-axis at $q > 0$. Given alarm, a time-0 rush of size q_0 solves $V_0(q_0, 0) = u(1, t^*(1))$, followed by an inaction period along the horizontal line, until time t_0 where $u(q_0, t_0) = u(1, t^*(1))$, and then gradual play along Γ_P (right). Proposition 3 proves that Γ_W is convex and Γ_P is concave, as drawn.

both $u_t > 0$ and $u_q > 0$. Since the Nash payoff is $u(t^*(0), 0)$, equilibrium play follows a *war of attrition gradual play locus* Γ_W that must obey the implicit equation:

$$u\big(t, \Gamma_W(t)\big) = u\big(t^*(0), 0\big). \tag{2}$$

Similarly, when $u_q < 0$ always, each quantile has an absolute advantage over later quantiles: $u(t, q') > u(t, q)$ for all $q' < q$ and $t \geq 0$. This is a *pure pre-emption game*. In this case, any gradual play interval ends at harvest time $t^*(1)$. By parallel logic to the war of attrition case, the equilibrium value is $u(t^*(1), 1)$. Thus, during gradual play, Q must satisfy the *pre-emption gradual play locus* Γ_P:

$$u\big(t, \Gamma_P(t)\big) = u\big(t^*(1), 1\big). \tag{3}$$

In Section C.1, we prove the following characterization of the gradual play loci Γ_W and Γ_P for the case of monotone payoffs u, and the later case with non-monotone payoffs u (seen in Figure 1).

LEMMA 1—Gradual Play Loci: *If* $q^*(\cdot) > 0$, *there is a finite* $t_W > t^*(0)$ *with* $\Gamma_W : [t^*(0), t_W] \mapsto [0, q^*(t_W)]$ *well-defined, continuous, and increasing. If* $q^*(\cdot) < 1$, *there exists* $t_P \in [0, t^*(1))$ *with* Γ_P *well-defined, continuous and increasing on* $[t_P, t^*(1)]$, *where* $\Gamma_P([t_P, t^*(1)]) = [q^*(t_P), 1]$ *when* $u(0, q^*(0)) \leq u(t^*(1), 1)$, *and otherwise* $\Gamma_P([0, t^*(1)]) = [\bar{q}, 1]$, *for some* $\bar{q} \in (q^*(0), 1]$.

When $u_q < 0$, we can no longer a priori rule out rushes. A rush at time $t > 0$ is impossible, since players would gain by pre-empting it. But since a time-zero rush cannot be pre-empted, and any gradual play phase ends at the harvest time $t^*(1)$, equilibrium entails either a *unit mass rush* (i.e., of all quantiles) at $t = 0$ or gradual play at $t^*(1)$. In the first case, quantile $q = 1$ would secure payoff $u(t^*(1), 1)$ by deviating, while $u(t^*(1), 1)$ is the Nash payoff in the second case. So no rush can ever occur if stopping as quantile $q = 1$ at time $t^*(1)$ dominates stopping in any time-zero rush. In terms of the *running average payoff function*, $V_0(t, q) \equiv q^{-1} \int_0^q u(t, x)\, dx$, no rush can ever occur when:

$$u\big(t^*(1), 1\big) \geq \max_q V_0(0, q). \tag{4}$$

Inequality (4) may fail a little—there is *alarm* if $V_0(0,1) < u(t^*(1),1) < \max_q V_0(0,q)$. Or it may fail a lot: *panic* arises for lower harvest time payoffs: $u(t^*(1),1) \le V_0(0,1)$. Since $u_q < 0$, panic implies $V_0(0,q) > u(t^*(1),1)$ for all $q < 1$, ruling out all but a unit mass rush at time zero; any equilibrium with gradual play has Nash payoff $u(t^*(1),1)$. Given alarm, equilibrium includes a size $q_0 < 1$ *alarm rush* at $t = 0$ obeying $V_0(0,q_0) = u(t^*(1),1)$. For wholly gradual play is impossible, as stopping at time 0 yields payoff $u(0,0) = \max_q V_0(0,q)$, exceeding the Nash payoff $u(t^*(1),1)$. Also, a unit mass rush is not an equilibrium, since a rush must occur at time 0, but $V_0(0,1) < u(t^*(1),1)$ with alarm. Last, as claimed, q_0 obeys $V_0(0,q_0) = u(t^*(1),1)$ in any time 0 rush of size $q_0 < 1$, for players must be indifferent between the rush and later gradual play. Since $V_0(0,q_0) > u(0,q_0)$ when $u_q < 0$, post-empting the time-zero rush is strictly dominated. This forces an *inaction phase*—a time interval $[t_1,t_2]$ with no stopping: $0 < Q(t_1) = Q(t_2) < 1$.

We now offer a complete characterization of equilibria, as illustrated in Figure 1. Using Lemma 1, we construct the unique quantile function in Appendices C.2 and C.3.

PROPOSITION 1: *Assume the stopping payoff is strictly monotone in quantile. There is a unique Nash equilibrium. If $u_q > 0$, a war of attrition for all quantiles starts at $t^*(0)$. If $u_q < 0$:*

(a) *with neither alarm nor panic, there is a pre-emption game for all quantiles ending at $t^*(1)$;*

(b) *with alarm, there is a time-0 rush of size q_0 obeying $V_0(q_0,0) = u(1,t^*(1))$, followed by an inaction phase, and then a pre-emption game ending at $t^*(1)$;*

(c) *with panic, there is a unit mass rush at time $t = 0$.*

4. GREED, FEAR, AND NON-MONOTONE PAYOFFS IN QUANTILE

This section characterizes Nash equilibria when the stopping payoff has an interior peak quantile, which is our key novelty. In Section 3, an equilibrium rush was only possible at time $t = 0$. With an interior peak quantile, we can have either an *initial rush* at time $t \ge 0$ of size $q_0 = Q(t)$, or a *terminal rush* at time $t > 0$ of size $q_1 = 1 - Q(t-)$. We introduce two types of Nash equilibria:

- A *pre-emption equilibrium* includes exactly one pre-emption phase, necessarily ending at harvest time $t^*(1)$, preceded by at most one rush, necessarily an initial rush.
- A *war of attrition equilibrium* involves at most one rush, necessarily a terminal rush, preceded by exactly one war of attrition starting at harvest time $t^*(0)$.

We next formalize our core insight that purely gradual play is impossible when preferences are hump-shaped in time *and* quantile, for that requires that early quantiles stop after later quantiles.

PROPOSITION 2A—Necessity: *If the peak quantile q^* is inside* $(0,1)$, *then any Nash equilibrium includes exactly one rush, and is either*: (a) *a pre-emption equilibrium*; (b) *a war of attrition equilibrium*; *or* (c) *a unit mass rush*.

PROOF:

Step 1: *Gradual Play Boundary Conditions*. We claim that whenever stopping begins with a gradual play phase, it begins at $t^*(0)$, and whenever stopping ends with a gradual play phase, it ends at $t^*(1)$. To see this, assume stopping begins with gradual play at $t_0 > t^*(0)$. Then quantile 0 can strictly gain by deviating to $t^*(0)$. Next, if $t_0 < t^*(0)$, then $u_t(t_0,0) > 0$, and local indifference requires $u_q(t_0,0) < 0$. But this violates our assumption of an interior peak quantile $q^*(t_0) > 0$, and the quasi-concavity of u. Likewise,

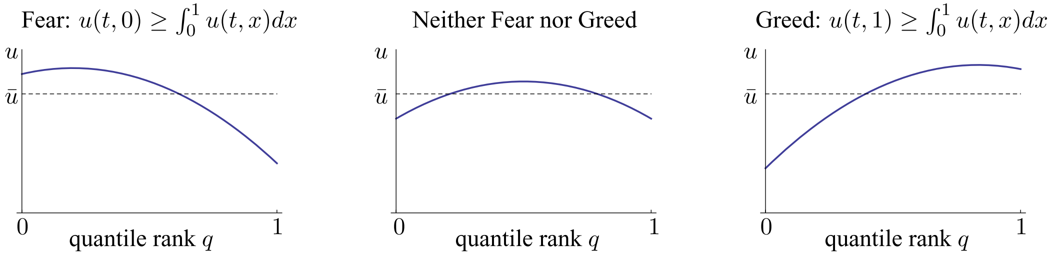

FIGURE 2.—*Fear and Greed*. Payoffs at any time t cannot exhibit both greed and fear, with first and last quantile factors better than average, but might exhibit neither (middle panel).

gradual play cannot end at time $t_1 \neq t^*(1)$. For if $t_1 < t^*(1)$, then quantile $q = 1$ profits from deviating to $t^*(1)$. But if $t_1 > t^*(1)$, then $u_t(t_1, 1) < 0$, and local indifference requires $u_q(t_1, 1) > 0$, violating $q^*(t_1) < 1$.

Step 2: *Exactly One Rush*. For a contradiction, assume no rush, that is, all quantiles stop in gradual play. By Step 0, stopping begins with gradual play at $t^*(0)$ and ends with gradual play at $t^*(1)$. That is, $Q(t^*(0)) = 0 < 1 = Q(t^*(1))$. But this violates $t^*(q)$ weakly decreasing.

Next, assume a Nash equilibrium Q with rushes at times t_1 and $t_2 > t_1$. Then $Q(t_2-) < q^*(t_2)$, or else players can strictly gain from pre-empting the rush, since u falls in q after the peak quantile. Likewise, $Q(t_1) > q^*(t_1)$, or there is a strict gain from post-empting the rush at t_1. Altogether, $q^*(t_1) < Q(t_1) \leq Q(t_2-) < q^*(t_2)$, contradicting $q^*(t)$ weakly decreasing.

Step 3: *No Interior Quantile Rush*. A rush at a quantile $q \in (0, 1)$ cannot occur at time 0. For a contradiction, assume a rush at time $t > 0$ at an interior quantile $q = Q(t)$ with $0 < Q(t-) < Q(t) < 1$. Since it is the unique rush, all other quantiles must stop in gradual play. And the last logic in Step 1 implies that gradual play begins at time $t^*(0)$ and ends at $t^*(1)$, so that $Q(t^*(0)) = 0 < Q(t^*(1)) = 1$. This violates Q weakly increasing and t^* weakly decreasing.

Step 4: *At Most One Gradual Play Phase*. Assume instead gradual play on two intervals $[t_1, t_2]$ and $[t_3, t_4]$, with $t_2 < t_3$. Then $Q(t_2) = Q(t_3)$ by Step 2. By Steps 1–3, either stopping begins with gradual play at $t^*(0)$ and ends in a rush, or gradual play ending at $t^*(1)$ follows a rush. In the first case, $t_1 \geq t^*(0)$, and since t^* is non-increasing, $u_t(t, Q(t)) < 0$ for all $t > t_1 \geq t^*(0) \geq t^*(Q(t))$. But then $u(t_2, Q(t_2)) > u(t_3, Q(t_3))$, contradicting indifference between t_2 and t_3. In the second case, $t_4 \leq t^*(1)$, and since t^* is non-increasing, $u_t(t, Q(t)) > 0$ for all $t < t_4$. So $u(t_2, Q(t_2)) < u(t_3, Q(t_3))$, contradicting indifference between t_2 and t_3. *Q.E.D.*

To best understand equilibria, we now generalize the first and last mover advantage notions (see Figure 2).

DEFINITION 1: There is fear at time t if $u(t, 0) \geq \int_0^1 u(t, x)\, dx$ and greed at t if $u(t, 1) \geq \int_0^1 u(t, x)\, dx$.

Fear relaxes the first mover advantage ($u_q < 0$), asking that the least quantile beat the average; greed relaxes the last mover advantage ($u_q > 0$), asking just that the last quantile payoff exceed the average. Naturally, *strict fear* and *strict greed* entail strict inequalities.

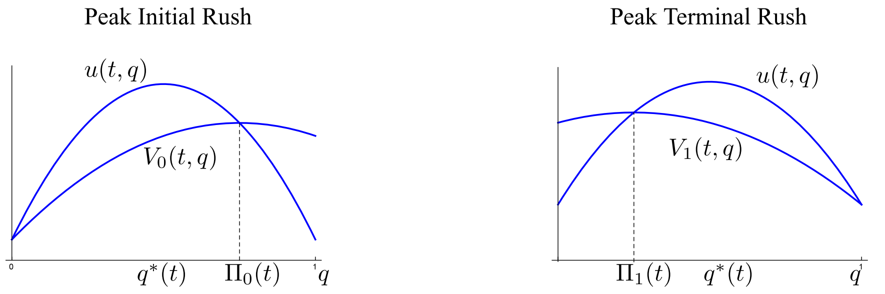

FIGURE 3.—*Rushes Include the Quantile Peak*. The time-t peak rush maximizes the average rush payoff $V_i(t,q)$, and so equates the average and adjacent marginal payoffs $V_i(t,q)$ and $u(t,q)$.

Since u is single-peaked in the quantile q, *greed and fear are mutually exclusive at the same time* t.

We define the early and late *peak rush loci* $\Pi_i(t) \equiv \arg\max_q V_i(t,q)$ for $i = 0, 1$, as seen in Figure 3.[5] Whenever the peak quantile is inside $(0,1)$, the running average integral V_i coincides with its marginal u at the peak of the average. So

$$u\big(t, \Pi_i(t)\big) = V_i\big(t, \Pi_i(t)\big). \tag{5}$$

In addition to the gradual play loci in Lemma 1, we now describe the associated peak rush loci.

LEMMA 2: *The loci $\Pi_i(t)$ are unique, continuous, and non-increasing.*[6] *Absent greed at time $t^*(1)$, $\Pi_0(t) \in (q^*(t), 1)$ for $t \leq t^*(1)$. Absent fear at time $t^*(0)$, $\Pi_1(t) \in (0, q^*(t))$ for $t \geq t^*(0)$.*

The proof is in Section C.1. Proposition 2A rules out all but a small set of quantile functions as possible equilibria. We now provide sufficient conditions for when the remaining quantile functions are indeed equilibria. So together, Propositions 2A and 2B completely characterize the equilibrium set.

PROPOSITION 2B: *A Nash equilibrium exists if the stopping payoff has an interior peak quantile*:

(a) *A pre-emption equilibrium exists if and only if there is no greed at time $t^*(1)$ and no panic.*

(b) *A war of attrition equilibrium exists if and only if there is no fear at time $t^*(0)$.*

(c) *A unit mass rush at time $t = 0$ is an equilibrium if and only if there is panic.*

(d) *A unit mass rush at time $t > 0$ is an equilibrium if and only if there is no fear and no greed at time t and provided*: (i) $V_0(t,1) \geq u(t^*(0), 0)$ *if* $t \geq t^*(0)$; *or* (ii) $V_0(t,1) \geq u(t^*(1), 1)$ *if* $t \leq t^*(1)$.

[5]The argmax in q of V_i is unique for $i = 0, 1$ because each V_i is a running average integral of a single-peaked and log-concave, and therefore strictly quasi-concave, function $u(t,\cdot)$ of q.

[6]Each locus $\Pi_i(t)$ is decreasing when $u(t,q)$ is strictly log-submodular. But in the log-modular (or multiplicative) case that we assume in the example in Section 9.2, the locus $\Pi_i(t)$ is constant in time t.

PROOF:
Step 1: *Parts* (a) *and* (b). Appendix C.4 proves part (a); we prove part (b) here. We show that fear at time $t^*(0)$ precludes a war of attrition equilibrium. By Proposition 2A, stopping in such an equilibrium would begin at $t^*(0)$ and have payoff $\bar{w} = u(t^*(0), 0)$. Since $t^*(q)$ is non-increasing, $u_t(t, q) < 0$ if $t > t^*(0)$. Thus, the Nash payoff falls short of the average payoff $\bar{w} < \int_0^1 u(t^*(0), x)\,dx$. So $u(t^*(0), 0) = \bar{w} < \int_0^1 u(t^*(0), x)\,dx$, contradicting fear at $t^*(0)$.

Assume no fear at time $t^*(0)$. We construct a war of attrition equilibrium. Since the peak quantile $q^* \in (0, 1)$, for some time $t_W > t^*(0)$, the war of attrition gradual play locus $\Gamma_W(t)$ is a continuously increasing map of $[t^*(0), t_W]$ onto $[0, q^*(t_W)]$, by Lemma 1. By Lemma 2, the late peak rush locus $\Pi_1(t)$ is continuous and non-increasing from $\Pi_1(t^*(0)) > 0 = \Gamma_W(t^*(0))$ to $\Pi_1(t_W) < q^*(t_W) = \Gamma_W(t_W)$. So $\Pi_1(t_1) = \Gamma_W(t_1) \in (0, q^*(t_1))$ for a unique time $t_1 \in (t^*(0), t_W)$.

Finishing the proof of part (b), we now prove that the following quantile function is a Nash equilibrium: $Q(t) = 0$ for $t < t^*(0)$, $Q(t) = \Gamma_W(t)$ on $[t^*(0), t_1)$, and $Q(t) = 1$ for $t \geq t_1$. By construction, payoffs are constant along the gradual play locus $\Gamma_W(t)$, while (5) holds at the terminal rush time t_1 since $\Pi_1(t_1) \in (0, 1)$, that is, the terminal rush payoff equals the gradual play payoff. This is an equilibrium, since no one can gain from stopping before $t^*(0)$. For $Q(t) = 0$ at such times, and $t^*(0)$ is the harvest time for quantile 0. And no one can gain from stopping after the rush at t_1, since $t_1 > t^*(0) \geq t^*(1)$, yielding a falling payoff $u_t(t, 1) < 0$ for all $t \geq t_1$.

Step 2: *Parts* (c) *and* (d). The payoff in a time-t unit mass rush is $V_0(t, 1)$. The most profitable deviation from a unit mass rush at time $t = 0$ is to the harvest time $t^*(1)$ with payoff $u(t^*(1), 1)$. So a unit mass rush at time $t = 0$ is an equilibrium iff $V_0(0, 1) \geq u(t^*(1), 1)$, that is, panic. Consider a unit mass rush at time $t > 0$, as in part (d). With greed at time t, players gain from post-empting a unit mass rush at time t, while pre-empting the rush is a profitable deviation given fear at time t. Conditions (i) and (ii) ensure that the best deviation is unprofitable.

Step 3: *A Nash Equilibrium Exists*. Assume no premise for (a)–(d) holds. Define

$$f(t) = \int_0^1 \big[u(t, x)/u(t, 0)\big]\,dx \quad \text{and} \quad g(t) = \int_0^1 \big[u(t, x)/u(t, 1)\big]\,dx. \tag{6}$$

By (c), there is no panic, and so greed obtains at time $t^*(1)$ by (a) and (b), and fear at $t^*(0)$, that is, $g(t^*(1)) \leq 1$ and $f(t^*(0)) \leq 1$. As greed and fear are mutually exclusive, there is no fear at $t^*(1)$ and no greed at $t^*(0)$, that is, $f(t^*(1)) > 1$ and $g(t^*(0)) > 1$. We conclude $t^*(1) \neq t^*(0)$. Then $t^*(1) < t^*(0)$, as $t^*(q)$ is non-increasing. So $f(t^*(1)) > 1 \geq f(t^*(0))$ and $g(t^*(1)) \leq 1 < g(t^*(0))$. By continuity of f, g, there exists $\bar{t} \in [t^*(1), t^*(0)]$ with $f(\bar{t}) = g(\bar{t})$. As greed and fear are mutually exclusive, $f(\bar{t}), g(\bar{t}) > 1$. So there is neither greed nor fear at $\bar{t}$, and conditions (i) and (ii) of (d) vacuously hold as $t^*(1) < \bar{t} < t^*(0)$, a unit mass rush at $\bar{t}$ is an equilibrium. *Q.E.D.*

We now deduce a lower bound on the size of equilibrium rushes using Propositions 2A and 2B. Since players only stop in a rush if gradual play is not more profitable, we have $V_i(t, q) \geq u(t, q)$, for $i = 0, 1$. Given how marginals u and the averages V_0, V_1 interact (see Figure 3), we have the following:

COROLLARY 1: *The early rush has size at least* Π_0, *and the late rush has size at least* $1 - \Pi_1$.

5. STOPPING RATES IN GRADUAL PLAY

In this section, we characterize stopping rates during any equilibrium gradual play phase. Let Q be a Nash equilibrium with payoff $\bar{w}$. By Propositions 1 and 2A, Q includes either a war of attrition phase or a pre-emption phase, but not both. Hence, $u_q(t, q)$ has a constant nonzero sign on the gradual play time interval, and so $u(t, q)$ can be inverted, yielding $Q(t) = u^{-1}(\bar{w}|t)$ in this interval.[7] As the inverse of a C^2 function, *the quantile function Q must be C^2 on the gradual play interval.* Easily, differentiating the indifference condition $\bar{w} = u(t, Q(t))$ yields the fundamental differential equation:

$$u_q\big(t, Q(t)\big)Q'(t) + u_t\big(t, Q(t)\big) = 0. \tag{7}$$

PROPOSITION 3—Stopping During Gradual Play: *Assume the payoff function is log-concave in t. In any Nash equilibrium, the stopping rate $Q'(t)$ is strictly increasing in time from zero during a war of attrition phase, and decreasing down to zero during a pre-emption game phase.*

PROOF: Wars of attrition begin at $t^*(0)$ and pre-emption games end at $t^*(1)$, by Propositions 1 and 2A. Since $u_t(t^*(q), q) = 0$ at the harvest time, the first term of (7) vanishes at the start of a war of attrition and end of a pre-emption game. Consequently, $Q'(t^*(0)) = 0$ and $Q'(t^*(1)) = 0$ in these two cases, since $u_q \neq 0$ at the two quantile extremes $q = 0, 1$.

Since Q'' exists, and $u(t, q)$ is log-concave in t and log-submodular, as $u_t \gtrless 0 \gtrless u_q$, we have

$$\big[\log Q'(t)\big]' = \big[\log(-u_t/u_q)\big]' = \big[\log(\pm u_t/u)\big]_t - \big[\log(\mp u_q/u)\big]_t \geq 0 - 0. \quad \textit{Q.E.D.}$$

We see that stopping waxes in war of attrition equilibria, climaxing in a rush when payoffs are not monotone in quantile, whereas pre-emption equilibria begin with a rush in this hump-shaped case, and continue into a waning gradual play phase. So wars of attrition intensify towards a rush, whereas pre-emption games taper off from a rush. Figure 4 reflects these facts, since the stopping indifference curve is (i) concave after the initial rush during any pre-emption equilibrium, and (ii) convex prior to the terminal rush during any war of attrition equilibrium.

6. SAFE EQUILIBRIA

Proposition 1 asserts a unique equilibrium when the stopping payoff is monotone in quantiles. With a non-monotone stopping payoff, equilibrium need not be unique. This section refines the set of Nash equilibria with a trembling argument, finding that at most one pre-emption equilibrium and at most one war of attrition equilibrium survives this refinement. The surviving equilibrium is natural, and lends itself most naturally to comparative statics. But in Section 8, we argue that these comparative statics findings apply to the whole Nash equilibrium set. So this refinement serves a dual pedagogical purpose, and our theory is ultimately robust to all Nash equilibria.

Now, our base model assumes perfectly timed actions. But one might venture that even the best timing technology is imperfect. If so, agents may be wary of equilibria in which tiny timing mistakes incur significant payoff losses—as happens when there are rushes.

[7] A proof by the analytic implicit function theorem is also possible, but requires attention to different details.

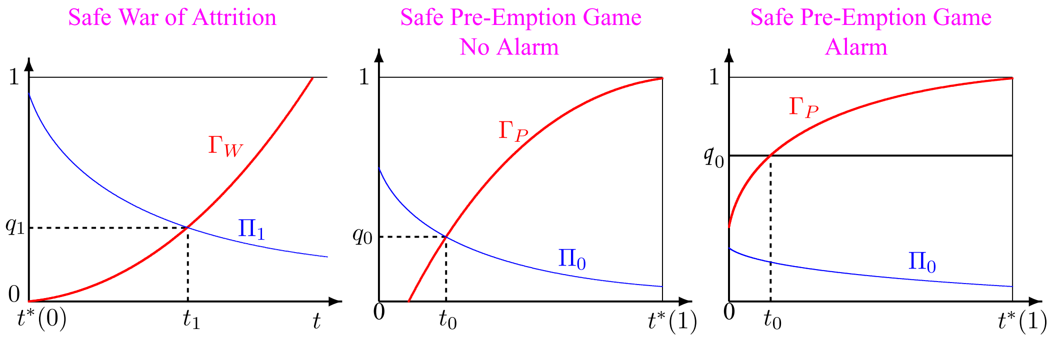

FIGURE 4.—*Safe Equilibria With Hump-shaped Payoffs*. In the safe war of attrition equilibrium (left), gradual play begins at $t^*(0)$, following the upward sloping gradual play locus (2), and ends in a terminal rush of quantiles $[q_1, 1]$ at time t_1 where the loci cross. In the pre-emption equilibrium without alarm (middle), an initial rush q_0 at time t_0 occurs where the upward sloping gradual play locus (3) intersects the downward sloping peak rush locus (5). Gradual play in the pre-emption phase then follows the gradual play locus Γ_P. With alarm (right), the alarm rush q_0 at $t = 0$ is followed by an inaction phase $(0, t_0)$, and then a pre-emption game follows Γ_P. With monotone payoffs, the peak rush loci are fixed at $\Pi_1 \equiv 1$ at left, and $\Pi_0 \equiv 0$ in the middle/right.

Let $w(t; Q) \equiv u(t, Q(t))$ be the payoff to stopping at time $t \geq 0$ given c.d.f. Q. The ε-*safe payoff* at t is therefore

$$w_\varepsilon(t; Q) = \max\Big\langle \inf_{\max(t-\varepsilon,0)\leq s<t} w(s; Q), \inf_{s\in[t,t+\varepsilon)} w(s; Q)\Big\rangle.$$

This can be understood as the minmax payoff in the richer model when individuals have access to two different ε-accurate timing technologies: One clock never runs late, and one never runs early. A Nash equilibrium Q is *safe* if there exists $\bar{\varepsilon} > 0$ so that $w_\varepsilon(t; Q) = w(t; Q)$ for all t in the support of Q, for all $\varepsilon \in (0, \bar{\varepsilon})$. We prove the following characterization result in Appendix C.5.

LEMMA 3: *A Nash equilibrium Q is safe if and only if its support is either a single nonempty interval of time or the union of time $t = 0$ and such a later time interval.*

In light of Propositions 1, 2A, and 2B, we see that safety rules out unit mass rushes at strictly positive times. More subtly, it precludes wars of attrition with a period of inaction preceding the terminal rush, or pre-emption equilibria with an initial rush at $t > 0$ followed by a period of inaction. In the case of fear at the harvest time $t^*(0)$, the war of attrition equilibrium in Step 2 of the proof of Proposition 2B is safe, since it involves no period of inaction. Because we proved that the peak rush locus Π_1 intersects the gradual play locus Γ_W at a unique time t_1, it is the unique safe war of attrition equilibrium. Likewise, the proof for the pre-emption case in Section C.4 separately constructs the unique safe pre-emption equilibrium given no greed at $t^*(1)$ for the cases of alarm, and neither alarm nor panic. In addition, unit mass rushes at time $t = 0$ are safe. Summarizing, we have the following:

PROPOSITION 4: *Absent fear at the harvest time $t^*(0)$, a unique safe war of attrition equilibrium exists. Absent greed at time $t^*(1)$, a unique safe equilibrium with an initial rush exists*:

(a) *with neither alarm nor panic, a pre-emption equilibrium with a rush at time $t > 0$*;

(b) *with alarm, a rush at $t = 0$ followed by an inaction phase and then a pre-emption phase*;

(c) *with panic, a unit mass rush at time* $t = 0$.
With fear at $t^*(0)$ *and greed at* $t^*(1)$, *Nash equilibria are unit mass rushes in* $[t^*(1), t^*(0)]$.

Now, $t^*(1) \leq t^*(0)$ since $t^*(q)$ is weakly decreasing. Since we cannot have greed and fear at the same time t, safe equilibria always exist when $t^*(1) = t^*(0)$ by Proposition 4—as with multiplicative payoffs $u(t, q) = \pi(t)v(q)$. But assume strictly log-submodular preferences. Then $t^*(1) < t^*(0)$. Recalling (6), fear at time t is $f(t) \leq 1$ and greed at time t is $g(t) \leq 1$. Both inequalities cannot hold at any t. But since g is increasing, and f decreasing, we can have $g(t^*(1)) \leq 1 < f(t^*(1))$, that is, greed at $t^*(1)$ and not fear at $t^*(1)$, *and also* $f(t^*(0)) \leq 1 < g(t^*(0))$, that is, fear at $t^*(0)$ and not greed at $t^*(0)$. This scenario is not ruled out by our assumptions,[8] whereupon no safe equilibrium exists by Proposition 4, since fear at $t^*(0)$ rules out a safe war of attrition, and greed at $t^*(1)$ rules out a safe equilibrium with an initial rush. In this case, absent safe equilibria, by Proposition 2A and Step 3 in the proof of Proposition 2B, the only Nash equilibria entail a unit mass rush at any time $t \in [t^*(1), t^*(0)]$. *Safe equilibria always exist when there is not greed for all times, or not fear for all times*, as holds in our examples in Section 9.

With this result, we see that panic and alarm have the same implications as in the monotone decreasing case, $u_q < 0$, analyzed by Proposition 1. In contrast, when neither panic nor alarm obtains, safe equilibria must include both a rush and a gradual play phase and no inaction: either an initial rush at $0 < t_0 < t^*(1)$, followed by a pre-emption phase on $[t_0, t^*(1)]$, or a war of attrition phase on $[t^*(0), t_1]$ ending in a terminal rush at t_1. In each case, the safe equilibrium is fully determined by the gradual play locus and peak rush locus (Figure 4).

7. PREDICTIONS ABOUT CHANGES IN GRADUAL PLAY AND RUSHES

This section explores how the equilibria evolve as: (a) fundamentals adjust that postpone the harvest time, or (b) the strategic interaction alters to change quantile rewards, increasing fear or greed. Index stopping payoffs as $u(t, q|\varphi)$ in C^2, where $\varphi \in \mathbb{R}$. To isolate the effect of time on payoffs, let $u(t, q|\varphi)$ be strictly log-supermodular in (t, φ) and log-modular in (q, φ). Then greater φ raises the marginal of payoffs in time, but leaves unaffected the marginal in quantile. An increase in φ is a *harvest time delay*, since $t^*(q|\varphi)$ rises in φ, by log-supermodularity in (t, φ).

We next argue that a harvest time delay postpones stopping, but it *intensifies* stopping rates once gradual play starts in a pre-emption game. We also find an inverse relation between stopping rates and rush size, with higher stopping rates during gradual play associated to smaller rushes.

PROPOSITION 5—Harvest Time Delay: *Let* Q_H *and* Q_L *be safe equilibria for* $\varphi_H > \varphi_L$.

(a) *If* Q_H, Q_L *are wars of attrition, then* $Q_H(t) \leq Q_L(t)$; *the rush for* Q_H *is later and no smaller*; *gradual play for* Q_H *starts later*; *and* $Q'_H(t) < Q'_L(t)$ *in the common gradual play interval*.

(b) *If* Q_H, Q_L *are pre-emption equilibria*, $Q_H(t) \leq Q_L(t)$; *the rush for* Q_H *is later and no larger*; *gradual play for* Q_H *ends later*; *and* $Q'_H(t) > Q'_L(t)$ *in the common gradual play interval*.

[8]A payoff function with greed at $t^*(1) = 1/4$ and fear at $t^*(0) = 1$ is $u(t, q) = (1 + q)(1 + 4t)e^{-(1+2q)t}$.

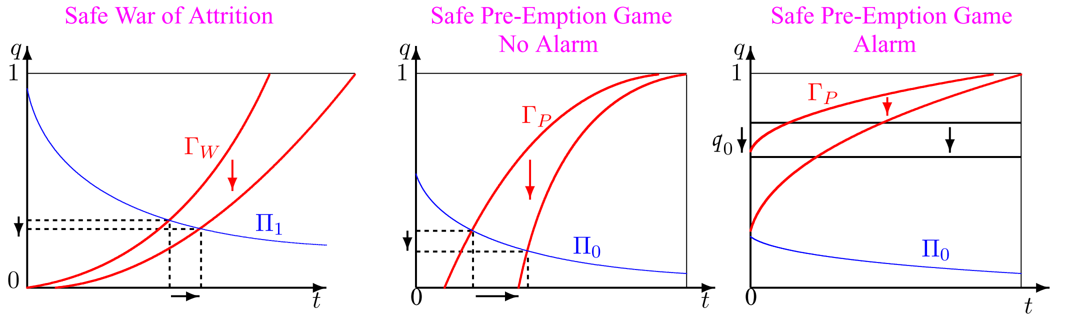

FIGURE 5.—*Harvest Time Delay*. The gradual play locus shifts down in φ. In the safe war of attrition (left): A larger terminal rush occurs later, while stopping rates fall during gradual play. In the safe pre-emption game without alarm (middle): A smaller initial rush occurs later and stopping rates rise during gradual play. With alarm (right), the rush occurs at $t=0$, but shrinks, gradual play ends later, and stopping rates rise. With monotone payoffs, the peak rush loci are fixed at $\Pi_1 \equiv 1$ at left, and $\Pi_0 \equiv 0$ in the middle/right; the graphs are otherwise unchanged.

PROOF: We focus on the safe pre-emption equilibrium with an interior peak quantile, proving that the pre-emption gradual play locus $\Gamma_P(t)$ shifts down and steepens with a harvest time delay, but the peak rush locus $\Pi_0(t)$ is unchanged, as in Figure 5. The logic for the war of attrition case is symmetric. The initial rush q_0 falls for the safe pre-emption equilibrium with alarm (Lemma C.2).

Since the marginal payoff u is log-modular in (t, φ), so, too, is the average. The peak rush locus $\Pi_0(t) \in \arg\max_q V_0(t, q|\varphi)$ is then constant in φ. Now, rewrite the pre-emption gradual play locus (3) as

$$\frac{u\big(t, \Gamma_P(t)|\varphi\big)}{u(t, 1|\varphi)} = \frac{u\big(t^*(1|\varphi), 1|\varphi\big)}{u(t, 1|\varphi)}. \tag{8}$$

The LHS of (8) falls in Γ_P, since $u_q < 0$ during a pre-emption game, and is constant in φ, by log-modularity of u in (q, φ). Log-differentiating the RHS in φ, and using the Envelope theorem:

$$\frac{u_\varphi\big(t^*(1|\varphi), 1|\varphi\big)}{u\big(t^*(1|\varphi), 1|\varphi\big)} - \frac{u_\varphi(t, 1|\varphi)}{u(t, 1|\varphi)} > 0,$$

since u is log-supermodular in (t, φ) and $t < t^*(1|\varphi)$ during a pre-emption game. Since the RHS of (8) increases in φ and the LHS decreases in Γ_P, the gradual play locus $\Gamma_P(t)$ obeys $\partial\Gamma_P/\partial\varphi < 0$. Next, differentiate the gradual play locus in (3) in t and φ, to get

$$\frac{\partial \Gamma_P'(t)}{\partial\varphi} = -\left[\left(\frac{\partial[u_t/u]}{\partial\varphi} + \frac{\partial[u_t/u]}{\partial\Gamma_P}\frac{\partial\Gamma_P}{\partial\varphi}\right)\frac{u}{u_q} + \frac{u_t}{u}\left(\frac{\partial[u/u_q]}{\partial\Gamma_P}\frac{\partial\Gamma_P}{\partial\varphi} + \frac{\partial[u/u_q]}{\partial\varphi}\right)\right] > 0. \tag{9}$$

The first parenthesized term is negative. Indeed, $\partial[u_t/u]/\partial\varphi > 0$ since u is log-supermodular in (t, φ), and $\partial[u_t/u]/\partial\Gamma_P < 0$ since u is log-submodular in (t, q), and $\partial\Gamma_P/\partial\varphi < 0$ (as shown above), and finally $u_q < 0$ during a pre-emption game. The second term is also negative because $u_t > 0$ during a pre-emption game, and $\partial[u/u_q]/\partial\Gamma_P \geq 0$ by log-concavity of $u(t, q)$ in q. *Q.E.D.*

Next consider pure changes in quantile preferences, by assuming the stopping payoff $u(t, q|\varphi)$ is log-supermodular in (q, φ) and log-modular in (t, φ). So greater φ inflates

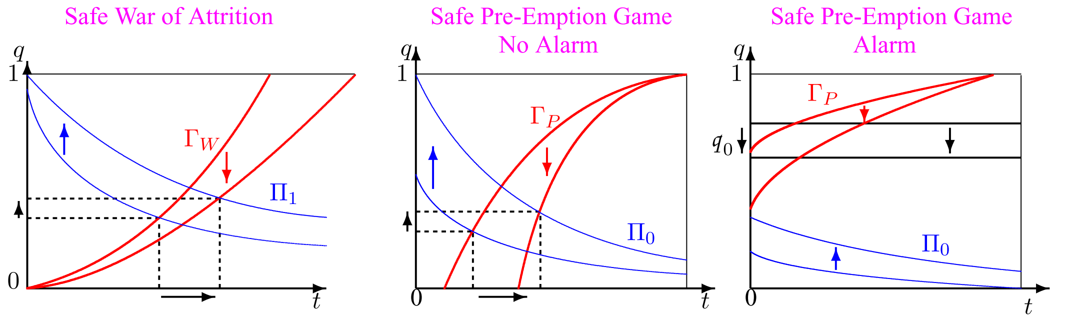

FIGURE 6.—*Monotone Quantile Payoff Changes.* An increase in greed (or a decrease in fear) shifts the gradual play locus down and the peak rush locus up. In the safe war of attrition (left): Smaller rushes occur later and stopping rates *fall* during longer wars of attrition. In the safe pre-emption game without alarm (middle): Larger rushes occur later and stopping rates *rise* on shorter pre-emption games. With alarm (right), the initial rush at $t = 0$ is smaller and stopping rates also rise. With monotone payoffs, the peak rush loci are fixed at $\Pi_1 \equiv 1$ at left, and $\Pi_0 \equiv 0$ in the middle/right, but the graphs are otherwise unchanged.

the relative return to a quantile delay, but leaves unchanged the relative return to a time delay. Hence, the peak quantile $q^*(t|\varphi)$ rises in φ. We say that *greed increases* when φ rises, since payoffs shift towards later ranks as φ rises; this relatively diminishes the potential losses of pre-emption, and relatively inflates the potential gains from later ranks. Also, if there is greed at time t, then this remains true if greed increases. Likewise, we say *fear increases* when φ falls.

PROPOSITION 6—Quantile Changes: *Let Q_H and Q_L be safe equilibria for $\varphi_H > \varphi_L$.*

(a) *If Q_H, Q_L are war of attrition equilibria, then $Q_H \le Q_L$; the rush for Q_H is later and smaller; and $Q'_H(t) < Q'_L(t)$ in the common gradual play interval.*

(b) *If Q_H, Q_L are pre-emption equilibria without alarm, then $Q_H \le Q_L$; the rush for Q_H is later and larger; and $Q'_H(t) > Q'_L(t)$ in the common gradual play interval.*

(c) *If Q_H, Q_L are pre-emption equilibria with alarm, then $Q_H \le Q_L$; the rush for Q_H is smaller; and $Q'_H(t) > Q'_L(t)$ in the common gradual play interval.*

Observe the pivotal role of alarm in the comparative statics of pre-emption equilibria. With alarm, the rush happens at time zero, and to maintain indifference, $V_0(0, q_0) = u(t^*(1), 1)$, the initial rush size q_0 shrinks. In the no alarm case, the effects on gradual play time and rush size are opposite (see Figure 6). For example, with multiplicative payoffs $u(t, q|v) = \pi(t)v(q|\varphi)$, the peak rush locus Π_0 solely determines the initial rush size, and Π_0 shifts up in φ, by log-supermodularity in (q, φ). Moreover, the rush occurs later because the relative payoff to early quantiles falls, forcing early quantiles to stop later to maintain indifference during gradual play.

PARTIAL PROOF: As with Proposition 5, our proof covers the pre-emption case with no alarm. Parallel logic establishes the results for the safe war of attrition equilibrium. Lemma C.2 completes the proof for the safe pre-emption equilibrium with alarm.

Define $\mathbb{I}(q, x) \equiv q^{-1}$ for $x \le q$ and 0 otherwise, and thus $V_0(t, q|\varphi) = \int_0^1 \mathbb{I}(q, x)u(t, x|\varphi)\,dx$. Easily, $\mathbb{I}$ is log-supermodular in (q, x), and so the product $\mathbb{I}(\cdot)u(\cdot)$ is log-supermodular in (q, x, φ). Thus, V_0 is log-supermodular in (q, φ) since log-supermodularity is preserved by integration by Karlin and Rinott (1980). So the peak rush locus $\Pi_0(t) = \arg\max_q V_0(t, q|\varphi)$ rises in φ.

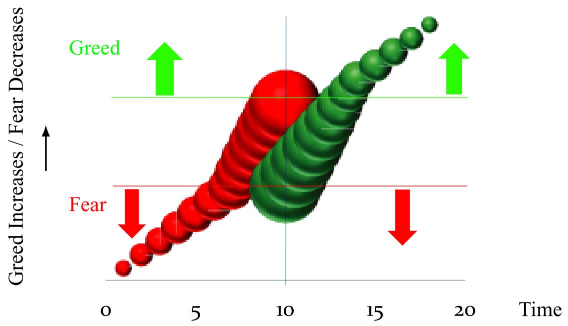

FIGURE 7.—*Rush Size and Timing With Increased Greed*. Circles at rush times are proportional to rush sizes. As fear falls, the unique safe pre-emption equilibrium has a larger initial rush, closer to the harvest time $t^* = 10$, and a shorter pre-emption phase. As greed rises, the unique safe war of attrition equilibrium has a longer war of attrition, and a smaller terminal rush (Proposition 6).

Now consider the gradual play locus (8). Its RHS is constant in φ since $u(t, q|\varphi)$ is log-modular in (t, φ), and $t^*(q)$ is constant in φ. The LHS falls in φ since u is log-supermodular in (q, φ), and falls in Γ_P since $u_q < 0$ during a pre-emption game. All told, the gradual play locus obeys $\partial\Gamma_P/\partial\varphi < 0$. To see how the slope Γ_P' changes, consider (9). The first term in brackets is negative. For $\partial[u_t/u]/\partial\varphi = 0$ since u is log-modular in (t, φ), and $\partial[u_t/u]/\partial\Gamma_P < 0$ since u is log-submodular in (t, q), and $\partial\Gamma_P/\partial\varphi < 0$ (as shown above), and finally $u_q < 0$ in a pre-emption game. The second term is also negative: $\partial[u/u_q]/\partial\varphi < 0$ as u is log-supermodular in (q, φ), and $\partial[u/u_q]/\partial\Gamma_P > 0$ as u is log-concave in q, and $\partial\Gamma_P/\partial\varphi < 0$, and $u_t > 0$ in a pre-emption game.

All told, an increase in φ: (i) has no effect on the harvest time; (ii) shifts the gradual play locus (3) down and makes it steeper; and (iii) shifts the peak rush locus (5) up (see Figure 6). Lemma C.3 proves that the peak rush locus shift determines whether rushes grow or shrink. *Q.E.D.*

Figure 7 summarizes an overarching take out message of Propositions 4 and 6. As we shift from fear to greed, the rushes delay: They grow and shift closer to the harvest time during the pre-emption phase (with no greed), and shrink during the war of attrition phase (with no fear), moving away from the harvest time. There is an overlap with neither greed nor fear in which both safe equilibria exist, rushes are maximal, and these move oppositely in size.

Finally, consider a general monotone shift, in which the payoff $u(t, q|\varphi)$ is log-supermodular in both (t, φ) and (q, φ). We call an increase in φ a *co-monotone delay* in this case, since the harvest time $t^*(q|\varphi)$ and the peak quantile $q^*(t|\varphi)$ both increase in φ. Intuitively, greater φ intensifies the game, by proportionally increasing the payoffs in time and quantile space. By the logic used to prove Propositions 5 and 6, such a co-monotone delay shifts the gradual play locus (3) down and makes it steeper, and shifts the peak rush locus (5) up (see Figure 6).

COROLLARY 2—Covariate Implications: *Assume safe equilibria with a co-monotone delay. Then stopping shifts stochastically later, and stopping rates fall in a war of attrition and rise in a pre-emption game. Given alarm, the time-zero initial rush shrinks.*

The effect on the rush size depends on whether the interaction between (t, φ) or (q, φ) dominates.

8. THE SET OF NASH EQUILIBRIA WITH NON-MONOTONE PAYOFFS

In any Nash equilibrium with gradual play and a rush, players must be indifferent between stopping in the rush and during gradual play. Thus, we introduce the associated *initial rush locus* $\mathcal{R}_P$ and the *terminal rush locus* $\mathcal{R}_W$, which are the largest, respectively smallest, solutions to

$$V_0\big(t, \mathcal{R}_P(t)\big) = u\big(t^*(1), 1\big) \quad \text{and} \quad V_1\big(t, \mathcal{R}_W(t)\big) = u\big(t^*(0), 0\big). \tag{10}$$

No player can gain by immediately pre-empting the initial peak rush $\Pi_0(t)$ in the safe pre-emption equilibrium, nor from stopping immediately after the peak rush $\Pi_1(t)$ in the safe war of attrition. For with an interior peak quantile, the maximum average payoff exceeds the extreme stopping payoffs (Figure 3). But for larger rushes, this constraint may bind. An initial time-t rush is *undominated* if $V_0(t, q) \geq u(t, 0)$, and a terminal time-t rush is *undominated* if $V_1(t, q) \geq u(t, 1)$.

If players can strictly gain from pre-empting any initial rush at $t > 0$, there is at most one pre-emption equilibrium. Since $u(t^*(1), 1)$ equals the initial rush payoff by (10) and $u_t(0, t) > 0$ for $t < t^*(1)$, the following inequality is necessary for multiple pre-emption equilibria:

$$u(0, 0) < u\big(t^*(1), 1\big). \tag{11}$$

We now define the time domain on which each rush locus is defined. Recall that Proposition 4 asserts a unique safe war of attrition equilibrium exactly when there is no fear at time $t^*(0)$. Let its rush include the terminal quantiles $[\bar{q}_W, 1]$ and occur at time $\bar{t}_1$. Likewise, let $\underline{q}_P$ and $\underline{t}_0$ be the initial rush size and time in the unique safe pre-emption equilibrium, when it exists.

LEMMA 4—Rush Loci: *Given no fear at $t^*(0)$, there exist $\underline{t}_1 \leq \underline{t}$, both in $(t^*(0), \bar{t}_1)$, such that $\mathcal{R}_W$ is a continuously increasing map from $[\underline{t}_1, \bar{t}_1]$ onto $[0, \bar{q}_W]$, with $\mathcal{R}_W(t) < \Gamma_W(t)$ on $[\underline{t}_1, \bar{t}_1)$, and $\mathcal{R}_W$ undominated exactly on $[\underline{t}, \bar{t}_1] \subseteq [\underline{t}_1, \bar{t}_1]$. With no greed at $t^*(1)$, no panic, and* (11), *there exist $\bar{t} \leq \bar{t}_0$ both in $(\underline{t}_0, t^*(1))$, such that $\mathcal{R}_P$ is a continuously increasing map from $[\underline{t}_0, \bar{t}_0]$ onto $[\underline{q}_P, 1]$, with $\mathcal{R}_P(t) > \Gamma_P(t)$ on $(\underline{t}_0, \bar{t}_0]$, and $\mathcal{R}_P(t)$ undominated exactly on $[\underline{t}_0, \bar{t}] \subseteq [\underline{t}_0, \bar{t}_0]$.*

Figure 8 graphically depicts the message of this result, with rush loci starting at $\underline{t}_0$ and $\bar{t}_1$.

We now construct two sets of *candidate quantile functions*: $\mathcal{Q}_W$ and $\mathcal{Q}_P$. The set $\mathcal{Q}_W$ is empty given fear at $t^*(0)$. Without fear at $t^*(0)$, $\mathcal{Q}_W$ contains all quantile functions Q such that (i) $Q(t) = 0$ for $t < t^*(0)$, and for any $t_1 \in [\underline{t}, \bar{t}_1]$: (ii) $Q(t) = \Gamma_W(t)$ $\forall t \in [t^*(0), t_W]$ where t_W uniquely solves $\Gamma_W(t_W) = \mathcal{R}_W(t_1)$; (iii) $Q(t) = \mathcal{R}_W(t_1)$ on (t_W, t_1); and (iv) $Q(t) = 1$ for all $t \geq t_1$. The set $\mathcal{Q}_P$ is empty with greed at $t^*(1)$ or panic. Given greed at $t^*(1)$, no panic, and not (11), $\mathcal{Q}_P$ contains a single quantile function: the safe pre-emption equilibrium by Proposition 4. And with no greed at $t^*(1)$, no panic, and inequality (11), then $\mathcal{Q}_P$ contains all quantile functions Q with (i) $Q(t) = 0$ for $t < t_0$, and for some $t_0 \in [\underline{t}_0, \bar{t}]$: (ii) $Q(t) = \mathcal{R}_P(t_0)$ $\forall t \in [t_0, t_P)$ where t_P solves $\Gamma_P(t_P) = \mathcal{R}_P(t_0)$; (iii) $Q(t) = \Gamma_P(t)$ on $[t_P, t^*(1)]$; and (iv) $Q(t) = 1$ for all $t > t^*(1)$. By Proposition 2B and Lemma 4, $\mathcal{Q}_W$ is non-empty *iff* there is not fear at $t^*(0)$, while $\mathcal{Q}_P$ is non-empty *iff* there is not greed at $t^*(1)$ and no panic.

Note that: (a) *there is a one-one map from undominated rush times in the domain of $\mathcal{R}_W$ ($\mathcal{R}_P$) to quantile functions in the sets $\mathcal{Q}_W$ ($\mathcal{Q}_P$); and* (b) *all quantile functions in $\mathcal{Q}_W$ ($\mathcal{Q}_P$) share the same gradual play locus Γ_W* (*resp. Γ_P*) *on the intersection of their gradual play*

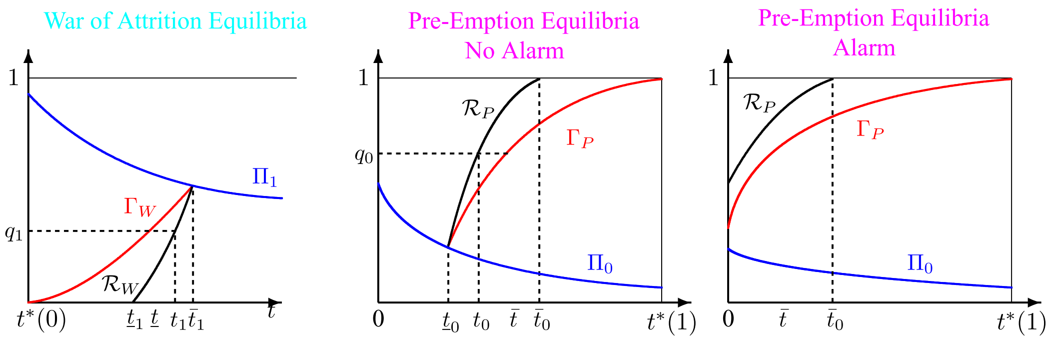

FIGURE 8.—*All Nash Equilibria With Gradual Play*. Left: All wars of attrition start at time $t^*(0)$, and are given by $Q(t) = \Gamma_W(t)$: Any terminal rush time $t_1 \in [\underline{t}, \bar{t}_1]$ determines a rush size $q_1 = \mathcal{R}_W(t_1)$, which occurs after an inaction phase $(\Gamma_W^{-1}(q_1), t_1)$ following the war of attrition. Middle: All pre-emption games start with an initial rush of size $Q(t) = q_0$ at time $t_0 \in [\underline{t}_0, \bar{t}]$, followed by inaction on $(t_0, \Gamma_P^{-1}(q_0))$, and then a slow pre-emption phase given by $Q(t) = \Gamma_P(t)$, ending at time $t^*(1)$. Right: The set of pre-emption equilibria with alarm is constructed similarly, but using the interval of allowable rush times $[0, \bar{t}]$.

intervals. Among all pre-emption (war of attrition) equilibria, the safe equilibrium has the smallest rush.

PROPOSITION 7—Nash Equilibria: *The set of war of attrition equilibria is the candidate set $\mathcal{Q}_W$. As the rush time postpones, the rush shrinks, and the gradual play phase lengthens. The set of pre-emption equilibria is the candidate set $\mathcal{Q}_P$. As the rush time postpones, the rush shrinks, and the gradual play phase shrinks. Gradual play intensity is unchanged on the common gradual play support.*

Across both pre-emption and war of attrition equilibria: larger rushes are associated with shorter gradual play phases. The covariate predictions of rush size, timing, and gradual play length coincide with Proposition 6 for all wars of attrition and pre-emption equilibria without alarm. The correlation between the length of the phase of inaction and the size of the rush implies that the safe war of attrition (pre-emption) equilibrium has the smallest rush and longest gradual play phase among all war of attrition (pre-emption) equilibria. In the knife-edge case when payoffs are log-modular in (t, q), the inaction phase is monotone in the time of the rush, as in Figure 8.

We now connect our theory of hump-shaped payoffs, for which there is a continuum of Nash equilibria each with a rush, to the standard case when payoffs are strictly monotone in quantile, for which there is a unique equilibrium with no rush absent alarm or panic. The next result joins Proposition 1 with our novel theory with rushes emerging from Propositions 2A and 2B.

COROLLARY 3: *Fix a stopping payoff u that is strictly monotone in quantile and for which there is no alarm or panic. For any sequence of hump-shaped stopping payoffs $u^\eta \to u$ in the L^1 norm as $\eta \downarrow 0$, the largest rush across all Nash equilibria vanishes as $\eta \downarrow 0$.*

PROOF: Let $u_q > 0$ always (the logic for $u_q < 0$ is symmetric). Fix $q > 0$. There exists $\eta^* > 0$ with $u^\eta(t, 1) > (1-q)^{-1} \int_q^1 u^\eta(t, x)\, dx > \int_0^1 u^\eta(t, x)\, dx$ for all $\eta \le \eta^*$ and all t. Since $u^\eta(t, 1) > \int_0^1 u^\eta(t, x)\, dx$, there is greed. By Proposition 2B, only war of attrition equilibria exist for u^η, that is, stopping in all equilibria ends in a terminal rush. By the first

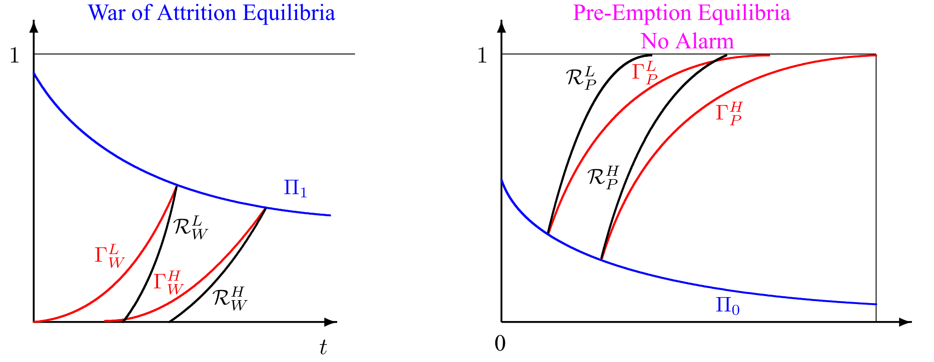

FIGURE 9.—*Harvest Time Delay Revisited*. With a harvest delay, from low L to high H, the gradual play loci $\Gamma_i^j = \Gamma_i(\cdot|\varphi_j)$ and rush loci $\mathcal{R}_i^j \equiv \mathcal{R}_i(\cdot|\varphi_j)$ shift down, where $i = W, P$ and $j = L, H$.

inequality, post-empting a terminal rush of size q beats stopping in the rush. So no size-q rush exists for u^η. *Q.E.D.*

Comparative statics prediction with sets of equilibria is problematic: Milgrom and Roberts (1994) resolved this by focusing on extremal equilibria. Here, safe equilibria are extremal—the safe pre-emption equilibrium starts the earliest, and the safe war of attrition equilibrium ends the latest. Our comparative statics predictions Propositions 5 and 6 for safe equilibria extend to suitably chosen selections from the Nash correspondences $\mathcal{Q}_W(\varphi)$ and $\mathcal{Q}_P(\varphi)$ for the indexed payoffs $u(t, q|\varphi)$ in Section 7. Figures 9 and 10 illustrate how the key loci characterizing the set of Nash equilibria shift with a harvest time delay and an increase in greed, respectively. In summary, the following hold:

Fundamentals Change. Assume a harvest time delay with $\varphi_H > \varphi_L$ and no panic at φ_L. For all $Q_L \in \mathcal{Q}_W(\varphi_L)$, there exists $Q_H \in \mathcal{Q}_W(\varphi_H)$ such that ordering (a) in Proposition 5 holds, and also gradual play for Q_H ends later. For all $Q_H \in \mathcal{Q}_P(\varphi_H)$, there exists $Q_L \in \mathcal{Q}_P(\varphi_L)$ such that ordering (b) in Proposition 5 holds, and also gradual play for Q_H starts later.

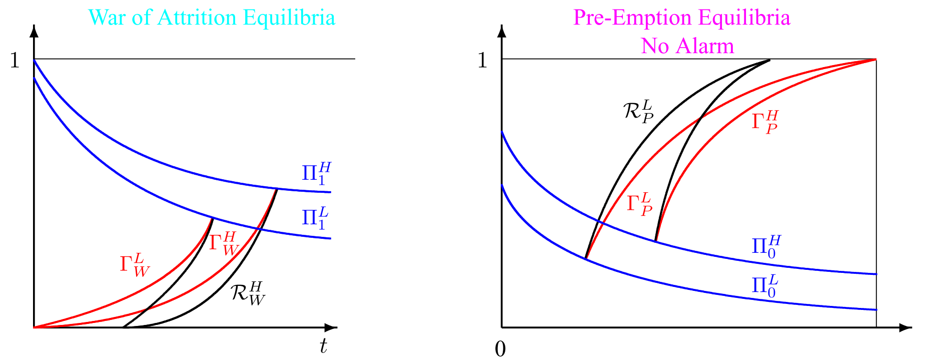

FIGURE 10.—*More Greed: Quantile Changes Revisited*. With an increase in greed or decrease in fear, from low L to high H, the gradual play loci $\Gamma_i^j = \Gamma_i(\cdot|\varphi_j)$ and rush loci $\mathcal{R}_i^j \equiv \mathcal{R}_i(\cdot|\varphi_j)$ shift down, where $i = W, P$ and $j = L, H$, while the peak rush loci Π_0 and Π_1 shift up.

Quantile Change. Assume an increase in greed with $\varphi_H > \varphi_L$ and no panic at φ_L. For all $Q_L \in \mathcal{Q}_W(\varphi_L)$, there exists $Q_H \in \mathcal{Q}_W(\varphi_H)$ such that ordering (a) in Proposition 6 holds, and also gradual play for Q_H ends later. For all $Q_H \in \mathcal{Q}_P(\varphi_H)$, there exists $Q_L \in \mathcal{Q}_P(\varphi_L)$ such that ordering (b) in Proposition 6 holds given no alarm at φ_L, and also gradual play for Q_H starts later, while ordering (c) holds with alarm at φ_L with the rush for Q_H occurring no later.

For the safe wars of attrition explored by Propositions 5 and 6, gradual play ends with an immediate terminal rush. But since we now allow for inaction phases between the war of attrition and the terminal rush, the rush no longer occurs when the war of attrition ends. Nonetheless, our earlier predictions robustly hold in all Nash equilibria: Wars of attrition end later and terminal rushes occur later with a harvest time delay or increase in greed. Similarly, for safe pre-emption equilibria with no alarm, the rush no longer occurs at the outset of gradual play, but still shifts later with a harvest time delay or increase in greed. Both predictions extend for our equilibrium selections.

9. ECONOMIC APPLICATIONS DISTILLED FROM THE LITERATURE

To illustrate our equilibrium predictions, we devise reduced form models for several well-studied timing games. Each reflects the subgame perfect equilibrium interpretation of our model.

9.1. *Land Runs, Sales Rushes, and Tipping Models*

The Oklahoma Land Rush of 1889 saw the allocation of the Unassigned Lands. High noon on April 22, 1889 was the clearly defined time zero, with no pre-emption allowed, just as we assume. Since the earliest settlers naturally claimed the best land, the stopping payoff was monotonically decreasing in quantile. This early mover advantage was strong enough to overwhelm any temporal gains from waiting, and so the panic or alarm cases in Proposition 1 applied.

Next consider the notion of a "tipping point" in sociology—the moment when a mass of people dramatically changes behavior, such as flight from a neighborhood (Grodzins (1957)). In his 1969, 1971 papers, Schelling showed that with a small threshold preference for neighbors of the same type, myopic adjustment eventually and suddenly tips into complete segregation. All told, Schelling's logic is a tatonnement explanation and mostly assumes a lattice structure. Later on, Granovette (1978) explored social settings explicitly governed by "threshold behavior," where individuals differ in the number or proportion of others who must act before one optimally follow suit. He showed that a small change in the threshold distribution may lead aggregate behavior to tip—for example, a large enough number of revolutionaries can eventually tip everyone into a revolution.

Seen through the lens of our model, if players had standard hump-shaped preferences over their neighborhood composition in Schelling (1969, 1971), our theory would offer an equilibrium explanation for the tipping—and therefore be robust to rationally forward-looking players. Our theory would offer predictions for the timing and size of these tipping rushes, and the speed of non-tipping behavior, too. If players prefer the initial over the average stopping payoff, then there is fear, and Proposition 2B predicts a tipping rush, and explains why it occurs early, before preference fundamentals might suggest. Given greed—for example, the last revolutionary does better than the average—tipping still occurs, but one might expect a revolution later than expected from fundamentals. We omit the detailed analysis, as it is similar to our matching model in Section 9.2.

9.2. *The Rush to Match*

We now consider assignment rushes. As in the entry-level gastroenterology labor market in Niederle and Roth (2004) [NR2004], early matching costs include "loss of planning flexibility," whereas the penalty for late matching owes to market thinness. For a cost of early matching, we simply follow Avery, Jolls, Posner, and Roth (2001), who alluded to the condemnation of early match agreements. So we posit a negative stigma to early matching relative to peers.

For a model of this, assume an equal mass of two worker varieties, A and B, each with a continuum of uniformly distributed abilities $\alpha \in [0, 1]$. Firms have an equal chance of needing a type A or B. For simplicity, we assume that the payoff of hiring the wrong type is zero, and that each firm learns its need at fixed exponential arrival rate $\delta > 0$. Thus, the chance that a firm chooses the right type if it waits until time t to hire is $e^{-\delta t}/2 + \int_0^t \delta e^{-\delta s}\, ds = 1 - e^{-\delta t}/2$.[9] Assume that an ability α worker of the right type yields flow payoff α, discounted at rate r. Thus, the present value of hiring the right type of ability α worker at time t is $(\alpha/r)e^{-rt}$.

Consider the quantile effect. Assume an initial ratio $2\theta \in (0, 2)$ of firms to workers (market *tightness*). If a firm chooses before knowing its type, it naturally selects each type with equal chance; thus, the best remaining worker after quantile q of firms has already chosen is $1 - \theta q$. We also assume a *stigma* $\sigma(q)$, with payoffs from early matching multiplicatively scaled by $1 - \sigma(q)$, where $1 > \sigma(0) \geq \sigma(1) = 0$, and $\sigma' < 0$. All told, the payoff is multiplicative in time and quantile concerns:

$$u(t, q) \equiv r^{-1}\big(1 - \sigma(q)\big)(1 - \theta q)\big(1 - e^{-\delta t}/2\big)e^{-rt}. \tag{12}$$

This payoff is log-concave in t, and initially increasing provided the learning effect is strong enough ($\delta > r$). This stopping payoff is concave in quantile q if σ is convex.

The match payoff (12) is log-modular in t and q, and so always exhibits greed, or fear, or neither. Specifically, there is fear when $\int_0^1 (1 - \sigma(x))(1 - \theta x)\, dx \leq 1 - \sigma(0)$, that is, when the stigma σ of early matching is low relative to the firm demand (tightness) θ. In this case, Proposition 2B predicts a pre-emption equilibrium, with an initial rush followed by gradual play; Proposition 3 asserts a waning matching rate, as payoffs are log-concave in time. Likewise, there is greed iff $\int_0^1 (1 - \sigma(x))(1 - \theta x)\, dx \leq 1 - \theta$. This holds when the stigma σ of early matching is high relative to the firm demand θ. Here, Proposition 2B predicts a war of attrition equilibrium, namely, gradual play culminating in a terminal rush, and Proposition 3 asserts rising matching rates nearing that rush. When neither inequality holds, neither fear nor greed obtains, and so both types of gradual play as well as unit mass rushes are equilibria, by Proposition 3.

For an application, NR2004 chronicled the gastroenterology market. The offer distribution in their reported years (see their Figure 1) is consistent with the pattern we predict for a pre-emption equilibrium as in the left panel of our Figure 4—that is, a rush and then gradual play. NR2004 highlighted how the offer distribution advances in time ("unraveling") between 2003 and 2005, and proposed that an increase in the relative demand for fellows precipitated this shift. Proposition 6 replicates this offer distribution shift. Specifically, assume the market exhibits fear, owing to early matching stigma. Since the match payoff (12) is log-submodular in (q, θ), fear rises in market tightness θ. So the rush for

[9]We assume firms unilaterally choose the start date t. One can model worker preferences over start dates by simply assuming the actual start date T is stochastic with a distribution $F(T|t)$.

workers occurs earlier by Proposition 6, and is followed by a longer gradual play phase (left panel of Figure 6). This predicted shift is consistent with the observed change in match timing reported in Figure 1 of NR2004.[10]

Next consider comparative statics in the interest rate r. Since the match payoff is log-submodular in (t, r), lower interest rates entail a harvest time delay, and a delayed matching distribution, by Proposition 5. In the case of a pre-emption equilibrium, the initial rush occurs later and matching is more intense, whereas for a war of attrition equilibrium, the terminal rush occurs later, and stopping rates fall. Since the match payoff is multiplicative in (t, q), the peak rush loci Π_i are constant in t; therefore, rush sizes are unaffected by the interest rate. The sorority rush environment of Mongell and Roth (1991) is one of extreme urgency, and so corresponds to a high interest rate. Given a low stigma of early matching and a tight market (for the best sororities), this matching market exhibits fear, as noted above; therefore, we have a pre-emption game, for which we predict an early initial rush, followed by a casual gradual play as stragglers match.

9.3. *The Rush to Sell in a Bubble*

We parallel Abreu and Brunnermeier (2003) [AB2003], dispensing with asymmetric information. A continuum of investors each owns a unit of an asset and chooses the time t to sell. A fraction $Q(t)$ sells by time t. There is common knowledge among these investors that the asset price is a bubble. As long as the bubble persists, the asset price $p(t|\xi)$ rises smoothly and deterministically in time t; once the bubble bursts, the price drops to the fundamental value, normalized to 0.

The bubble explodes once $Q(t)$ exceeds a threshold $\kappa(t + t_0)$, where t_0 is a random variable with log-concave c.d.f. F common across investors: Investors know the length of the "fuse" κ, but do not know how long the fuse had been lit before they became aware of the bubble at time 0. We assume that κ is log-concave, with $\kappa'(t + t_0) < 0$ and $\lim_{t\to\infty} \kappa(t) = 0$.[11] So the burst chance is the probability $1 - F(\tau(t, q))$ that $\kappa(t + t_0) \leq q$, where $\tau(t, q)$ uniquely satisfies $\kappa(t + \tau(t, q)) \equiv q$, and so falls in q. *The expected stopping price $F(\tau(t, q))p(t|\xi)$ is decreasing in the quantile q.*[12]

Unlike AB2003, we allow for an interior peak quantile by admitting relative performance concerns. Indeed, institutional investors, acting on behalf of others, are often paid for their performance relative to their peers. This imposes an extra cost to leaving a growing bubble early relative to other investors. For a simple model of this peer effect, scale stopping payoffs by $1 + \rho q$, where $\rho \geq 0$ measures *relative performance concern*.[13] All told, the payoff from stopping at time t as quantile q is

$$u(t, q) \equiv (1 + \rho q)F\big(\tau(t, q)\big)p(t|\xi). \tag{13}$$

[10]One can reconcile a tatonnement process playing out over several years, by assuming that early matching in the current year leads to lower stigma in the next year. Specifically, if the ratio $(1 - \sigma(x))/(1 - \sigma(y))$ for $x < y$ falls in response to earlier matching in the previous year, then a natural feedback mechanism emerges. The initial increase in θ stochastically advances match timing, further increasing fear; the rush to match occurs earlier in each year.

[11]By contrast, AB2003 assumed a constant function κ, but that the bubble eventually bursts exogenously even with no investor sales. Moreover, absent AB2003's asymmetric information of t_0, with a constant threshold κ, players could perfectly infer the burst time $Q(t_\kappa) = \kappa$, and so strictly gain by stopping before t_κ.

[12]A rising price is tempered by the bursting chance in bubble models (Brunnermeier and Nagel (2004)).

[13]When a fund does well relative to its peers, it often experiences cash inflows (Berk and Green (2004)). In particular, Brunnermeier and Nagel (2004) documented that during the tech bubble of 1998–2000, funds that rode the bubble longer experienced higher net inflow and earned higher profits than funds that sold significantly earlier.

In Appendix C.8, we argue that this payoff is log-submodular in (t, q), and log-concave in t and q.

In AB2003, the bubble bursts for sure once all insiders have sold. While we allow the bubble to persist after all investors sell, we assume that when $q = 1$, the burst chance is large enough so that

$$(1+\rho)F\bigl(\tau(t,1)\bigr) < \int_0^1 (1+\rho x)F\bigl(\tau(t,x)\bigr)\,dx. \tag{14}$$

By Definition 1, this assumption rules out greed and so, by Proposition 4, a safe equilibrium exists. For ρ near zero, the stopping payoff (13) is monotonically decreasing in the quantile, and Proposition 1 predicts either a pre-emption game for all quantiles, or a pre-emption game preceded by a time $t = 0$ rush, or a unit mass rush at time $t = 0$. For higher values of ρ, the stopping payoff initially rises in q, the peak quantile q^* is interior,[14] implying that a rush obtains, and that the unique safe initial rush may occur at a later time $t > 0$. With ρ high enough, there is no fear at $t^*(0)$, since $F(\tau(0, t^*(0))) < \int_0^1 (1+\rho x)F(\tau(x, t^*(0)))\,dx$. In this case, Proposition 4 implies that a war of attrition climaxing in a late rush at time $t > t^*(0)$ is also a safe equilibrium.[15]

Turning to our comparative statics in the fundamentals, recall that as long as the bubble survives, the price is $p(t|\xi)$. Since it is log-supermodular in (t, ξ), if ξ rises, then so does the rate p_t/p at which the bubble grows, and thus there is a harvest time delay. This stochastically postpones sales, by Proposition 5, and so not only does the bubble inflate faster, but it also lasts longer, since the selling pressure diminishes. Both findings are consistent with the comparative static derived in AB2003 that lower interest rates lead to stochastically later sales and a higher undiscounted bubble price. To see this, simply write our present value price as $p(t|\xi) = e^{\xi t}\hat{p}(t)$, that is, let $\xi = -r$ and let $\hat{p}$ be their undiscounted price. Then the discounted price is log-submodular in (t, r): A decrease in the interest rate corresponds to a harvest delay, which delays sales, leading to a higher undiscounted price, while selling rates fall in a war of attrition and rise in a pre-emption game.

For a quantile comparative static, AB2003 assumed the bubble deterministically grows until the rational trader sales exceed a threshold $\kappa > 0$. They showed that if κ increases, bubbles last stochastically longer, and price crashes are larger. Consider this exercise here. Assume any two quantiles $q_2 > q_1$. We found in Section 9.3 that the bubble survival chance $F(\tau(t, q))$ is log-submodular in (t, q), so that $F(\tau(t, q_2))/F(\tau(t, q_1))$ falls in t. Since the threshold $\kappa(t)$ falls in time, lower t is tantamount to an upward shift in κ. All told, an upward shift in κ increases the bubble survival odds ratio $F(\tau(t, q_2))/F(\tau(t, q_1))$. So the stopping payoff (13) is log-supermodular in q and κ—greater κ leading to more greed. Proposition 6 then finds a stochastic delay in sales when κ rises: The bubble bursts stochastically later, and the price drop stochastically increases, as in AB2003.[16] Our model also predicts intensifying selling during gradual play in a pre-emption phase (low ρ or κ), and

[14]Since inequality (14) rules out greed, the stopping payoff is not always rising in quantile. But since $u_q(t, 0) = F'(\tau(t, 0))\tau_q(t, 0) + \rho F(\tau(t, 0))$, the stopping payoff is initially rising for ρ large enough. Altogether, the peak quantile is interior for sufficiently large ρ.

[15]Griffin, Harris, and Topaloglu (2011) asserted this.

[16]Shleifer and Vishny (1997) found a related result in a model with noise traders. Their prices diverge from true values, and this divergence increases in the level of noise trade. This acts like greater κ in our model, since prices grow less responsive to rational trades, and in both cases, we predict a larger gap between price and fundamentals.

selling slows in a war of attrition phase. Finally, since our payoff (13) is log-supermodular in q and relative performance concerns ρ, greater ρ is qualitatively similar to greater κ.

9.4. *Bank Runs*

Bank runs are among the most fabled of rushes in economics. In the benchmark model of Diamond and Dybvig (1983) [DD1983], these arise because banks make illiquid loans or investments, but simultaneously offer liquid demandable deposits to individual savers. So if they try to withdraw their funds at once, a bank might be unable to honor all demands. In their elegant model, savers deposit money into a bank in period 0. Some consumers are unexpectedly struck by liquidity needs in period 1, and withdraw their money plus an endogenous positive return. In an efficient Nash equilibrium, all other depositors leave their money untouched until period 2, whereupon the bank finally realizes a fixed positive net return. But an inefficient equilibrium also exists, in which all depositors withdraw in period 1 in a bank run that over-exhausts the bank savings, since the bank is forced to liquidate loans, and forego the positive return.[17]

We adapt the withdrawal timing game, abstracting from optimal deposit contract design.[18] Given our homogeneous agent model, we ignore private liquidity shocks. A unit continuum of players $[0,1]$ have deposited their money in a bank. The bank divides deposits between a safe and a risky asset, subject to the constraint that at least fraction R be held in the safe asset as reserves. The safe asset has log-concave discounted expected value $p(t)$, satisfying $p(0)=1$, $p'(0)>0$, and $\lim_{t\to\infty} p(t)=0$. The present value of the risky asset is $p(t)(1-\zeta)$, where the shock $\zeta \le 1$ has twice differentiable c.d.f. $H(\zeta|t)$ that is log-concave in ζ and t and log-supermodular in (ζ, t). To balance the risk, we assume this shock has positive expected value: $E[-\zeta]>0$.

As long as the bank is solvent, depositors can withdraw $\alpha p(t)$, where the *payout rate* $\alpha<1$, that is, the bank makes profit $(1-\alpha)p(t)$ on safe reserves. Since the expected return on the risky asset exceeds the safe return, the profit maximizing bank will hold the minimum fraction R in the safe asset, while fraction $1-R$ will be invested in the risky project. Altogether, the bank will pay depositors as long as total withdrawals $\alpha q p(t)$ fall short of total bank assets $p(t)(1-\zeta(1-R))$, that is, as long as $\zeta \le (1-\alpha q)/(1-R)$. The stopping payoff to withdrawal at time t as quantile q is

$$u(t,q) = H\big((1-\alpha q)/(1-R)|t\big)\alpha p(t). \tag{15}$$

Clearly, $u(t,q)$ is decreasing in q, log-concave in t, and log-submodular (since $H(\zeta|t)$ is).

Since the stopping payoff (15) weakly falls in the quantile q, bank runs occur at once or never, by Proposition 1, in the spirit of Diamond and Dybvig (1983) [DD1983]. But unlike there, Proposition 1 predicts a unique equilibrium that may or may not entail a bank run. Specifically, a bank run is avoided *iff* fundamentals $p(t^*(1))$ are strong enough, since (4) is equivalent to

$$u\big(t^*(1),1\big) = H\big((1-\alpha)/(1-R)|t^*(1)\big)p\big(t^*(1)\big) \ge u(0,0) = H\big(1/(1-R)|0\big) = 1. \tag{16}$$

Notice how bank runs do not occur with a sufficiently high reserve ratio or low payout rate. When (16) is violated, the size of the rush depends on the harvest time payoff $u(t^*(1),1)$. When the harvest time payoff is low enough, panic obtains and all depositors run. For

[17] As DD1983 admitted, with a first-period deposit choice, depositors avoid a rationally anticipated run.
[18] Thadden (1998) showed that the efficient contract is impossible in a continuous time version of DD1983.

intermediate harvest time payoffs, there is alarm. In this case, Proposition 1(b) fixes the size q_0 of the initial run via

$$q_0^{-1}\int_0^{q_0} H\big((1-\alpha x)/(1-R)|0\big)\,dx = H\big((1-\alpha)/(1-R)|t^*(1)\big)\,p\big(t^*(1)\big). \tag{17}$$

Since the left side of (17) falls in q_0, the run shrinks in the peak asset value $p(t^*(1))$ or return hazard rate H'/H.

Appendix C.8 establishes a log-submodular payoff interaction between the payout α and both time and quantiles. Hence, Corollary 2 predicts three consequences of a higher payout rate: withdrawals shift stochastically earlier, the bank run grows (with alarm), and withdrawal rates fall during any pre-emption phase. Next consider changes in the reserve ratio. The stopping payoff is log-supermodular in (t, R), since $H(\zeta|t)$ is log-supermodular, and log-supermodular in (q, R) provided the elasticity $\zeta H'(\zeta|t)/H(\zeta|t)$ is weakly falling in ζ (proven in Appendix C.8).[19] Corollary 2 then predicts that a reserve ratio increase shifts the distribution of withdrawals later, shrinks the bank run, and increases the withdrawal rate during any pre-emption phase.[20]

10. CONCLUSION

We have developed a novel and unifying theory of large timing games that subsumes pre-emption games and wars of attrition. If individuals have hump-shaped preferences over their stopping quantile, then a rush is inevitable. When the game tilts towards rewarding early or late ranks compared to the average—fear or greed, respectively—this rush happens early or late, and is adjacent to a pre-emption game or a war of attrition, respectively. Stopping in this gradual play phase monotonically intensifies approaching this rush when payoffs are log-concave in time. We derive robust monotone comparative statics with many realistic and testable implications. Our theory is tractable and identifiable, and rationalizes predictions in several classic timing games.

APPENDIX A: DYNAMIC EQUILIBRIUM REFINEMENTS

Our Nash equilibria have assumed a single information set. We now argue that this is purely for simplicity, and that our results are in fact subgame perfect both in a weak and a strong sense.

A.1. *All Nash Equilibria Are Subgame Perfect*

Assume a history of play at which an *arbitrary fraction* $x \in [0, 1)$ of players stop by an *arbitrary time* $\tau \geq 0$. The induced payoff function for the subgame starting at time τ over the remaining $1 - x$ quantiles is thus

$$u_{(\tau,x)}(t, q) \equiv u\big(t+\tau, x+q(1-x)\big).$$

[19] Equivalently, the stochastic return $1-\zeta$ has an increasing generalized failure rate, a property satisfied by most commonly used distributions (see Table 1 in Banciu and Mirchandani (2013)).

[20] An increase in the reserve ratio increases the probability of being paid at the harvest time, but it also increases the probability of being paid in any early run. Log-concavity of H is necessary, but not sufficient, for the former effect to dominate: This requires our stronger monotone elasticity condition.

Thus, we may define a Nash equilibrium following any such history (τ, x) as in the original game.

We first check incentives on the equilibrium path. Any equilibrium quantile function $Q(t)$ induces a continuation quantile function $Q_{(\tau,x)}(t) \equiv [Q(t+\tau) - Q(\tau)]/[1-x]$ on the equilibrium path. By indifference, if Q is a Nash equilibrium, then $Q_{(\tau,x)}$ is a Nash equilibrium for $u_{(\tau,x)}$ on the equilibrium path, that is, for all (τ, x) with $x = Q(\tau)$.

We next define equilibrium strategies after any out of equilibrium history $(\tau, x) \in [0, \infty) \times [0, 1)$. We claim that $u_{(\tau,x)}$ inherits all the model assumptions of u: boundedness, continuity, quasi-concavity in t for fixed q, and monotonicity or log-concavity in q for fixed t, and log-submodularity in (t, q). Finally, $u_{(\tau,x)}$ satisfies inequality (1), since

$$\lim_{s\to\infty} \max_q u_{(\tau,x)}(s, q) = \lim_{s\to\infty} \max_q u\big(s, x + q(1-x)\big) \le \lim_{s\to\infty} \max_q u(s, q)$$
$$\equiv \lim_{s\to\infty} u\big(s, q^*(s)\big) < u(t, q).$$

Since $u_{(\tau,x)}$ satisfies all the model assumptions for any $(\tau, x) \in [0, \infty) \times [0, 1)$, the set of Nash equilibria after any history (τ, x) is as characterized in Section 3 and Section 8, but for the induced payoff function $u_{(\tau,x)}$. In particular, it is non-empty by Propositions 1 and 2B. Since no player from the continuum can unilaterally alter the quantile,[21] subgame payoffs are irrelevant for incentives; and we can therefore choose *any* continuation from the set of Nash equilibria given $u_{(\tau,x)}$.

PROPOSITION A.1—Subgame Perfection: *All Nash equilibria are subgame perfect.*

A.2. *All Nash Equilibria Are Nearly Strict Subgame Perfect*

Taking inspiration from Harsanyi (1973), we show that any Nash equilibrium is arbitrarily closely approximated by a nearby (Bayesian) Nash equilibrium of a slightly perturbed game. Index the players by types ε having C^1 c.d.f. Y_δ in δ with density Y'_δ on $[-\delta, \delta]$ of uniformly bounded variation, so that stopping during slow play at time t as quantile q yields payoff $\mathfrak{u}(t, q, \varepsilon)$ to a type ε. The stopping payoff $\mathfrak{u}$ obeys all properties of u in (t, q) for fixed ε, and is log-supermodular in (q, ε) and strictly so in (t, ε), C^1 in (t, q, ε) with $\mathfrak{u}(t, q, 0) = u(t, q)$, $\mathfrak{u}_t(t, q, 0) = u_t(t, q)$, and $\mathfrak{u}_q(t, q, 0) = u_q(t, q)$. So type $\varepsilon = 0$ enjoys the payoff function just as in the original model.[22] This formulation includes as special cases both pure differences in time preferences, such as $\mathfrak{u}(t, q, \varepsilon) = u((1-\varepsilon)t, q)$, or in quantile preferences, like $\mathfrak{u}(t, q, \varepsilon) = u(t, (1-\varepsilon)q)$, so that lower ε will stop weakly earlier in t and q space.[23]

[21] For some context, our assumption that strategic interaction is embodied in the quantile implies the maintained assumption in Gul, Sonnenschein, and Wilson (1986) on page 159 that "measure-zero" deviations do not affect play.

[22] This is not required, and we could simply assume that payoffs collapse to the original one as δ vanishes.

[23] Prompted by a referee, we note that a special case of this payoff structure arises in an asymmetric information model. Let the realized stopping payoff be multiplicative in time and quantile, $\pi(t, z)v(q, y)$, where π and v are log-supermodular. A player's type ε is a signal she has observed of the unobserved scalars z and y. The conditional densities $\mu_z(z, \varepsilon)$ and $\mu_y(y, \varepsilon)$ are affiliated, and so log-supermodular. The expected stopping payoff for type ε is log-modular in (t, q), and also log-supermodular in (t, ε) and (q, ε) by Karlin and Rinott (1980), because

$$\mathfrak{u}(t, q, \varepsilon) = \iint \pi(t, z)v(q, y)\mu_z(z, \varepsilon)\mu_y(y, \varepsilon)\,dz\,dy = \left[\int \pi(t, z)\mu_z(z, \varepsilon)\,dz\right]\left[\int v(q, y)\mu_y(y, \varepsilon)\,dy\right].$$

A *strategy* is now a function $s : [-\delta, \delta] \mapsto [0, \infty)$ mapping realized types ε into stopping times, yielding a quantile function $Q_\delta(t) = \Pr[s(\varepsilon) \le t]$. Let $w_\delta(t|\varepsilon, s)$ be the expected payoff for type ε stopping at time t, given Q_δ generated by strategy s. A strategy s is then a Nash equilibrium if $s(\varepsilon) \in \arg\max_t w_\delta(t|\varepsilon, s)$. Since u is log-supermodular, $s(\varepsilon)$ is monotone.

The Lévy–Prohorov metric measures the distance between quantile functions:

$$\varrho(Q_1, Q_2) \equiv \inf\bigl\{d > 0 | Q_1(x - d) - d \le Q_2(x) \le Q_1(x + d) + d \ \forall x \in [0, 1]\bigr\}.$$

PROPOSITION A.2—Approximation: *Fix a Nash equilibrium Q of the original game. For all $\Delta > 0$, there exists $\delta^* > 0$ such that, for all $\delta \le \delta^*$, a Nash equilibrium Q_δ exists with $\varrho(Q, Q_\delta) \le \Delta$.*

PROOF OVERVIEW: In Steps 1 and 2, we generalize the gradual play and peak rush loci in Figure 4 to accommodate payoff heterogeneity (delaying one technical step to Section C.9), and verify that these generalized loci converge to the homogeneous payoff loci as payoff heterogeneity vanishes. In Step 3, we show how these generalized loci can be used to define a quantile function Q_δ approximating any safe equilibrium of the original game involving gradual play. In Step 4, we verify that the quantile function Q_δ is a Nash equilibrium of the heterogeneous payoff game. In Step 5, we generalize the rush loci of Figure 8 to approximate the full set of Nash equilibria involving gradual play. Step 6 considers approximating a unit mass rush equilibrium.

Step 1: *Gradual Play Loci*. We define the *gradual play type interval* as an open interval $(\varepsilon_1, \varepsilon_2)$ on which $s' > 0$, so that gradual play happens on the time interval $(s(\varepsilon_1), s(\varepsilon_2))$. Since any type ε can secure payoff $u(s(\hat\varepsilon), Y_\delta(\hat\varepsilon), \varepsilon)$ by mimicking any type $\hat\varepsilon \in (\varepsilon_1, \varepsilon_2)$, the Relevation Principle gives the equilibrium gradual play differential equation on $(\varepsilon_1, \varepsilon_2)$:

$$u_t\bigl(s(\varepsilon), Y_\delta(\varepsilon), \varepsilon\bigr)s'(\varepsilon) + u_q\bigl(s(\varepsilon), Y_\delta(\varepsilon), \varepsilon\bigr)Y'_\delta(\varepsilon) = 0.$$

Since $s(\varepsilon)$ is invertible on this interval, we have $\varepsilon \equiv Y_\delta^{-1}(q)$ and $Q_\delta(s(\varepsilon)) = Y_\delta(\varepsilon)$, whereupon $Q'_\delta(s(\varepsilon))s'(\varepsilon) = Y'_\delta(\varepsilon)$. In sum, defining $\mathcal{E}_\delta(q) \equiv Y_\delta^{-1}(q)$,

$$u_t\bigl(t, Q_\delta(t), \mathcal{E}_\delta\bigl(Q_\delta(t)\bigr)\bigr) + u_q\bigl(t, Q_\delta(t), \mathcal{E}_\delta\bigl(Q_\delta(t)\bigr)\bigr)Q'_\delta(t) = 0. \tag{18}$$

As in the original game, we argue that there is a unique ending time t_P for any pre-emption phase. The largest type $\varepsilon = \delta$ is the last player to stop. Then $u_t(t_P, 1, \delta) \le 0$, for this player cannot profit from further delay. Since $u_q \le 0$ in a pre-emption phase, (18) implies that the passage of time cannot be strictly harmful to this type, and so $u_t(t_P(\delta), 1, \delta) = 0$. Then $t_P(\delta) \equiv \arg\max_t u(t, 1, \delta)$ is the unique harvest time for the type δ who stops last. Similarly, define the harvest time for the type $-\delta$ stopping first $t_W(-\delta) \equiv \arg\max_t u(t, 0, -\delta)$. Since u is continuous, by the theorem of the maximum, the harvest time functions $t_P(\delta)$ and $t_W(-\delta)$ are continuous in δ, and thus obey $t_P(0) = \arg\max_t u(t, 1, 0) = \arg\max_t u(t, 1) = t^*(1)$, likewise $t_W(0) = t^*(0)$.

We now construct approximations to the gradual play phase for any equilibrium of the original game, using the gradual play differential equation (18) and harvest time functions $t_P(\delta)$ and $t_W(-\delta)$. First, let us approximate the pre-emption phase for an equilibrium Q of the original game. By Propositions 1 and 2B, pre-emption equilibria cannot survive greed at $t^*(1)$. Thus, whenever Q involves a pre-emption phase, we have no greed at $t^*(1)$. In Section C.9, we show that in this case, for small enough $\delta > 0$, there exists a unique solution to (18) with terminal condition $Q_\delta(t_P(\delta)) = 1$—called the *perturbed pre-emption locus* $\Gamma_P(t|\delta)$—and that this solution is continuous in δ. Likewise, if Q includes a war of attrition, then for small enough $\delta > 0$, the *perturbed war of attrition locus* $\Gamma_W(t|\delta)$ uniquely solves (18) with initial condition $Q_\delta(t_W(\delta)) = 0$ and is continuous in δ. Given the extremal conditions $u_t(t, q, 0) = u_t(t, q)$, $u_q(t, q, 0) = u_q(t, q)$, $t_P(0) = t^*(1)$, and $t_W(0) = t^*(0)$, continuity of the loci $\Gamma_P(t|\delta)$ and $\Gamma_W(t|\delta)$ in δ, and $|\mathcal{E}_\delta(Q_\delta)| \leq \delta$, we have $\Gamma_P(t|\delta) \to \Gamma_P(t)$ and $\Gamma_W(t|\delta) \to \Gamma_W(t)$ as $\delta \to 0$.

Step 2: *Peak Rush Loci*. Define the payoff to type ε in an initial/terminal rush:

$$\mathcal{V}_0(t, q, \varepsilon) \equiv q^{-1} \int_0^q u(t, x, \varepsilon)\, dx \quad \text{and} \quad \mathcal{V}_1(t, q, \varepsilon) \equiv (1 - q)^{-1} \int_q^1 u(t, x, \varepsilon)\, dx. \qquad (19)$$

Consider an initial rush of quantiles $[0, q]$. Since the strategy $s(\varepsilon)$ is non-decreasing, all types $\varepsilon \in [-\delta, \mathcal{E}_\delta(q)]$ participate in such a rush. Likewise, all types on $[\mathcal{E}_\delta(q), \delta]$ participate in any terminal rush of quantiles $[q, 1]$. Generalizing the peak rush loci $\Pi_i(t|\delta) \equiv \arg\max_q \mathcal{V}_i(t, q, \mathcal{E}_\delta(q))$, when $\Pi_i(t|\delta) \in (0, 1)$, the marginal type is indifferent between the rush payoff and adjacent gradual play payoff:

$$u\big(t, \Pi_i(t|\delta), \mathcal{E}_\delta\big(\Pi_i(t|\delta)\big)\big) \equiv \mathcal{V}_i\big(t, \Pi_i(t|\delta), \mathcal{E}_\delta\big(\Pi_i(t|\delta)\big)\big). \qquad (20)$$

Since $|\mathcal{E}_\delta| \leq \delta$ and $u(t, q, 0) = u(t, q)$, our original peak rush loci satisfying (5) also solve (20) at $\delta = 0$. Since $Y_\delta(q)$ is continuous in δ, so is $\mathcal{E}_\delta(q) \equiv Y_\delta^{-1}(q)$. Given u, $\mathcal{V}_i$, and $\mathcal{E}_\delta$ continuous in δ, the maximum $\Pi_i(t|\delta)$ is well-defined and continuous near $\delta = 0$, by Berge's theorem.

Step 3: *An Approximate Quantile Function, Q_δ*. We can use the perturbed gradual play and peak rush loci to approximate any safe equilibrium with gradual play. In particular, consider a safe pre-emption equilibrium Q with an initial rush at $t > 0$ (similar steps apply to any other type of safe equilibria involving gradual play). By Proposition 4, since Q is a safe equilibrium with an initial rush at $t > 0$, the stopping payoff u is hump-shaped in quantiles and displays no greed at $t^*(1)$, no alarm, and no panic, and the initial safe rush is of size $q_0 = \Pi_0(t_0)$ at the unique time $t_0 \in (0, t^*(1))$ obeying $\Pi_0(t_0) = \Gamma_P(t_0)$, followed at once by a gradual pre-emption game along $\Gamma_P(t)$ ending at time $t^*(1)$ (as in Figure 4). Likewise, in the heterogeneous type model, construct the quantile function $Q_\delta(t)$ with an initial rush of size $q_\delta = \Pi_0(t_\delta|\delta)$ at the unique *rush time* $t_\delta \in (0, t_P(\delta))$ obeying $\Pi_0(t_\delta|\delta) = \Gamma_P(t_\delta|\delta)$, followed at once by a gradual pre-emption game along $\Gamma_P(t|\delta)$ ending at time $t_P(\delta)$. This quantile function Q_δ is well-defined and arbitrarily close to Q for δ small enough, by continuity of Γ_P, Π_0, and t_P in δ.

Step 4: *Q_δ is a Nash Equilibrium*. We sequentially rule out all deviations.

4.A: First, no type can gain from stopping after the harvest time $t_P(\delta)$. For by construction, the highest type $\varepsilon = \delta$ cannot gain by delay to $t > t_P(\delta)$, since $t_P(\delta)$ is the harvest time for this type. But then by complementarity in (t, ε), no type can gain from stopping after $t_P(\delta)$.

4.B: Next, we claim that all types in the rush weakly prefer rushing to the adjacent gradual play payoff. This follows because the highest rushing type is indifferent by (20), and since $\mathcal{V}_0(t, q, \varepsilon)/u(t, q, \varepsilon)$ is non-increasing in ε, by log-supermodularity of u in (q, ε).

4.C: Next, we rule out profitable deviations to some time in the gradual pre-emption phase $(t_\delta, t_P(\delta)]$ from the rush or from another time in gradual play. Fix type ε, and consider his optimal stopping time on the gradual play interval $\arg\max_{t \in [t_\delta, t_P(\delta)]} u(t, Q_\delta(t), \varepsilon)$.

Log-differentiating equilibrium payoffs $w_\delta(t|\varepsilon, s) = u(t, Q_\delta(t), \varepsilon)$ in t yields

$$\frac{w'_\delta(t|\varepsilon, s)}{w_\delta(t|\varepsilon, s)} = \frac{u_t(t, Q_\delta(t), \varepsilon)}{u(t, Q_\delta(t), \varepsilon)} + \frac{u_q(t, Q_\delta(t), \varepsilon)}{u(t, Q_\delta(t), \varepsilon)} Q'_\delta(t).$$

Since u is log-supermodular in (t, ε) and (q, ε), the ratio $w'_\delta(t|\varepsilon, s)/w_\delta(t|\varepsilon, s)$ is nondecreasing in ε, and is identically zero for $\varepsilon = \mathcal{E}_\delta(Q_\delta(t))$, since stopping is locally optimal during gradual play by construction. Thus, $w'_\delta(t|\varepsilon, s) \leq 0$ whenever $\mathcal{E}_\delta(Q_\delta(t)) \geq \varepsilon$, while $w'_\delta(t|\varepsilon, s) \geq 0$ when $\mathcal{E}_\delta(Q_\delta(t)) \leq \varepsilon$. Then no type stopping during gradual play can strictly gain from deviating to another gradual play time. Also, $w'_\delta(t|\varepsilon, s) \leq 0$ throughout the gradual play time interval for all types $[0, Q_\delta(t_\delta)]$ that take part in the initial rush, and so they weakly prefer stopping at t_δ to elsewhere in gradual play, and so weakly prefer to rush, by step 4.B.

4.D: Pre-empting the rush is strictly dominated. First, this is true in our original game because payoffs $u(t, q)$ are hump-shaped in the quantile q and $u_t(t, q) > 0$ for all $t < t_0$, and thus $V_0(t_0, q_0) \equiv \max_q V_0(t_0, q) > u(t_0, 0) > u(t, 0)$. Next, we argue that this payoff wedge remains for small enough $\delta > 0$. For by assumption, preferences in our game obey $u(t, q, \varepsilon) \to u(t, q)$ as $\varepsilon \to 0$. Also, $(t_\delta, q_\delta) \to (t_0, q_0)$ as $\delta \to 0$ by continuity of $\Pi_0(t|\delta)$ and $\Gamma_P(t|\delta)$ in δ. Altogether, for small enough $\delta > 0$, the rush payoff $\mathcal{V}_0(t_\delta, q_\delta, \varepsilon)$ strictly exceeds the best possible payoff from stopping before the rush time t_δ, namely, $\max_{t \leq t_\delta} u(t, 0, \varepsilon)$ for all $\varepsilon \in [-\delta, \delta]$.

Step 5: *Approximating All Nash Equilibria Involving Gradual Play*. We now generalize the initial/terminal rush loci (10) for preference heterogeneity, which then can be used to approximate the full set of Nash equilibria by parallel logic to that used for the homogeneous payoff case in Section 8. To be concrete, let us approximate an equilibrium Q of the original game with an initial rush followed by an inaction phase, and then a pre-emption phase (as in Figure 8, middle panel). What remains is to define the *perturbed initial rush locus* $\mathcal{R}_P(t|\delta)$, that is, the type $\mathcal{E}_\delta(\mathcal{R}_P(t|\delta))$ indifferent between rushing with quantiles $[0, \mathcal{R}_P(t|\delta)]$ and stopping in the gradual play phase along the pre-emption locus at time $\Gamma_P^{-1}(\mathcal{R}_P(t|\delta)|\delta) > t$. Define the function

$$\Delta(t, q, \varepsilon) \equiv \mathcal{V}_0(t, q, \varepsilon) - u(\Gamma_P^{-1}(q|\delta), q, \varepsilon).$$

We implicitly define $\mathcal{R}_P(t|\delta)$ by $\Delta(t, \mathcal{R}_P(t|\delta), \mathcal{E}_\delta(\mathcal{R}_P(t|\delta))) = 0$. Since this equation collapses to (10), $\mathcal{R}_P(t|\delta)$ reduces to the gradual play locus $\mathcal{R}_P(t)$ at $\delta = 0$. Next, $\mathcal{R}_P(t|\delta)$ is well-defined and continuous near $\delta = 0$ if the implicit function theorem applies. Now, Δ is a composition of continuous functions. Next, the derivative of $\Delta(t, q, \mathcal{E}_\delta(q))$ in $q = R_P(t|\delta)$ is positive near $\delta = 0$. For it equals $q^{-1}[\mathcal{V}_0(t, q, \mathcal{E}_\delta(q)) - u(t, q, \mathcal{E}_\delta(q))]$, that is, the partial derivative of $\mathcal{V}_0$ in q by (19), plus

$$-\left[u_t(\Gamma_P^{-1}(q|\delta), q, \mathcal{E}_\delta(q)) \frac{\partial \Gamma_P^{-1}(q|\delta)}{\partial q} + u_q(\Gamma_P^{-1}(q|\delta), q, \mathcal{E}_\delta(q))\right] + \Delta_\varepsilon(t, q, \mathcal{E}_\delta(q))\mathcal{E}'_\delta(q).$$

The bracketed term is identically zero, by (18). We argue that the last term vanishes. Since the density Y'_δ integrates to one on $[-\delta, \delta]$, and has uniformly bounded variation, as $\delta \downarrow 0$, the minimum of the density $Y'_\delta(\mathcal{E}_\delta(q))$ explodes, and so $\mathcal{E}'_\delta(q) = 1/Y'_\delta(\mathcal{E}_\delta(q))$ vanishes. All told, as $\delta \downarrow 0$, the q derivative of $\Delta(t, q, \mathcal{E}_\delta(q))$ converges to $q^{-1}[V_0(t, \mathcal{R}_P(t)) - u(t, \mathcal{R}_P(t))]$ at $q = \mathcal{R}_P(t|\delta)$. This is strictly positive for any initial rush followed by inaction (as shown in Section 8).

We can similarly define a perturbed terminal rush locus $\mathcal{R}_W(t|\delta)$ leaving the marginal type indifferent between the terminal rush payoff and his type's gradual play payoff along $\Gamma_W(t|\delta)$. This will be continuous near $\delta = 0$, by symmetric reasoning, thereby establishing robustness.

Step 6: *Approximating Unit Mass Rush Equilibria*. Assume a unit mass rush equilibrium for the original game. If this rush occurs at $t = 0$, then, by Proposition 2B, panic obtains in the original game. There are two possibilities for the heterogeneous payoff game for small δ: either u also displays panic, in which case a unit mass rush at $t = 0$ remains an equilibrium, or u displays alarm, in which case there exists a pre-emption equilibrium Q_δ involving a $t = 0$ rush of size $\mathcal{R}_P(0|\delta)$ by Steps 1–5. But since the original game displays panic, we must have $\mathcal{R}_P(0|\delta) \to 1$ as $\delta \to 0$. And thus, Q_δ converges to Q in the Lévy–Prohorov metric.

If, instead, the original equilibrium is a unit mass rush at $0 < t_r \leq t^*(0)$ (the logic for $t_r \geq t^*(0)$ is symmetric), then by Proposition 2B, the original game obeys not greed and not fear at t_r. And since both inequalities are strict, there exists $\delta^* > 0$ such that $\mathcal{V}_0(t_r, 1, \varepsilon) > \max\{u(t_r, 0, \varepsilon), u(t_r, 1, \varepsilon)\}$ for all $\varepsilon \in [-\delta^*, \delta^*]$. Now, consider the case of $t_r = t^*(1)$. Since u and $\mathcal{V}_0$ are continuous in t, $\mathcal{V}_0(t_r, 1, \varepsilon) > u(t_r, 1, \varepsilon)$, and $t_P(\delta) \to t^*(1)$, no type can gain from post-empting a rush at t_r when δ is sufficiently small. And since $u_t > 0$ for $t < t^*(1)$, we have $V_0(t_r, 1) > u(t, 0)$ for all $t \leq t_r$ and no type can gain from pre-empting a rush at t_r. Thus, $t_r = t^*(1)$ remains an equilibrium for sufficiently small δ in the heterogeneous payoff game. If, instead, $t_r < t^*(1)$, then there exists $\delta^\dagger > 0$ such that for all $\delta \leq \delta^\dagger$: (i) there exists $\mathcal{T}(\delta) > 0$, obeying $\mathcal{T}(\delta) \to 0$ as $\delta \to 0$ and (ii) a unit mass rush at $t_r + \mathcal{T}(\delta)$ is an equilibrium in the heterogeneous payoff game. We already established above that there exists $\delta^* > 0$ such that $\mathcal{V}_0(t_r, 1, \varepsilon) > u(t_r, 0, \varepsilon)$ for all $\varepsilon \in [-\delta^*, \delta^*]$, and thus by continuity in t, there exists $\mathcal{T}(\delta) > 0$ obeying $\mathcal{T}(\delta) \to 0$ as $\delta \to 0$ such that for all $\delta \leq \delta^*$, $\mathcal{V}_0(t_r + \mathcal{T}(\delta), 1, \varepsilon) > u(t_r + \mathcal{T}(\delta), 0, \varepsilon)$ for all $\varepsilon \in [-\delta, \delta]$: No type can gain from pre-empting a rush at $t_r + \mathcal{T}(\delta)$. We claim that no type can gain from post-empting such a rush. Indeed, $V_0(t, 1) > V_0(t_r, 1) \geq u(t^*(1), 1)$ for all $t \in (t_r, t^*(1)]$ by Proposition 2B part (d)(ii) and $u_t(t, x) > 0$ for $t < t^*(1)$. Then by continuity of $u(t, q, \varepsilon)$ in (t, ε), for any $t \in (t_r, t^*(1))$, there exists δ^0 such that $\mathcal{V}_0(t, 1, \varepsilon) > u(t^*(1|\varepsilon), 1, \varepsilon)$ for all $\varepsilon \in [-\delta^0, \delta^0]$: No type can gain from post-empting any unit mass rush at $t \in (t_r, t^*(1)]$ for $\delta \leq \delta^0$. Now set $\delta^\dagger = \min\{\delta^*, \delta^0\}$ to conclude that for $\delta \leq \delta^\dagger$, a unit mass rush at $t_r + \mathcal{T}(\delta)$ is an equilibrium with heterogeneous preferences for some $\mathcal{T}(\delta) > 0$ obeying $\mathcal{T}(\delta) \to 0$. *Q.E.D.*

APPENDIX B: GEOMETRIC PAYOFF TRANSFORMATIONS

We have formulated greed and fear in terms of quantile preference in the strategic environment. It is tempting to consider their heuristic use as descriptions of individual risk preference—for example, as a convex or concave transformation of the stopping payoff. For example, if the stopping payoff is an expected payoff, then concave transformations of expected payoffs correspond to ambiguity aversion (Klibanoff, Marinacci, and Mukerji (2005)).

We can show (♣): *for the specific case of a geometric transformation of payoffs $u(t, q)^\beta$, if $\beta > 0$ rises, then rushes shrink, any pre-emption equilibrium advances in time, and any war of attrition equilibrium postpones, while the quantile function is unchanged during gradual play*. A comparison to Proposition 6 is instructive. One might muse that greater risk (ambiguity) aversion corresponds to more fear. We see instead that concave geometric transformations mimic decreases in fear for pre-emption equilibria, and decreases in greed for war of attrition equilibria. Our notions of greed and fear are therefore observationally distinct from risk preference.

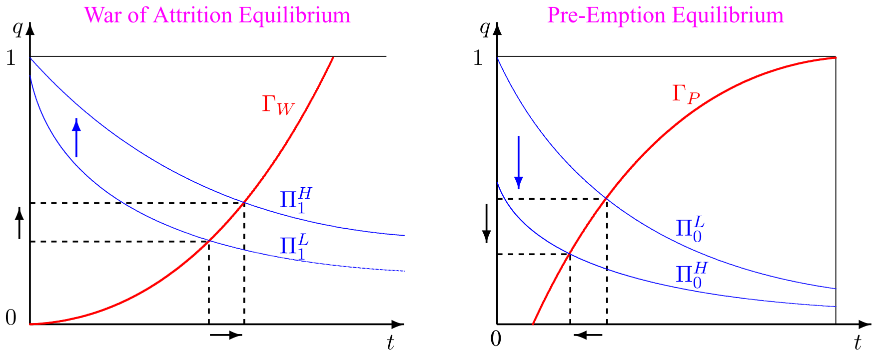

FIGURE 11.—*Geometric Payoff Transformations*. Assume a payoff transformation $u(t,q)^\beta$. The (thick) gradual play locus is constant in β, while the (thin) peak rush locus shifts up in β for a war of attrition equilibrium (right) and down in β for a pre-emption equilibrium (left).

To prove (♣), consider any C^2 transformation $v(t,q) \equiv f(u(t,q))$ with $f' > 0$. Then $v_t = f'u_t$ and $v_q = f'u_q$ and $v_{tq} = f''u_tu_q + f'u_{tq}$. So $v_tv_q - vv_{tq} = [(f')^2 - ff'']u_tu_q - ff'u_{tq}$ yields

$$v_tv_q - vv_{tq} = \left[(f')^2 - ff'' - ff'/u\right]u_tu_q - ff'[u_{tq} - u_tu_q/u]. \tag{21}$$

Since the term u_tu_q changes sign, given $u_tu_q \geq uu_{tq}$, expression (21) is always nonnegative when $(f')^2 - ff'' - ff'/u = 0$, which requires our geometric form $f(u) = cu^\beta$, with $c, \beta > 0$. So the proposed transformation preserves log-submodularity. Log-concavity is proven similarly.

Clearly, $f' > 0$ ensures a fixed gradual play locus (3) in a safe pre-emption equilibrium. Now consider the peak rush locus (5). Given any convex transformation f, Jensen's inequality implies

$$f\big(u(t,\Pi_0)\big) = f\big(V_0(\Pi_0,t)\big) \equiv f\left(\Pi_0^{-1}\int_0^{\Pi_0} u(t,x)\,dx\right) \leq \Pi_0^{-1}\int_0^{\Pi_0} f\big(u(t,x)\big)\,dx.$$

So to restore equality, the peak rush locus $\Pi_0(t)$ must decrease. (Figure 11 summarizes the equilibrium comparative statics in β.) Finally, any two geometric transformations with $\beta_H > \beta_L$ are also related by a geometric transformation $u^{\beta_H} = (u^{\beta_L})^{\beta_H/\beta_L}$.

APPENDIX C: OMITTED PROOFS

C.1. *Gradual Play and Peak Rush Loci: Proofs of Lemmas 1 and 2*

PROOF OF LEMMA 1: *Step* 1: Γ_W. First, there exists finite $t_W > t^*(0)$ such that $u(t_W, q^*(t_W)) = u(t^*(0), 0)$. For $q^* > 0$ implies $u(t^*(0), q^*(t^*(0))) > u(t^*(0), 0)$, while (1) asserts the opposite inequality for t sufficiently large: existence of t_W then follows from continuity of $u(t, q^*(t))$. Next, since $u_t < 0$ for all $t > t^*(0)$, we have $u(t,0) < u(t^*(0), 0)$ and $u(t, q^*(t_W)) > u(t^*(0), 0)$, there exists a unique $\Gamma_W(t) \in [0, q^*(t_W)]$ satisfying (2) for all $t \in [t^*(0), t_W]$. Since $u_q > 0$, $u_t < 0$ on $(t^*(0), t_W] \times [0, q^*(t_W)]$, and u is c^2, $\Gamma_W'(t) > 0$ by the implicit function theorem.

Step 2: Γ_P. First assume $u(0, q^*(0)) \leq u(t^*(1), 1)$. Then $q^*(t) < 1 \Rightarrow u(t^*(1), q^*(t^*(1))) > u(t^*(1), 1)$, while the continuous function $u(t, q^*(t))$ is strictly increasing in $t \leq t^*(1)$. So there exists a unique $t_P \in [0, t^*(1))$ such that $u(t_P, q^*(t_P)) \equiv u(t^*(1), 1)$, with $u(t, q^*(t)) > u(t^*(1), 1)$ for all $t \in (t_P, t^*(1)]$. Also, $u(t, 1) < u(t^*(1), 1)$ and $u_t(t, q) > 0$ for all $t < t^*(1) \leq t^*(q)$. In sum, there is a unique $\Gamma_P(t) \in (q^*(t), 1)$ solving (3), for $t \in (t_P, t^*(1))$. If, instead, the reverse inequality $u(0, q^*(0)) > u(t^*(1), 1)$ holds, then $u(t, q^*(t)) \geq u(0, q^*(0)) > u(t^*(1), 1) > u(t, 1)$, for all $t \leq t^*(1)$. Again by $u_t > 0$, there is a unique $\Gamma_P(t) \in (q^*(t), 1]$ satisfying (3) for all $t \in [0, t^*(1)]$, that is, $t_P = 0$. All told, $\Gamma_P(t) \geq q^*(t)$, so that $u_q(t, \Gamma_P(t)) < 0 < u_t(t, \Gamma_P(t))$, while u is C^2, so that $\Gamma_P'(t) > 0$ by the implicit function theorem. *Q.E.D.*

PROOF OF LEMMA 2: *Step* 1: *Greed and Fear Obey Single Crossing*. Since $u(t, q)$ is log-submodular, $u(t, y)/u(t, x)$ is non-increasing in t for all $y \geq x$. So without greed at $t^*(1)$, there is no greed at any $t \leq t^*(1)$, and without fear at $t^*(0)$, there is no fear at any $t \geq t^*(0)$.

Step 2: *Π_i is Continuous.* Since u is log-concave with unique peak quantile $q^*(t) \in (0, 1)$, the running integral $V_i(t)$ for $i \in \{0, 1\}$ is strictly quasi-concave, and thus the maximizer $\Pi_i(t) = \arg\max_q V_i(t)$ is unique. Continuity of $\Pi_i(t)$ follows from the theorem of the maximum.

Step 3: *Π_i is Non-increasing.* Put $\mathbb{I}(q, x) \equiv q^{-1}$ for $x \leq q$ and 0 otherwise, and $\ell \equiv t^*(1) - t$, and thus $V_0(t^*(1) - \ell, q) = \int_0^1 \mathbb{I}(q, x) u(t^*(1) - \ell, x)\, dx$. Easily, $\mathbb{I}$ is log-supermodular in (q, x), and so the product $\mathbb{I}(\cdot)u(\cdot)$ is log-supermodular in (q, x, ℓ). So V_0 is log-supermodular in (q, ℓ) as integration preserves log-supermodularity (Karlin and Rinott (1980)). So the peak rush locus $\Pi_0(t^*(1) - \ell) = \arg\max_q V_0(t^*(1) - t, q)$ rises in ℓ (falls in t). The logic for Π_1 is symmetric.

Step 4: *No Greed at t Implies $\Pi_0(t) \in (q^*(t), 1)$.* With an interior peak quantile, $\Pi_0(t) > 0$. By continuity, the solution $\Pi_0(t) < 1$ if and only if $\Pi_0(t)$ obeys the marginal equals the average equality (5), which holds *iff* $u(t, 1) < V_0(t, 1)$ (i.e., no greed at t). Since u is single-peaked in q, the solution obeys $u_q(t, \Pi_0(t)) < 0$. Finally, $u_q(t, q) \gtrless 0$ as $q \lessgtr q^*(t)$ implies $\Pi_0(t) > q^*(t)$. *Q.E.D.*

C.2. *A Nash Equilibrium With Alarm*

DEFINITION 2: Quantile function Q is *secure* if it is a Nash equilibrium whose support is either a single non-empty interval of time or the union of $t = 0$ and such an interval.

Assuming no greed at $t^*(1)$ and alarm, we construct a secure quantile function with a rush at $t = 0$ and a pre-emption phase. To this end, let quantile $q_0 \in (\Pi(0), 1)$ be the largest solution to $V_0(0, q_0) = u(t^*(1), 1)$. First, $\Pi(0) = \arg\max_q V_0(0, q) < 1$ is well-defined by Lemma 2. Given alarm, $V_0(0, 1) < u(t^*(1), 1) < \max_q V_0(0, q) \equiv V_0(0, \Pi(0))$. So the unique $q_0 \in (\Pi(0), 1)$ follows from $V_0(0, q)$ continuously decreasing in $q > \Pi(0)$ (Lemma 2). Then, given q_0, define $\Gamma_P(t_A) = q_0$. To see that such a time $t_A \in (0, t^*(1))$ uniquely exists, observe that $u(0, q^*(0)) \geq \max_q V_0(0, q) > u(t^*(1), 1)$ (by alarm); so that the premise of Lemma 1 part (b) is met. Thus, Γ_P is continuously increasing on $[0, t^*(1)]$ with endpoints $\Gamma_P(t^*(1)) = 1$ and $\Gamma_P(0) < q_0$, where this latter inequality follows from $q_0 > \Pi(0) \Rightarrow u(0, q_0) < V_0(0, q_0) = u(t^*(1), 1) = u(0, \Gamma_P(0))$. Finally, define the candidate quantile function Q_A as: (i) $Q_A(t) = q_0$ for all $t \in [0, t_A)$; (ii) $Q_A(t) = \Gamma_P(t)$ on $[t_A, t^*(1)]$; and (iii) $Q_A(t) = 1$ for all $t > t^*(1)$.

LEMMA C.1: *Assume alarm and no greed at $t^*(1)$. Then Q_A is a secure equilibrium. It is the unique equilibrium with a $t = 0$ rush, one gradual play phase ending at $t^*(1)$, and no other rush.*

PROOF: By construction, the stopping payoff is $u(t^*(1), 1)$ on the support $\{0\} \cup [t_A, t^*(1)]$ of quantile function Q_A. The payoff on the inaction region $(0, t_A)$ is strictly lower, since $u_t(t, q_0) > 0$ for $t < t^*(1) \leq t^*(q_0)$. Finally, since $u_t(t, 1) < 0$ for $t > t^*(1)$, no player can gain from stopping after $t^*(1)$. Altogether, Q_A is a secure Nash equilibrium.

This is the unique Nash equilibrium with the stated characteristics. In any such equilibrium, the expected payoff is $u(t^*(1), 1)$, and q_0 is the unique $t = 0$ rush with this stopping value. Given q_0, the time t_A at which the pre-emption game begins follows uniquely from Γ_P, which in turn is the unique gradual pre-emption locus given payoff $u(t^*(1), 1)$ by Lemma 1. *Q.E.D.*

C.3. *Monotone Payoffs in Quantile: Proof of Proposition 1*

Case 1: $u_q > 0$. In the text, we proved that any equilibrium must involve gradual play for all quantiles beginning at $t^*(0)$, satisfying (2), which defines a unique quantile function by Lemma 1 and $q^*(\cdot) = 1$. This is an equilibrium. No agent can gain by pre-empting gradual play, since $t^*(0)$ maximizes $u(t, 0)$. Further, since $t^*(q)$ is decreasing, we have $u_t(t, 1) < 0$ for all $t \geq t^*(0)$; thus, no agent can gain by delaying until after the war of attrition ends.

Case 2: $u_q < 0$. The text proved that gradual play ends at $t^*(1)$ and rushes occur at $t = 0$.

Step 1: *A* $t = 0$ *Unit Mass Rush* iff *Panic*. Without panic, $V_0(0, 1) < u(t^*(1), 1)$: Deviating to $t^*(1)$ offers a strict improvement over stopping in a unit mass rush at $t = 0$. Now assume panic, but gradual play, necessarily with expected payoff $u(t^*(1), 1)$. The payoff for stopping at $t = 0$ is either $u(0, 0)$ (if no rush occurs at $t = 0$) or $V_0(0, q)$, given a rush of size $q < 1$. But since $u_q < 0$, we have $u(0, 0) > V_0(0, q) > V_0(0, 1) \geq u(t^*(1), 1)$ (by panic), a contradiction.

Step 2: *Equilibrium With Alarm*. First, alarm implies a $t = 0$ rush. Instead, assume alarm and gradual play for all q, necessarily with expected payoff $u(t^*(1), 1)$. Given $u_q < 0$, we have $u(0, 0) = \max_q V_0(0, q)$, which strictly exceeds $u(t^*(1), 1)$ by alarm; therefore, deviating to $t = 0$ results in a strictly higher payoff, a contradiction. But by Step 1, any equilibrium with alarm must also include gradual play, ending at $t^*(1)$. Altogether, we must have a $t = 0$ rush, followed by gradual play ending at $t^*(1)$. By Lemma C.1, there is a unique such equilibrium: Q_A.

Step 3: *Equilibrium With No Alarm or Panic*. First, a rush is impossible without alarm or panic. For by Step 1, we cannot have a unit mass rush; thus, the equilibrium must involve gradual play ending at $t^*(1)$. Then given $u_q < 0$ and the no alarm or panic inequality (4), we have $u(t^*(1), 1) \geq \max_x V_0(0, x) = u(0, 0) > V_0(0, q)$ for all $q > 0$: The payoff in any $t = 0$ rush is strictly lower than the equilibrium payoff $u(t^*(1), 1)$, a contradiction. Next, we construct the unique Nash equilibrium. Absent alarm $u(0, q^*(0)) = \max_q V_0(0, q) \leq u(t^*(1), 1)$, Lemma 1 states that Γ_P defined by (3) is unique, continuous, and increasing from $[t_P, t^*(1)]$ onto $[q^*(t_P), 1] = [0, 1]$ (by $u_q < 0$). Thus, the unique candidate equilibrium is $Q(t) = 0$ on $[0, t_P)$; $Q(t) = \Gamma_P(t)$ on $[t_P, t^*(1)]$; and $Q(t) = 1$ for $t > t^*(1)$. Since $u_t < 0$ for $t > t^*(1)$, no player can gain from delaying until after the gradual play phase, while $u_t(t, 0) > 0$ for $t < t^*(1) \leq t^*(0)$ implies that stopping before gradual play begins is not a profitable deviation. *Q.E.D.*

C.4. *Pre-Emption Equilibria: Proof of Proposition 2B, Part (a)*

Step 1: *No Pre-Emption With Greed at* $t^*(1)$ *or Panic*. Assume a pre-emption equilibrium. By Proposition 2A, stopping must end at $t^*(1)$, implying Nash payoff $\bar{w} =$

$u(t^*(1), 1)$. Also, since $t^*(q)$ is non-increasing, $u_t(t, q) > 0$ for all $(t, q) < (t^*(1), 1)$; and thus, $\bar{w}$ is strictly below the average payoff at $t^*(1)$, $\int_0^1 u(t^*(1), x)\,dx$. Altogether, $\bar{w} = u(t^*(1), 1) < \int_0^1 u(t^*(1), x)\,dx$, contradicting greed at $t^*(1)$. Pre-emption equilibria require no greed at $t^*(1)$.

Next, assume a pre-emption equilibrium with a rush of size q_0 at time t_0, no greed at $t^*(1)$, and panic. For this to be an equilibrium, $V_0(t_0, q_0) \geq u(t_0, q_0)$, which given $u(t_0, x)$ single-peaked in x implies that q_0 exceeds the peak of the average payoff $V_0(t_0, x)$, that is, $q_0 \geq \Pi_0(t_0)$; and thus, since $V_0(t_0, x)$ falls in x after the peak, $V_0(t_0, q_0) > V_0(t_0, 1)$. In addition, since $u_t(t, q) > 0$ for all $(t, q) \leq (t^*(1), 1)$, we have $V_0(t_0, 1) > V_0(0, 1)$. Altogether, the rush payoff obeys $V_0(t_0, q_0) > V_0(0, 1)$, but then since the panic inequality is $V_0(0, 1) \geq u(t^*(1), 1)$, the rush payoff strictly exceeds the payoff during gradual play, that is, $V_0(t_0, q_0) > u(t^*(1), 1)$, a contradiction.

Step 2: *Pre-Emption Equilibrium With Alarm.* For later use, we prove a stronger result: Given no greed at $t^*(1)$ and alarm, there exists a unique secure pre-emption equilibrium. Assume no greed at $t^*(1)$ and alarm. By Proposition 2B part (c), we cannot have a unit rush at $t = 0$, while a unit rush at $t > 0$ is not secure by definition. Then, given Proposition 2A, any equilibrium with an initial rush must end with gradual play at $t^*(1)$. By Lemma C.1, there exists one such secure equilibrium with an initial rush at $t = 0$. We claim that any equilibrium with an initial rush at $t > 0$ is not secure. First, given an initial rush at $t > 0$, security requires that gradual play must immediately follow the rush. Thus, any initial rush at time t of size q must be on both the gradual play locus (3) and peak rush locus (5), that is, $q = \Gamma_P(t) = \Pi_0(t)$, which implies $u(t^*(1), 1) = \max_q V_0(t, q)$. But alarm states $u(t^*(1), 1) < \max_q V_0(0, q)$, while $\max_q V_0(0, q) < \max_q V_0(t, q)$ for $t \in (0, t^*(1)]$; so that $u(t^*(1), 1) < \max_q V_0(t, q)$, a contradiction. Altogether, the unique pre-emption equilibrium with alarm is that characterized by Lemma C.1.

Step 3: *Pre-Emption With No Alarm or Panic.* By Proposition 2A, pre-emption equilibria begin with an initial rush, followed by gradual play ending at $t^*(1)$. This step proves that there exists a unique secure pre-emption equilibrium given no greed at $t^*(1)$ and no alarm or panic.

First, consider the case when (4) holds with equality. Then $u(0, q^*(0)) > \max_q V_0(0, q) = u(t^*(1), 1)$, and Lemma 1 yields $\Gamma_P(t)$ well-defined on $[0, t^*(1)]$ with $u(0, \Gamma_P(0)) = u(t^*(1), 1) = V_0(0, \Pi_0(0))$. That is, $\Gamma_P(0) = \Pi_0(0)$. In fact, $t = 0$ is the only candidate for a secure initial rush. For if the rush occurs at any $t > 0$, security demands that t be on both the gradual play (3) and peak rush locus (5), that is, $\Gamma_P(t) = \Pi_0(t)$, but $\Gamma_P(t)$ is increasing and $\Pi_0(t)$ non-increasing on $[0, t^*(1)]$: $t_P = 0$ and $q_P = \Pi_0(0)$ is the only possible secure initial rush. Now assume (4) is strict, which trivially rules out a $t = 0$ rush, since the maximum rush payoff falls short of the gradual play payoff $u(t^*(1), 1)$. Given a rush at $t > 0$, security requires that $\Gamma_P(t) = \Pi_0(t)$, which we claim uniquely defines a rush time $t_P \in (0, t^*(1))$ and rush size $q_P \in (q^*(0), 1]$. We prove this separately for two exhaustive cases. First, assume $u(0, q^*(0)) \leq u(t^*(1), 1)$. In this case, combining Lemma 1(a) and Lemma 2, we find $\Gamma_P(t_P) = q^*(t_P) < \Pi_0(t_P)$, $\Gamma_P(t^*(1)) = 1 > \Pi_0(t^*(1))$, while Γ_P is increasing and Π_0 is non-increasing on $[t_P, t^*(1))$: There exists a unique solution $(t_P, q_P) \in (t_P, t^*(1)) \times (q^*(t_P), 1)$ with $q_P = \Gamma_P(t_P) = \Pi_0(t_P)$. For the second case, we assume the opposite $u(0, q^*(0)) > u(t^*(1), 1)$, then combine Lemma 1(b) and Lemma 2 to see that Γ_P is increasing and Π_0 non-increasing on $[0, t^*(1)]$, again with $\Gamma_P(t^*(1)) = 1 > \Pi_0(t^*(1))$. To get the reverse inequality at $t = 0$, use (4) to get $u(0, \Pi_0(0)) = V_0(0, \Pi_0(0)) < u(t^*(1), 1) = u(0, \Gamma_P(0))$, and thus $\Pi_0(0) > \Gamma_P(0)$, since both Π_0 and Γ_P exceed $q^*(0)$, which implies $u_q < 0$.

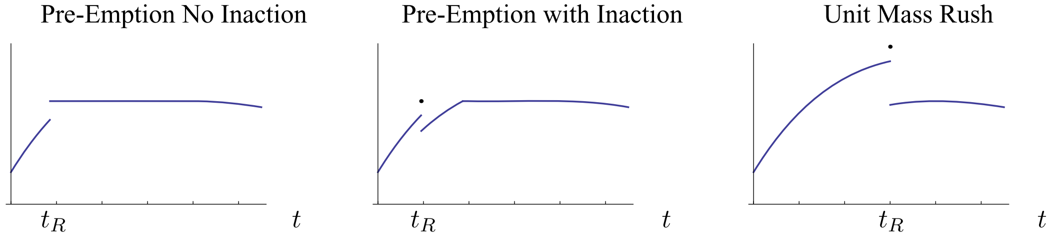

FIGURE 12.—*Equilibrium Payoffs*. We graph equilibrium payoffs as a function of stopping time. On the left is the unique safe pre-emption equilibrium with a flat payoff on an interval; in the middle, a pre-emption equilibrium with inaction; on the right, a unit rush equilibrium.

In all cases, the only possible secure pre-emption equilibrium is: (i) $Q(t)=0$ for $t<t_P$; (ii) $Q(t)=\Gamma_P(t)$ on $[t_P,t^*(1)]$; and (iii) $Q(t)=1$ for all $t>t^*(1)$. Since $u_t<0$ for $t>t^*(1)$, no player can gain from delaying until after the gradual play phase. To see that no player can gain by pre-empting the rush, note that $u_t>0$ prior to the rush, while the peak rush payoff $V_0(t_P,\Pi_0(t_P))>u(t_P,0)$. Altogether, Q is the unique secure pre-emption equilibrium. *Q.E.D.*

C.5. *Safe Equilibria: Proofs for Section 6*

PROOF OF LEMMA 3: By Definition 2, we must show: *Q is safe if and only if Q is secure.*

Step 1: *Secure* $\Rightarrow$ *Safe*. Clearly, $w_\varepsilon(0;Q)=w(0;Q)$ in a Nash equilibrium with a $t=0$ rush. Now, assume Q with constant payoff $\hat{\pi}$ on gradual play interval $[t_a,t_b]$. So for any $\varepsilon'<(t_b-t_a)/2$ and any $t\in[t_a,t_b]$, one of the two intervals $[t,t+\varepsilon')$ or $(t-\varepsilon',t]$ will be contained in $[t_a,t_b]$ and thus obtain payoff $\hat{\pi}$. Safety is maintained with a rush of payoff $\hat{\pi}$ at t_a or t_b. Figure 12 illustrates how equilibria that are not safe cannot be secure.

Step 2: *Safe* $\Rightarrow$ *Secure.* We show that if an equilibrium is not secure, then it is not safe. If u is monotone in q, then Proposition 1 states that there is a unique equilibrium, which involves a single gradual play phase, a rush at $t=0$, or both; and is thus secure. Now consider the hump-shaped case. By Proposition 2A, any equilibrium involves either an initial rush, perhaps followed by a single pre-emption phase, or a single war of attrition phase followed by a terminal rush. Assume an equilibrium with an initial rush of $\hat{q}\in(0,1]$ at time $\hat{t}$, necessarily with $Q(t)=0$ for all $t<\hat{t}$ (the terminal rush case follows similar logic). If this equilibrium is not secure, then $\hat{t}\in(0,t^*(1)]$ and $Q(t)=\hat{q}$ on an interval following $\hat{t}$. Since this is an equilibrium, $V_0(\hat{t},\hat{q})\geq u(\hat{t},0)$. Altogether, $\inf_{s\in(\hat{t}-\varepsilon,\hat{t}]}w(s;Q)=\inf_{s\in(\hat{t}-\varepsilon,\hat{t}]}u(s,0)<V_0(\hat{t},\hat{q})=w(\hat{t};Q)$ for all $\varepsilon\in(0,\hat{t})$, where the strict inequality follows from $u_t(t,q)>0$ for all $t<\hat{t}\leq t^*(1)\leq t^*(q)$.

Now consider an interval following the rush $[\hat{t},\hat{t}+\varepsilon)$. If $\hat{q}<1$, gradual play follows the rush after delay $\Delta>0$, and $V_0(\hat{t},\hat{q})=u(\hat{t}+\Delta,\hat{q})$. But since $t+\Delta<t^*(1)$, we have $u_t(t,\hat{q})>0$ during the delay, and $w(t;Q)=u(t,\hat{q})<V_0(\hat{t},\hat{q})$ for all $t\in(\hat{t},\hat{t}+\Delta)$. Thus, $\inf_{s\in[\hat{t},\hat{t}+\varepsilon)}w(s;Q)<w(\hat{t};Q)$ for all $\varepsilon\in(0,\Delta)$. Now assume $\hat{q}=1$ and consider the two cases $\hat{t}<t^*(1)$ and $\hat{t}=t^*(1)$. If $\hat{t}<t^*(1)$, then $V_0(\hat{t},1)>u(\hat{t},1)$, else stopping at $t^*(1)$ is strictly optimal. But then by continuity, there exists $\delta>0$ such that $w(\hat{t};Q)=V_0(\hat{t},1)>u(t,1)=w(t;Q)$ for all $t\in(\hat{t},\hat{t}+\delta)$. If $\hat{t}=t^*$, equilibrium requires the weaker condition $V_0(t^*(1),1)\geq u(t^*(1),1)$, but then we have $u_t(t,1)<0$ for all $t>\hat{t}$; and so, $w(\hat{t};Q)=V_0(\hat{t},1)>u(t,1)=w(t;Q)$ for all $t>\hat{t}$. *Q.E.D.*

PROOF OF PROPOSITION 4: When the stopping payoff is monotone in quantiles, there is a unique Nash equilibrium by Proposition 1. Further, in each case, the identified equilibrium is secure, and thus safe by Lemma 3. Now consider the non-monotone case. The stopping support for any secure war of attrition equilibrium must be a single interval, since the rush occurs at $t > 0$. To satisfy indifference, the rush payoff and adjacent gradual play payoff must coincide, that is, $V_0(t, \Gamma_W(t)) = u(t, \Gamma_W(t))$, or equivalently $\Gamma_W(t) = \Pi_1(t)$. In the proof of Proposition 2B part (b), we constructed the unique war of attrition equilibrium that obeyed this equality. Altogether, our constructed war of attrition is the unique secure war of attrition equilibrium.

Section C.4 Step 2 establishes the existence of a unique secure pre-emption equilibrium given no greed at $t^*(1)$ and alarm, while Step 3 establishes a unique secure pre-emption equilibrium given no greed at $t^*(1)$ and no alarm or panic. Finally, a unit mass rush at $t = 0$ is secure, and is the unique equilibrium with an initial rush given panic by Proposition 2B parts (a) and (c). *Q.E.D.*

C.6. *All Nash Equilibria: Proofs for Characterizing the Nash Set*

PROOF OF LEMMA 4: We consider $\mathcal{R}_P$, and thus assume no greed at $t^*(1)$ and no panic.

Step 1: $\mathcal{R}_P([\underline{t}_0, \bar{t}_0]) = [\underline{q}_P, 1]$ *is Continuous*, *Increasing*, *and Exceeds* Γ_P. By Propositions 2B and 4, the unique safe initial rush $(\underline{t}_0, \underline{q}_P)$ satisfies (10). And since any equilibrium initial rush includes the peak of V_0 (Corollary 1) with V_0 falling after this peak (Lemma 2), $\underline{q}_P$ must be the largest such solution at $\underline{t}_0$, that is, $\mathcal{R}_P(\underline{t}_0) = \underline{q}_P$. Now, for the upper endpoint $\bar{t}_0$, combine the inequalities for no greed at $t^*(1)$ and no panic: $V_0(0, 1) < u(t^*(1), 1) < V_0(t^*(1), 1)$, with $V_0(t, 1)$ continuously increasing for $t < t^*(1)$ to get a unique $\bar{t}_0 < t^*(1)$ satisfying $V_0(\bar{t}_0, 1) = u(t^*(1), 1)$. That is, $\mathcal{R}_P(\bar{t}_0) = 1$. Combining $\mathcal{R}_P(\underline{t}_0) = \underline{q}_P < 1 = \mathcal{R}_P(\bar{t}_0)$, with $V_0(t, q)$ smoothly increasing in $t \le t^*(1)$ and smoothly decreasing in $q \ge \underline{q}_P$, we discover: (i) $\underline{t}_0 < \bar{t}_0$; as well as (ii) the two inequalities $V_0(t, \underline{q}_P) > u(t^*(1), 1)$ and $V_0(t, 1) < u(t^*(1), 1)$ for all $t \in (\underline{t}_0, \bar{t}_0)$. So, by $V_0(t, q)$ smoothly increasing in $t \le t^*(1) \le t^*(q)$ and smoothly decreasing in $q \ge \underline{q}_P$, the largest solution $\mathcal{R}_P$ to (10) uniquely exists for all t, and is continuously increasing from $[\underline{t}_0, \bar{t}_0]$ onto $[\underline{q}_P, 1]$ by the implicit function theorem.

We claim that $\mathcal{R}_P(t) > \Gamma_P(t)$ on $(\underline{t}_0, \bar{t}_0]$. First, $V_0(\underline{t}_0, \underline{q}_P) \ge u(\underline{t}_0, \underline{q}_P)$, else players would not stop in the safe rush $(\underline{t}_0, \underline{q}_P)$. Combining this inequality with $V_0(t, q) \ge u(t, q)$ for $q \ge \Pi_0(t)$ by Lemma 2, we find $\mathcal{R}_P(\underline{t}_0) \ge \Pi_0(\underline{t}_0)$. But then since $\mathcal{R}_P$ is increasing and Π_0 non-increasing (Lemma 2), we have $\mathcal{R}_P(t) > \Pi_0(t)$ on $(\underline{t}_0, \bar{t}_0]$; and thus, $u(t, \mathcal{R}_P(t)) < V_0(t, \mathcal{R}_P(t)) = u(t^*(1), 1)$ by Lemma 2 and (10). Altogether, given (3), we have $u(t, \mathcal{R}_P(t)) < u(t, \Gamma_P(t))$; and thus, $\mathcal{R}_P(t) > \Gamma_P(t)$ by $u_q(t, q) < 0$ for $q \ge \Pi_0(t) > q^*(t)$.

Step 2: *Local Optimality*. Formally, the candidate initial rush $\mathcal{R}_P(t)$ is undominated *iff*:

$$V_0\big(t, \mathcal{R}_P(t)\big) \ge \max\big\{u(t, 0), u\big(t, \mathcal{R}_P(t)\big)\big\}. \tag{22}$$

Step 1 established $V_0(t, \mathcal{R}_P(t)) \ge u(t, \mathcal{R}_P(t))$ on $[\underline{t}_0, \bar{t}_0]$. When inequality (11) holds, we trivially have $V_0(0, \mathcal{R}_P(0)) \ge u(0, 0)$: a $t = 0$ rush of size $\mathcal{R}_P(0)$ is undominated. Thus, we henceforth assume that the lower bound on the domain of $\mathcal{R}_P$ is strictly positive: $\underline{t}_0 > 0$. But, since this lower bound is defined as the unique safe initial rush time, Proposition 4 asserts that we cannot have alarm or panic. In this case, the proof of

Proposition 2B part (a) Step 3 and Proposition 4 establish that the safe rush size obeys $\mathcal{R}_P(\underline{t}_0) = \arg\max_q V_0(\underline{t}_0, q)$, but then $V_0(\underline{t}_0, \mathcal{R}_P(\underline{t}_0)) > u(\underline{t}_0, 0)$. Now, since $V_0(t, \mathcal{R}_P(t))$ is constant in t on $[\underline{t}_0, \bar{t}_0]$ by (10) and $u(t, 0)$ is increasing in t on this domain, we either have $V_0(t, \mathcal{R}_P(t)) \geq u(t, 0)$ for all $t \in [\underline{t}_0, \bar{t}_0]$, in which case we set $\bar{t} \equiv \bar{t}_0$, or there exists $\bar{t} < \bar{t}_0$ such that $V_0(t, \mathcal{R}_P(t)) \gtreqless u(t, 0)$ as $t \lesseqgtr \bar{t}$ for $t \in [\underline{t}_0, \bar{t}_0]$. In either case, $\mathcal{R}_P(t)$ satisfies (22) for all $t \in [\underline{t}_0, \bar{t}]$, but for any $t \in (\bar{t}, \bar{t}_0]$, inequality (22) is violated. *Q.E.D.*

PROOF OF PROPOSITION 7: Let $\mathcal{Q}_{\mathrm{NP}}$ be the set of pre-emption equilibria. With greed at $t^*(1)$ or panic, $\mathcal{Q}_{\mathrm{NP}}, \mathcal{Q}_P = \emptyset$ by Proposition 2B. Henceforth, assume no greed at $t^*(1)$ and no panic.

Step 1: $\mathcal{Q}_{\mathrm{NP}} \subseteq \mathcal{Q}_P$. By Proposition 2A, pre-emption equilibria share Nash payoff $u(t^*(1), 1)$ and involve an initial rush. Thus, any equilibrium rush must satisfy $u(t^*(1), 1) = V_0(t, q) \geq u(t, q)$, that is, be of size $q = \mathcal{R}_P(t)$ at a time $t \in [\underline{t}_0, \bar{t}_0]$. Further, by Proposition 2A, there can only be a single inaction phase separating this rush from an uninterrupted gradual play phase obeying (3), which Lemma 1 establishes uniquely defines Γ_P. Finally, by Lemma 4, the interval $[\underline{t}_0, \bar{t}]$ are the only times for which stopping in the initial rush $\mathcal{R}_P(t)$ is undominated.

Step 2: $\mathcal{Q}_P \subseteq \mathcal{Q}_{\mathrm{NP}}$. Recall that $w(t; Q)$ is the payoff to stopping at time $t \geq 0$ given quantile function Q. Let $Q_S \in \mathcal{Q}_P$ be the unique safe pre-emption equilibrium and consider an arbitrary $Q \in \mathcal{Q}_P$ with an initial rush at t_0, inaction on (t_0, t_P), and gradual play following $\Gamma_P(t)$ on $[t_P, 1]$. By construction, the stopping payoff is $u(t^*(1), 1)$ for all $t \in \mathrm{supp}(Q)$. Further, since Q_S is an equilibrium and $Q(t) = Q_S(t)$ on $[0, \underline{t}_0)$ and $[t_P, \infty)$, we have $w(t, Q) = w(t, Q_S) \leq u(t^*(1), 1)$ on these intervals: No player can gain by deviating to either $[0, \underline{t}_0)$ or $[t_P, \infty)$. By Lemma 4, $\mathcal{R}_P(t) \geq \Gamma_P(t)$ on the inaction interval (t_0, t_P) and thus $V_0(t, \mathcal{R}_P(t)) \geq u(t, q)$: No player can gain from deviating to the inaction interval. Finally, consider the interval $[\underline{t}_0, t_0)$ on which $Q(t) = 0$. Since $u(t, 0)$ is increasing on this interval, no player can gain from pre-empting the rush at t_0 provided $V_0(t_0, \mathcal{R}_P(t_0)) > u(t_0, 0)$, which is ensured by (22).

Step 3: *Covariate Predictions*. Consider $Q_1, Q_2 \in \mathcal{Q}_P$, with rush times $t_1 < t_2$. The rush sizes obey $\mathcal{R}_P(t_1) < \mathcal{R}_P(t_2)$ by $\mathcal{R}_P$ increasing (Lemma 4). Gradual play start times are ordered $\Gamma_P^{-1}(\mathcal{R}_P(t_1)) < \Gamma_P^{-1}(\mathcal{R}_P(t_2))$ by Γ_P^{-1} increasing (Lemma 1). Thus, gradual play durations obey $t^*(1) - \Gamma_P^{-1}(\mathcal{R}_P(t_1)) > t^*(1) - \Gamma_P^{-1}(\mathcal{R}_P(t_2))$. Finally, by construction, $Q_1(t) = Q_2(t) = \Gamma_P(t)$ on the intersection of the gradual play intervals $[\Gamma_P^{-1}(\mathcal{R}_P(t_2)), t^*(1)]$. *Q.E.D.*

C.7. *Comparative Statics: Proofs for Changes in Payoffs (Section 7 and Section 8)*

LEMMA C.2—Rush Loci Changes: *Assume a co-monotone delay. The initial and terminal rush loci $\mathcal{R}_P(t), \mathcal{R}_W(t)$ fall. As a consequence, the initial rush with alarm $q_0 = \mathcal{R}_P(0)$ falls.*

PROOF: Rewriting (10), we see that any initial rush is defined by the indifference equation:

$$\mathcal{R}_P(t)^{-1} \int_0^{\mathcal{R}_P(t)} \frac{u(t, x|\varphi)}{u(t, 1|\varphi)}\, dx = \frac{u\big(t^*(1|\varphi), 1|\varphi\big)}{u(t, 1|\varphi)}. \tag{23}$$

Since the initial rush $\mathcal{R}_P(t) \geq \Pi_0(t)$ (by Corollary 1), the LHS of (23) falls in $\mathcal{R}_P(t)$, while the LHS falls in φ by log-supermodularity of u in (q, φ). Now, (23) shares the RHS of (8), shown increasing in φ in the proof of Proposition 5. So, the initial rush locus obeys $\partial\mathcal{R}_P/\partial\varphi < 0$. *Q.E.D.*

Final Steps for Proposition 6

The following lemma completes Proposition 6.

LEMMA C.3: *In the safe war of attrition equilibrium, the terminal rush shrinks in greed. In the safe pre-emption equilibrium with no alarm, the initial rush shrinks in fear.*

We prove the result for the pre-emption case; the war of attrition logic is symmetric.

Step 1: *Preliminaries*. First we claim that

$$V_0(t,q|\varphi) \geq u(t,q|\varphi) \quad \Rightarrow \quad q^{-1}\int_0^q u_t(t,x|\varphi)\,dx \geq u_t(t,q|\varphi). \tag{24}$$

Indeed, using $u(t,q|\varphi)$ log-submodular in (t,q),

$$\begin{aligned}\frac{1}{q}\int_0^q \frac{u_t(t,x|\varphi)}{u(t,q|\varphi)}\,dx &= \frac{1}{q}\int_0^q \frac{u_t(t,x|\varphi)}{u(t,x|\varphi)}\frac{u(t,x|\varphi)}{u(t,q|\varphi)}\,dx \\ &\geq \frac{u_t(t,q|\varphi)}{qu(t,q|\varphi)}\int_0^q \frac{u(t,x|\varphi)}{u(t,q|\varphi)}\,dx \geq \frac{u_t(t,q|\varphi)}{u(t,q|\varphi)}.\end{aligned}$$

Define $\nu(t,q,\varphi)\equiv u(t,q|\varphi)/u(t,1|\varphi)$, $\nu^*(t,\varphi)\equiv u(t^*(1),1|\varphi)/u(t,1|\varphi)$, and $\mathcal{V}(t,q,\varphi)=q^{-1}\int_0^q \nu(t,x,\varphi)\,dx$. By u log-modular in (t,φ), $\nu^*_\varphi=0$, while $\nu^*_t \lesseqgtr 0$ as $u_t \gtreqless 0$. By log-submodularity in (t,q) and log-supermodularity in (q,φ), $\nu_t>0$ and $\nu_\varphi<0$.

Step 2: $\mathcal{V}_t \geq \nu_t$ *and* $-\mathcal{V}_\varphi > -\nu_\varphi$ *for all* (t,q) *satisfying* (5), *that is*, $q=\Pi_0(t)$.

$$\begin{aligned}\mathcal{V}_t-\nu_t &= q^{-1}\int_0^q\left[\frac{u_t(t,x|\varphi)}{u(t,1|\varphi)}-\frac{u(t,x|\varphi)u_t(t,1|\varphi)}{u(t,1|\varphi)^2}\right]dx \\ &\quad -\left[\frac{u_t(t,q|\varphi)}{u(t,1|\varphi)}-\frac{u(t,q|\varphi)u_t(t,1|\varphi)}{u(t,1|\varphi)^2}\right] \\ &\geq -q^{-1}\int_0^q\left[\frac{u(t,x|\varphi)u_t(t,1|\varphi)}{u(t,1|\varphi)^2}\right]dx+\frac{u(t,q|\varphi)u_t(t,1|\varphi)}{u(t,1|\varphi)^2} \quad \text{by (24)} \\ &= \frac{u_t(t,1|\varphi)}{u(t,1|\varphi)^2}\left[u(t,q|\varphi)-q^{-1}\int_0^q u(t,x|\varphi)\,dx\right]=0 \quad \text{by (5)}.\end{aligned}$$

Since u is strictly log-supermodular in (q,φ), symmetric steps establish that $-\mathcal{V}_\varphi > -\nu_\varphi$.

Step 3: *A Difference in Derivatives.* Lemma 1 proved $\Gamma_P'(t)>0$, Lemma 2 established $\Pi_0'(t)\leq 0$, while the in-text proof of Proposition 6 showed $\partial\Gamma_P/\partial\varphi<0$ and $\partial\Pi_0/\partial\varphi>0$. We now finish the proof that the initial rush rises in φ by proving that starting from any (t,q,φ), satisfying $q=\Gamma_P(t)=\Pi(t)$ and holding q fixed, the change $dt/d\varphi$ in the gradual play locus (3) is smaller than the $dt/d\varphi$ in the peak rush locus (5). Evaluating both derivatives, this entails

$$\frac{\nu_\varphi(t,q,\varphi)-\mathcal{V}_\varphi(t,q,\varphi)}{\mathcal{V}_t(t,q,\varphi)-\nu_t(t,q,\varphi)} > \frac{-\nu_\varphi(t,q,\varphi)}{\nu_t(t,q,\varphi)-\nu^*_t(t)}. \tag{25}$$

Since $u_t>0$ during a pre-emption game, we have $\nu^*_t<0$, $\nu_t>0$, and $\nu_\varphi<0$ by Step 1; so that inequality (25) is satisfied if $\nu_t(\nu_\varphi-\mathcal{V}_\varphi)>-\nu_\varphi(\mathcal{V}_t-\nu_t)\Leftrightarrow -\mathcal{V}_\varphi\nu_t>-\nu_\varphi\mathcal{V}_t$, which follows from $\mathcal{V}_t\geq\nu_t>0$ and $-\mathcal{V}_\varphi>-\nu_\varphi>0$ as established in Steps 1 and 2. *Q.E.D.*

Set Comparative Statics

We now prove the comparative statics claims of Section 8.

LEMMA C.4: *Assume a harvest delay or increase in greed $\varphi_H > \varphi_L$, with $q_H = \mathcal{R}_P(t_H|\varphi_H)$ undominated. If $t_L = \mathcal{R}_P^{-1}(q_H|\varphi_L) \geq \underline{t}_0(\varphi_L)$, then $\mathcal{R}_P(t_L|\varphi_L)$ is undominated.*

Step 1: *Harvest Delay*. If $q_H = \mathcal{R}_P(t_H|\varphi_H)$ satisfies inequality (22), then

$$\begin{aligned} 1 &\leq \int_0^{q_H} \frac{u(t_H, x|\varphi_H)}{q_H u(t_H, 0|\varphi_H)}\, dx = \int_0^{q_H} \frac{u(t_H, x|\varphi_L)}{q_H u(t_H, 0|\varphi_L)}\, dx \\ &\leq \int_0^{q_H} \frac{u(t_L, x|\varphi_L)}{q_H u(t_L, 0|\varphi_L)}\, dx = \frac{V_0(t_L, q_H|\varphi_L)}{u(t_L, 0|\varphi_L)}, \end{aligned}$$

where the first equality follows from log-modularity in (q, φ) and the inequality owes to u log-submodular in (t, q) and $t_L < t_H$ by $\mathcal{R}_P$ falling in φ (Lemma C.2). We have shown $V_0(t_L, q_H|\varphi_L) \geq u(t_L, 0|\varphi_L)$, while $t_L \geq \underline{t}_0(\varphi_L)$ by assumption. Together, these two conditions are sufficient for $\mathcal{R}_P(t_L|\varphi_L)$ undominated, as shown in the proof of Lemma 4 Step 2.

Step 2: *Increase in Greed.* By Proof Step 2 for Lemma 4, $\mathcal{R}_P(t|\varphi)$ is undominated for $t \in [\underline{t}_0(\varphi), \bar{t}_0(\varphi)]$ *iff* $u(t, 0|\varphi) \leq V_0(t, \mathcal{R}_P(t|\varphi)|\varphi)$. Given $u(t, 0|\varphi)$ increasing in $t \leq \bar{t}_0(\varphi)$ and $V_0(t, \mathcal{R}_P(t|\varphi)|\varphi)$ constant in t by (10), if the largest undominated time $\bar{t}(\varphi) < \bar{t}_0(\varphi))$, it solves

$$\bar{V}\big(\bar{t}(\varphi), \mathcal{R}_P\big(\bar{t}(\varphi)|\varphi\big), \varphi\big) = 1, \quad \text{where} \quad \bar{V}(t, q, \varphi) \equiv \frac{V_0(t, q|\varphi)}{u(t, 0|\varphi)}. \tag{26}$$

By assumption, $t_L \geq \underline{t}_0(\varphi_L)$. For a contradiction, assume t_L is not undominated: $t_L > \bar{t}(\varphi_L)$.

We claim that starting from any $(\bar{t}, q, \varphi)$ satisfying both (10), that is, $q = \mathcal{R}_P(\bar{t}|\varphi)$, and (26), that is, $\bar{V}(\bar{t}, q, \varphi) = 1$, the change in the rush locus $dR_P^{-1}(q|\varphi)/d\varphi$, holding q fixed, exceeds the change along (26) $d\bar{t}/d\varphi$, holding q fixed. Indeed, defining $h(t, q, \varphi) \equiv u(t^*(1), 1|\varphi)/u(t, 0|\varphi)$ and differentiating, we discover $dR_P^{-1}(q|\varphi)/d\varphi - d\bar{t}/d\varphi = \bar{V}_\varphi/(h_t - \bar{V}_t) - \bar{V}_\varphi/(-\bar{V}_t) > 0$, where the inequality follows from $h_t < 0$ (by $u_t > 0$ for $t < t^*(1|\varphi)$), $h_\varphi = 0$ (by u log-modular in (t, φ)), $\bar{V}_t \leq 0$ (by u log-submodular in (t, q)), $\bar{V}_\varphi > 0$ (by u log-supermodular in (q, φ)), and $h_t - \bar{V}_t \geq 0$ (else $\mathcal{R}(t|\varphi)$ falls in t, contradicting Lemma 4). Altogether, given $\bar{q}_L \equiv \mathcal{R}_P(\bar{t}(\varphi_L)|\varphi)$, we have shown $\bar{t}(\varphi_L) = \mathcal{R}_P^{-1}(\bar{q}_L|\varphi_H) \geq \bar{t}(\varphi_H)$; and thus, $t_L > \bar{t}(\varphi_H)$, but this contradicts $t_H > t_L$ (by $\mathcal{R}_P(\cdot|\varphi)$ falling in φ) and $t_H \leq \bar{t}(\varphi_H)$ (by $q_H = \mathcal{R}_P(t_H|\varphi)$ undominated). *Q.E.D.*

Common Steps

Consider the sets $\mathcal{Q}_P(\varphi_H)$ and $\mathcal{Q}_P(\varphi_L)$ for a co-monotone delay $\varphi_H > \varphi_L$. The results vacuously hold if $\mathcal{Q}_P(\varphi_H)$ is empty. Henceforth assume not. By Proposition 2B, $\mathcal{Q}_P(\varphi_H)$ non-empty implies no greed at $t^*(1|\varphi_H)$, which in turn implies no greed at $t^*(1|\varphi_L)$ by $\int_0^1 [u(t, x|\varphi)/u(t, 1|\varphi)]\, dx$ falling in φ (by log-supermodularity in (q, φ)), rising in t (by log-submodularity in (t, q)), and $t^*(1|\varphi_H) \geq t^*(1|\varphi_L)$ (Propositions 5 and 6). Then, since we have assumed no panic at φ_L, $\mathcal{Q}_P(\varphi_L)$ is non-empty, containing at least the safe preemption equilibrium by Proposition 2B. Two results follow. First, by Proposition 2B, preemption games end at $t^*(1|\varphi)$, while Proposition 5 asserts $t^*(1|\varphi_H) > t^*(1|\varphi_L)$ for a harvest delay and Proposition 6 claims $t^*(1|\varphi_H) = t^*(1|\varphi_L)$ for an increase in greed. Thus, gradual play end times are ordered as claimed for *any* $Q_H \in \mathcal{Q}_P(\varphi_H)$ and $Q_L \in \mathcal{Q}_P(\varphi_L)$. Likewise, the exit rates are ordered as claimed for all $Q_H \in \mathcal{Q}_P(\varphi_H)$ and $Q_L \in \mathcal{Q}_P(\varphi_L)$, since Γ_P' rises in φ by Propositions 5 and 6 and $Q \in \mathcal{Q}_P(\varphi)$ share $Q'(t) = \Gamma_P'(t)$ on any common gradual play interval by Proposition 7.

By construction, choosing $Q_H \in \mathcal{Q}(\varphi_H)$ is equivalent to choosing a rush time t_H in the undominated interval $[\underline{t}_0(\varphi_H), \bar{t}(\varphi_H)]$ characterized by Lemma 4. Let $q_H \equiv \mathcal{R}_P(t_H|\varphi_H)$ be the associated rush. Let $\underline{t}_0(\varphi_L)$ and $\underline{q}_L = \mathcal{R}_P(\underline{t}_0(\varphi_L)|\varphi_L)$ be the safe rush time and size for φ_L.

Final Steps for Fundamental Changes

Assume a harvest delay $\varphi_H > \varphi_L$.

Case 1: $\underline{q}_L > q_H$. Let Q_L be the safe pre-emption equilibrium. By Proposition 5, the safe rush times obey $\underline{t}_0(\varphi_L) \leq \underline{t}_0(\varphi_H)$, while $\underline{q}_L > q_H$ by assumption: Q_L has a larger, earlier rush than Q_H, as claimed. Since $\Gamma_P(t|\varphi)$ is increasing in t by Lemma 1 and decreasing in φ by Proposition 5, the inverse function $\Gamma_P^{-1}(q|\varphi)$ is increasing in q and decreasing in φ. Thus, gradual play start times obey $\Gamma_P^{-1}(q_H|\varphi_H) > \Gamma_P^{-1}(\underline{q}_L|\varphi_L)$, as claimed. Altogether, $Q_L \geq Q_H$, since Q_L has a larger and earlier rush, an earlier start and end time to gradual play, and the gradual play c.d.f.s are ordered $\Gamma_P(t|\varphi_L) > \Gamma_P(t|\varphi_H)$ on the common gradual play support.

Case 2: $\underline{q}_L \leq q_H$. Since Q_H is an equilibrium, $q_H = \mathcal{R}_P(t_H|\varphi_H)$ is undominated. And by Lemma 4, $\mathcal{R}_P(\cdot|\varphi_L)$ is continuously increasing with domain $[\underline{q}_L, 1]$, which implies $t_L \equiv \mathcal{R}_P^{-1}(q_H|\varphi_L) \geq \underline{t}_0(\varphi_L)$ exists. Thus, $q_H = \mathcal{R}_P(t_L|\varphi)$ is undominated by Lemma C.4, and t_L defines an equilibrium $Q_L \in \mathcal{Q}_P(\varphi_L)$. Further, Q_L and Q_H have the same size rush by construction, while rush times are ordered $t_L < t_H \equiv \mathcal{R}_P^{-1}(q_H|\varphi_H)$ by $\mathcal{R}_P$ falling in φ (Lemma C.2). Now, since $\Gamma_P(t|\varphi)$ is increasing in t (Lemma 1) and falling in φ (Proposition 5), gradual play start times obey $\Gamma_P^{-1}(\mathcal{R}_P(t_L|\varphi_L)|\varphi_L) < \Gamma_P^{-1}(q_H|\varphi_H)$, as required. Altogether, $Q_L \geq Q_H$, since Q_L has the same size rush, occurring earlier, an earlier start and end time to gradual play, and the gradual play c.d.f.s are ordered $\Gamma_P(t|\varphi_L) > \Gamma_P(t|\varphi_H)$ on any common gradual play interval.

Final Steps for Quantile Changes

Since $\mathcal{Q}_P(\varphi_H) = \emptyset$ with panic (Proposition 2B), assume no panic at φ_H. The proof for alarm at φ_L parallels the above steps for fundamental changes. We henceforth assume no alarm at φ_L, and since the premise assumes no panic at φ_L, inequality (4) obtains at φ_L. But then, since $V_0(0, q|\varphi)/u(t^*(1|\varphi), 1|\varphi)$ falls in φ by u log-modular in (t, φ) and log-supermodular in (q, φ), inequality (4) also obtains at φ_H. Altogether, neither alarm nor panic obtain at φ_L and φ_H. Then, by Proposition 6, safe rush times obey $\underline{t}_0(\varphi_L) < \underline{t}_0(\varphi_H) \leq t_H$ with sizes $\underline{q}_L < \mathcal{R}_P(\underline{t}_0(\varphi_H)|\varphi_H) \equiv \underline{q}_H$. By Lemma 4, $\mathcal{R}_P(\cdot|\varphi_L)$ is continuously increasing onto domain $[\underline{q}_L, 1] \supset [\underline{q}_H, 1]$; and thus, $t_L \equiv \mathcal{R}_P^{-1}(q_H|\varphi_L) > \underline{t}_0(\varphi_L)$ uniquely exists, is undominated by Lemma C.4, and satisfies $t_L < t_H$ by $\mathcal{R}_P(\cdot|\varphi)$ falling in φ (Proposition 6). Altogether, t_L defines $Q_L \in \mathcal{Q}_P(\varphi_L)$ with an earlier rush of the same size as Q_H, as claimed. The function $\Gamma_P(t|\varphi)$ is increasing in t (Lemma 1) and decreasing in φ (Proposition 6): Gradual play start times obey $\Gamma_P^{-1}(\mathcal{R}_P(t_L|\varphi_L)|\varphi_L) < \Gamma_P^{-1}(q_H|\varphi_H)$, as required. Altogether, $Q_L \geq Q_H$ as claimed, since Q_L has the same size rush, occurring earlier, an earlier start and same end time to gradual play, and gradual play c.d.f.s obey $\Gamma_P(t|\varphi_L) > \Gamma_P(t|\varphi_H)$ on any common gradual play interval. *Q.E.D.*

C.8. *Asset Bubble and Bank Run Payoffs: Omitted Proofs (Section 9)*

LEMMA C.5: *The bubble payoff* (13) *is log-submodular in* (t,q), *and log-concave in* t *and* q.

PROOF: That $\kappa(t+\tau(t,q))\equiv q$ yields $\kappa'(t+\tau(t,q))(1+\tau_t(t,q))=0$ and $\kappa'(t+\tau(t,q))\tau_q(t,q)=1$. So, $\tau_t\equiv -1$ and $\tau_q<0$ given $\kappa'<0$. Hence, $\tau_{tq}=0$, $\tau_{tt}=0$, and $\tau_{qq}=-(\kappa''/\kappa')(\tau_q)^2$. Thus,

$$\frac{\partial^2\log(F(\tau(t,q)))}{\partial t\partial q}F(\tau(t,q))^2=[FF''-(F')^2]\tau_t\tau_q+FF'\tau_{tq}=[FF''-(F')^2]\tau_t\tau_q\leq 0.$$

Twice differentiating $\log(F(\tau(q,t)))$ in t likewise yields $[FF''-(F')^2]/F^2\leq 0$. Similarly,

$$\partial^2\log(F(\tau(t,q)))/\partial q^2=(\tau_q)^2[FF''-(F')^2-(\kappa''/\kappa')FF']/F^2\leq 0,$$

where $-\kappa''/\kappa'\leq 0$ follows since κ is decreasing and log-concave. *Q.E.D.*

LEMMA C.6: *The bank run payoff* (15) *is log-submodular in* (q,α). *This payoff is log-supermodular in* (q,R) *provided the elasticity* $\zeta H'(\zeta|t)/H(\zeta|t)$ *is weakly falling in* ζ.

PROOF: By Lemma 2.6.4 in Topkis (1998), u is log-submodular in (q,α), as H is monotone and log-concave in ζ, and $1-\alpha q$ is monotone and submodular in (q,α). It is log-supermodular in (q,R):

$$\frac{\partial^2\log(H(\cdot))}{\partial q\partial R}H(\cdot)^2(1-R)^2=\left(\frac{1-\alpha q}{1-R}\right)\left((H')^2-HH''\right)-HH'\geq 0,\tag{27}$$

that is, $x(H'(x)^2-H(x)H''(x))-H(x)H'(x)\geq 0$, namely, with $xH'(x)/H(x)$ weakly falling. *Q.E.D.*

C.9. *Payoff Heterogeneity: Unique Gradual Play Loci*

Let t_0 (t_1) be the initial (terminal) rush time in the safe pre-emption (war of attrition) equilibrium.

LEMMA C.7: *For any original game, there exist* $\delta^*>0$ *and* $\lambda\geq 0$, *such that, for all* $\delta\leq\delta^*$:
(a) *Given no fear at* $t^*(0)$ *in the original game, there exists a unique solution* $\Gamma_W(t|\delta)$ *to* (18) *on* $[t_W(\delta), t_1+\lambda]$ *satisfying* $Q_\delta(t_W(\delta))=0$, *which is continuous in* δ.
(b) *Given no greed at* $t^*(1)$ *and no alarm or panic in the original game, there exists a unique solution* $\Gamma_P(t|\delta)$ *to* (18) *on* $[t_0-\lambda, t_P(\delta)]$ *satisfying* $Q_\delta(t_P(\delta))=1$, *which is continuous in* δ.
(c) *Given no greed at* $t^*(1)$ *and alarm in the original game, there exists a unique solution* $\Gamma_P(t|\delta)$ *to* (18) *on* $[0, t_P(\delta)]$ *satisfying* $Q_\delta(t_P(\delta))=1$, *which is continuous in* δ.

We prove part (b). The proof for parts (a) and (c) follow similar steps.

Step 1: *The MRS is Smooth.* The marginal rate of substitution $u_t(t,q,\mathcal{E}_\delta(q))/u_q(t,q,\mathcal{E}_\delta(q))$ is continuous in t and δ, and Lipschitz in q on any non-empty set $[t_L,t_H]\times[q_L,q_H]$ for which u_q is uniformly bounded away from zero (in t). Indeed, continuity in t follows from u_t and u_q C^1 in t and u_q non-zero. Lipschitz continuity in q and continuity in δ follow from u_q uniformly bounded away from zero, u_t and u_q C^1 in $\varepsilon=\mathcal{E}_\delta(q)$, and the inverse $\mathcal{E}_\delta(q)$ of the C^1 (in (δ,q)) function $Y_\delta(q)$ C^1 in q and δ (by the implicit function theorem).

Step 2: *Uniformly Bounding* u_q. When u is monotone in q, we assumed $|u_q|$ uniformly bounded away from 0 (in t), and thus there exists $\delta^* > 0$ and $B > 0$ such that $|u_q(t, q, \mathcal{E}_\delta(q))| > B$ for all $\delta < \delta^*$ by $u_q(t, q, \varepsilon) \to u_q(t, q)$ as $\varepsilon \to 0$ and $\mathcal{E}_\delta(q) \to 0$ as $\delta \to 0$.

Next, consider the non-monotone case, that is, $q^*(t) \in (0, 1)$. In this case, u is log-concave in q, while u is always log-submodular in (t, q): The ratio u/u_q is non-decreasing in t and q. Thus, since $u > 0$ and $u_q(t, q) < 0$ for $q > q^*(t)$, if $u_q(t_L, q_L) < -B'$ for some $B' > 0$, t_L, and $q_L > q^*(t_L)$, then $u_q(t, q) < -B'$ for all $(t, q) > (t_L, q_L)$. Now, given no greed at $t^*(1)$ and no alarm or panic, the safe rush at $t_0 > 0$ is of size $\Pi_0(t_0) \in (q^*(t_0), 1)$. Thus, setting $-B' = u_q(t_0, \Pi_0(t_0))$, we have $u_q(t, q) \le -B' < 0$ on $[t_0, \infty) \times [\Pi_0(t_0), 1]$. Finally, u is continuous in (t, q) and $u_q(t, q, \mathcal{E}_\delta(q)) \to u_q(t, q)$ as $\delta \to 0$, thus there exist $(\delta^*, B, \lambda_1, \lambda_2) > 0$, such that $u_q(t, q, \mathcal{E}_\delta(q)) < -B$ on $[t_0 - \lambda_1, \infty) \times [\Pi_0(t_0) - \lambda_2, 1]$ for all $\delta \le \delta^*$.

Step 3: *Existence, Uniqueness, and Continuity.* First, there exist $\delta^* > 0$ and $M_1 > 0$, such that there exists a unique solution $\Gamma_P(t|\delta)$ to (18) on $[t_P(\delta) - M_1, t_P(\delta)]$ satisfying $Q_\delta(t_P(\delta)) = 1$ for all $\delta \le \delta^*$, and this solution is continuous in δ. Indeed, $t_P(\delta) \to t^*(1)$ as shown in the text after Proposition A.2, while $t_0 < t^*(1)$; and thus, by Steps 1 and 2, there exist $\delta^*, M_1, M_2 > 0$ such that $u_t(t, q, \mathcal{E}_\delta(q))/u_q(t, q, \mathcal{E}_\delta(q))$ is well-defined (u_q nonzero), continuous in t, and Lipschitz in q on $[t_P(\delta) - M_1, \infty] \times [1 - M_2]$ for all $\delta \le \delta^*$. Thus, there exists a unique solution $\Gamma_P(t|\delta)$ to (18) on $[t_P(\delta) - M_1, t_P(\delta)]$ by the Picard–Lindelof theorem. Further, since the MRS is continuous in δ on this interval and $t_P(\delta)$ is continuous in δ, Theorem 2.6 in Khalil (1992) yields $\Gamma_P(t|\delta)$ continuous in $\delta \le \delta^*$ on this interval.

If $t_P(\delta) - M_1 < t_0$, we are done. Otherwise, we recursively define $\Gamma_P(t|\delta)$ on the time intervals $I(n, \delta) \equiv [\max\{t_P(\delta) - nM_1, t_0 - \lambda_1\}, t_P(\delta) - (n-1)M_1]$ for $n = 1, \dots, N$, where N satisfies $t_P(\delta) - NM_1 < t_0 < t_P(\delta) - (N-1)M_1$. The prior paragraph proved that for sufficiently small δ, the solution $\Gamma_P(t|\delta)$ to (18) satisfying $Q_\delta(t_P(\delta)) = 1$ uniquely exists on $I(1, \delta)$ and is continuous in δ. For $n = 2, \dots, N$, let $\Gamma_P(t|\delta)$ be the unique solution to (18) on $I(n, \delta)$, obeying terminal condition $Q_\delta(t_P(\delta) - (n-1)M_1) = \Gamma_P(t_P(\delta) - (n-1)M_1|\delta)$. To see that $\Gamma_P(t|\delta)$ is uniquely defined and continuous in δ on $I(n, \delta)$ for $2 \le n \le N$, note that if $\Gamma_P(t|\delta)$ uniquely exists and is continuous in δ on $I(n-1, \delta)$, then the terminal condition $Q_\delta(t_P(\delta) - (n-1)M_1) = \Gamma_P(t_P(\delta) - (n-1)M_1|\delta)$ is well-defined and continuous in δ, converging to $\Gamma_P(t^*(1) - (n-1)M_1) > \Gamma_P(t_0) \ge \Pi_0(t_0)$. Thus, given $\lambda_2 > 0$ defined in Step 2, there exists $\delta^* > 0$ such that the terminal condition exceeds $\Pi_0(t_0) - \lambda_2$ for all $\delta \le \delta^*$, while the interval $I(n, \delta) \subset [t_0 - \lambda_1, t_P(\delta)]$. Thus, we may WLOG restrict attention to $(t, q) \in [t_0 - \lambda_1, \infty) \times [\Pi_0(t_0) - \lambda_2, 1]$ for which we establish in Steps 1 and 2 that the conditions for the Picard–Lindelof theorem and Theorem 2.6 in Khalil (1992) hold with Lipschitz constant B across all $I(n, \delta)$, justifying the use of the same constant M_1 for all n. Altogether, $\Gamma_P(t|\delta)$ uniquely exists and is continuous in δ on an interval $[t_0 - \lambda_1, t_P(\delta)]$ for all $\delta \le \delta^*$. *Q.E.D.*

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