Ordinal Aggregation Results via Karlin's Variation Diminishing Property[*]

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Abstract

When is the weighted sum of quasi-concave functions quasi-concave? We answer this, extending an analogous preservation of the single-crossing property in QS: Quah and Strulovici (2012). Our approach develops a general preservation of n-crossing properties, applying the variation diminishing property in Karlin (1956). The QS premise is equivalent to Karlin's total positivity of order two, while our premise uses total positivity of order three: The weighted sum of quasi-concave functions is quasi-concave if each has an increasing portion more risk averse than any decreasing portion.

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1 Introduction

We comment on Quah and Strulovici (2012) (QS), who posed an important question: When is the sum of two single-crossing functions also single-crossing (SC)? They deduced a property on pairs of functions, *signed-ratio monotonicity* (SRM), asking that whenever they have opposite signs, the positive one increases proportionately faster than the negative one. They showed that the sum of single-crossing functions obeying SRM is single-crossing; they applied this condition to Bayesian equilibria and decisions under uncertainty.

This paper builds on the classic *variation diminishing property* (VDP) of Karlin (1956, 1968). Assume that a kernel operator is *totally positive of order* n, meaning that it has n positive principal determinant signs. The VDP asserts that if a function has at most $m \leq$

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$n-1$ sign changes, then so too does its integral transform. Log-supermodularity, the simplest case of total positivity of order two, preserves single-crossing properties (Karlin and Rubin, 1955).[1] In fact, QS follows from the single-crossing $n=2$ case of the VDP. For aggregation can be written as a kernel transformation, and QS impose the same properties on the aggregated functions as Karlin puts on the kernel. Indeed, let f_+ and f_- be the positive and negative parts of a function f, so that $K(x,1) \equiv f_- + g_-$ and $K(x,2) \equiv f_+ + g_+$ are nonnegative. Consider single-crossing functions f and g, and assume WLOG $f(x) > 0 > g(x)$ for an interval of x. SRM requires $f_+(x)/g_-(x)$ to be non-decreasing in this interval. Since $K(x,2)/K(x,1) = f_+(x)/g_-(x)$ in this interval, SRM says that $K(x,i)$ is totally positive of order two, i.e. $K(x,2)/K(x,1)$ is non-decreasing in x. So the sum $f+g = -K(x,1)+K(x,2) = K(x,1)[K(x,2)/K(x,1)-1]$ single crosses.

Our novel contribution pursues the two-crossing $n=3$ case of the VDP. Finding that total positivity of order three admits a simple economic formulation using risk aversion, we derive a novel condition for quasi-concavity preservation under summation and integration: *the weighted integral of quasi-concave functions is quasi-concave if the increasing portion of each is more risk averse than any decreasing portion* (à la Arrow-Pratt). We conclude by noting that decompositions for preservation of single crossing and quasi-concavity are special cases of n-crossing preservation and n-turning point preservation.

2 Total Positivity and the Variation Diminishing Property

Let X and Z be linearly ordered space, namely endowed with a complete order relation, like the reals. Let σ be a sigma-finite positive measure on Z. A function $K : X \times Z \to \mathbb{R}$ is *totally positive of order* n (TP_n) if for each $k = 1, \ldots, n$, and each $x_1 < \cdots < x_k$ and $z_1 < \cdots < z_k$, the determinant of the $k \times k$ matrix $[K(x_i, z_j)]$ is nonnegative. In particular, K is a non-negative kernel. Any strictly positive bivariate function is TP_2 if and only if it is log-supermodular. If K is TP_n for all n, then K is *totally positive* (TP).

For $m = 1, 2, \ldots$, let (x_i) have $s(x_1, x_2, \cdots, x_m)$ sign changes, discarding zero terms. For example, $s(1,0,0,1,-1) = s(1,1,-1) = 1$, as the 0 terms are dropped. Let $S[f] = \sup s(f(x_1), \cdots, f(x_m))$ be the number of sign changes of any function $f : X \to \mathbb{R}$ over all $x_1 < \cdots < x_m$, where $m < \infty$ is arbitrary. If $S[f] \leq n$, call f an *n-crossing function*.

Assume that K and h are bounded Borel functions and that $f(x) = \int_Z K(x,z)h(z)d\sigma(z)$ is a finite kernel transformation. Then:

[1] Descartes' well-known "Rule of Sign" is an old manifestation of the VDP, but the VDP has seen applications in convolutions, mixture models (Lindsay and Roeder, 1993), and the comparative statics of risk aversion with many risk sources (Jewitt, 1988).

Theorem (Variation Diminishing Property)**.** *If $K(x, z)$ is TP_r and $S[h] = n \leq r - 1$, then $S[f] \leq n$. Also, if h is piecewise-continuous and $S[f] = S[h]$, then f and h have the same sign sequence when their arguments traverse their domains from left to right.*

Define $\gamma(x, y)$ as the *primary function* and μ a sigma-finite positive measure on an arbitrary measurable space Y. In many economic applications, we are confronted with aggregation problems of the form:

$$F(x) = \int_Y \gamma(x, y) d\mu(y) \tag{1}$$

where F is the *aggregation function*. For instance, μ may be a probability measure over an uncertain economic index or agent's type y, and $F(x)$ the agent's expected payoff. When γ is n-crossing in x for each y, when can we conclude that F is at most n-crossing?

Since the primary function γ is n-crossing, it is sometimes negative and thus it cannot represent any totally positive kernel. One might therefore deem the VDP inapplicable. This is not so. To apply the VDP, decompose γ into a sum $\gamma(x, y) = \Sigma_{i=1}^{n+1} \Gamma_i(x, y)(-1)^i$, for the non-negative *sign-enumerated function* $\Gamma : \{1, 2, \ldots, n + 1\} \times X \times Y \to \mathbb{R}_+$. Define the kernel K by

$$K(i, x) \equiv \int_Y \Gamma_i(x, y) d\mu(y). \tag{2}$$

Lemma 1. *If $K(i, x)$ is TP_{n+1}, then $F(x)$ in* (1) *has at most n sign changes. If $S[F] = n$, then $F(x)$ has the same sign sequence as $(-1)^i$ for $i = 1, \ldots, n + 1$.*

Proof. The function $(-1)^i$ is n-crossing in i over $\{1, 2, \ldots, n+1\}$, while $K(i, x)$ is TP_{n+1}. Since $F(x) = \int_Y \gamma(x, y) d\mu(y) = \int_Y [\Sigma_i (-1)^i \Gamma_i(x, y)] \, d\mu(y) = \Sigma_{i=1}^{n+1} K(i, x)(-1)^i$, the VDP gives the result by integrating in i using the counting measure. $\square$

Lemma 1 only imposes conditions on the endogenous kernel K. Depending on the goal, one can be creative in decomposing the primary function γ into a TP_{n+1} kernel. If $n = 1$, it is natural to decompose γ into positive and negative parts. This is essentially the decomposition in QS, since sign ratio monotonicity ensures that K is TP_2 (log-supermodular). Put differently, given $K(2, x)/K(1, x)$ non-decreasing by TP_2 and $K > 0$, the product $F(x) = [K(2, x)/K(1, x) - 1]K(1, x)$ has at most one sign change ($-$ to $+$). In §3, we preserve quasi-concavity by decomposing γ into a decreasing and an increasing part.

3 Quasi-Concavity Preservation

Quasi-concavity plays a crucial role in economics — from reducing optimization theory to first order conditions, modeling preference for diversification, to justifying the median voter

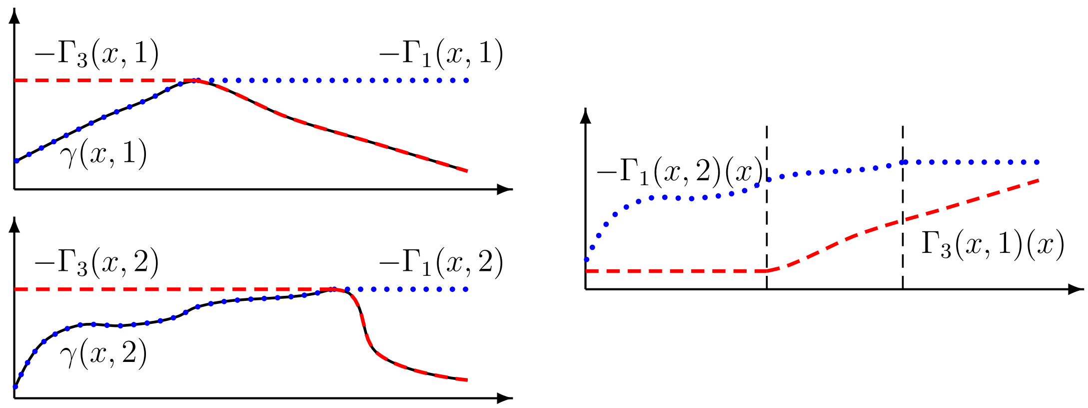

Figure 1: (Left) Decomposition of a QC function $\gamma(x,y)$ into $\Gamma_1(x,y)$ and $\Gamma_3(x,y)$. (Right) To preserve QC, it suffices to check that $\Gamma_3(x,1)$ grows proportionally faster than $-\Gamma_1(x,2)$. The proportional growth condition (5) is met because (a) when either $\Gamma_3(x,1)$ or $-\Gamma_1(x,2)$ is flat, condition (5) is satisfied, and (b) when both functions are rising (the area between the vertical dashed lines), $-\Gamma_1(x,2)$ is relatively more concave than $\Gamma_3(x,1)$.

theorem (see Jewitt (1988), Schmeidler (1989) and Black (1948)). A function $f:\mathbb{R}\to\mathbb{R}$ is *quasi-concave* (QC) if its upper contour set $f^{-1}[c,\infty)$ is convex for all $c\in\mathbb{R}$. Then f is QC if and only if $S[f(x)-c]\leq 2$ for all $c\in\mathbb{R}$, and where $f(x)-c$ has the sign sequence $-,+,-$ if $S[f(x)-c]=2$. Using the VDP with TP_3 kernels, we derive sufficient conditions for 2-crossing and sign sequence preservation, and thereby QC preservation.[2]

Assume $X,Y\subseteq\mathbb{R}$. For $y\in Y$, define the global supremum $\hat{\gamma}(y)\equiv\sup_{x'}\{\gamma(x',y)\}$ of the bounded QC primary function γ, the running max $\gamma^*(x,y)\equiv\sup_{x'}\{\gamma(x',y)|x'\leq x\}$, and continuation max $\gamma^{**}(x,y)\equiv\sup_{x'}\{\gamma(x',y)|x'\geq x\}$. Then $\gamma^*(x,y)=\gamma(x,y)$ when it increases in x, and is otherwise constant, and $\gamma^{**}(x,y)=\gamma(x,y)$ when it falls in x, and is otherwise constant. Parse γ into decreasing and increasing portions with the sign enumerated functions:

$$\Gamma_1(x,y)\equiv a_1-\gamma^*(x,y) \tag{3}$$

$$\Gamma_3(x,y)\equiv a_2-\gamma^{**}(x,y)+\hat{\gamma}(y) \tag{4}$$

and one constant portion $\Gamma_2(x,y)\equiv a_1+a_2-c$, for any $c\in\mathbb{R}$, and constants a_1,a_2 ensuring that $\Gamma_i(x,y)>0$ for all i,x and y, given c. Figure 1 (left panel) illustrates two examples of such decompositions. Note that $\gamma(x,y)-c=\Sigma_{i=1}^3\Gamma_i(x,y)(-1)^i$. By integrating over y, we have $F(x)-c\mu(Y)=\Sigma_{i=1}^3K(i,x)(-1)^i$ where K follows (2).

We now introduce a comparison of any two increasing functions, like $-\Gamma_1(x,y)$ and $\Gamma_3(x,y)$. Renaming a concept in Pratt (1964), say that a weakly increasing function g *grows*

[2] Jewitt (1988) used TP_3 for justifying the first order approach for solving moral hazard problems.

proportionately faster than f, another weakly increasing function, if for any $x_3 \geq x_2 \geq x_1$:

$$[f(x_2) - f(x_1)]\,[g(x_3) - g(x_2)] \geq [f(x_3) - f(x_2)]\,[g(x_2) - g(x_1)]\,. \tag{5}$$

This applies to weakly increasing, not necessarily continuous, functions. If f and g are strictly increasing and continuous, then (5) reduces to f *more concave than* g, i.e. $f(x) = \phi[g(x)]$, for some increasing and concave function ϕ. When f and g are twice differentiable, then (5) becomes $-f''(x)/f'(x) \geq -g''(x)/g'(x)$ for all x, or f *more risk averse than* g.[3]

Proposition 1. *If the primary function $\gamma(x, y)$ is bounded and QC in x, for $y \in Y$, then the aggregation function $F(x)$ in (1) is QC for all positive measures μ if and only if the sign-enumerated function $\Gamma_3(x, y')$ grows proportionately faster than $-\Gamma_1(x, y'')$, if $y', y'' \in Y$.*

In the only related result in the literature that we have found, Debreu and Koopmans (1982) and later Crouzeix and Lindberg (1986) explore conditions so that $f(x) = \Sigma_{i=1}^m f_i(x_i)$ is quasi-convex if the summand functions $f_i : X_i \to \mathbb{R}$ are quasi-convex. This differs from our problem in two key respects: We allow for general measures, and the domain of our aggregation function is scalar, but the domain here is the Cartesian product $\times_{i=1}^m X_i$.

Proof: ($\Leftarrow$) We use Lemma 1. Fix $c \in \mathbb{R}$. If the resulting $K(i, x)$ in (2) is TP_3, then we have $S[F(x) - c\mu(Y)] \leq 2$, and $F(x) - c\mu(Y)$ has sign sequence $-, +, -$ if $S[F(x) - c\mu(Y)] = 2$.

Now, $K(i, x) \equiv \int_Y \Gamma_i(x, y) d\mu(y)$ is respectively decreasing, constant and increasing in x, as $i = 1, 2, 3$, since this is true of $\Gamma_i(x, y)$ for all y. As a result, $K(3, x)/K(2, x)$ and $K(2, x)/K(1, x)$ are both non-decreasing in x, and so $K(i, x)$ is TP_2. To verify that K is TP_3, it thus suffices to verify the next determinant condition, that for any $x_3 \geq x_2 \geq x_1$:[4]

$$[K(1, x_1) - K(1, x_2)][K(3, x_3) - K(3, x_2)] \geq [K(1, x_2) - K(1, x_3)][K(3, x_2) - K(3, x_1)].$$

For a proof, put $f(x) = -\Gamma_1(x, y')$ and $g(x) = \Gamma_3(x, y'')$ in (5), and integrate in y', y'':

$$\begin{aligned} & \int_Y [\Gamma_1(x_1, y) - \Gamma_1(x_2, y)] d\mu(y) \int_Y [\Gamma_3(x_3, y) - \Gamma_3(x_2, y)] d\mu(y) \\ \geq \; & \int_Y [\Gamma_1(x_2, y) - \Gamma_1(x_3, y)] d\mu(y) \int_Y [\Gamma_3(x_2, y) - \Gamma_3(x_1, y)] d\mu(y). \end{aligned}$$

($\Rightarrow$) If the proportional growth proviso fails at some $x_3 > x_2 > x_1$ and $y', y'' \in Y$, then for some $\alpha > 0$:

[3]Theorem $1(e)$ in Pratt (1964) adds an extra degree of freedom to (5). Pratt states the next two definitions, but his equivalence proofs assume differentiability. We can dispense with differentiability (proof omitted).

[4]The determinant condition is independent of the constant $K(2, x)$ because the sign of the determinant of the 3×3 matrix $[K(i, x_j)]$ is unaffected by the value of any such positive constant.

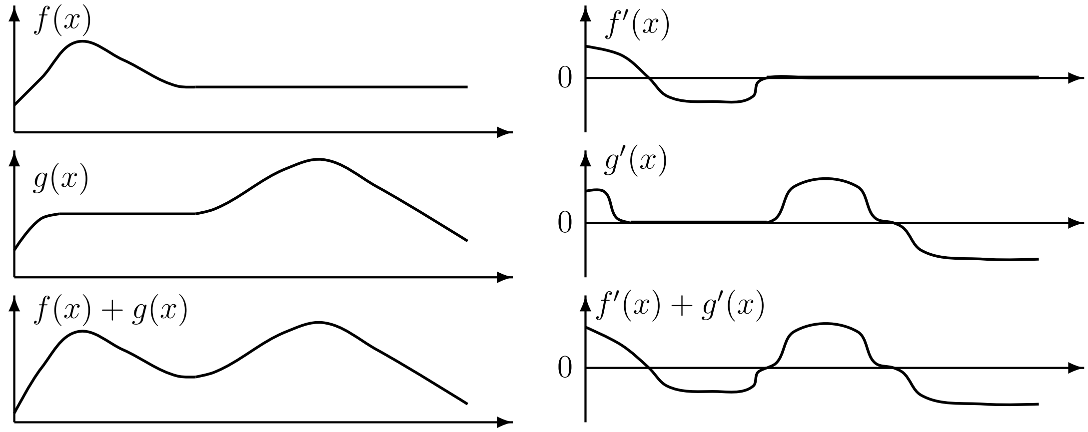

Figure 2: As QC functions may have flats, SC preservation of slopes does not imply QC preservation. Here, f and g are QC, and f' and g' obey SRM, but $f+g$ is not quasi-concave.

$$\frac{\Gamma_3(x_2,y')-\Gamma_3(x_1,y')}{\Gamma_1(x_1,y'')-\Gamma_1(x_2,y'')} > \alpha > \frac{\Gamma_3(x_3,y')-\Gamma_3(x_2,y')}{\Gamma_1(x_2,y'')-\Gamma_1(x_3,y'')}. \tag{6}$$

We claim that the aggregation function $F(x) \equiv \gamma(x,y') + \alpha\gamma(x,y'')$ obeys $F(x_2) < F(x_1)$. Now, $\Gamma_1(x_2,y'') > \Gamma_1(x_3,y'')$ implies $\gamma(x_3,y'') > \gamma(x_2,y'')$, by (3). Then $\gamma(x,y'')$ is increasing in x for all $x < x_3$, since γ is QC in x. Thus, $\gamma(x_2,y'') - \gamma(x_1,y'') = \Gamma_1(x_1,y'') - \Gamma_1(x_2,y'') > 0$ because $x_1 < x_2 < x_3$. Similarly, $\Gamma_3(x_2,y') > \Gamma_3(x_1,y')$ implies $\gamma(x_2,y') - \gamma(x_1,y') = \Gamma_3(x_1,y') - \Gamma_3(x_2,y')$, by (4). By the left inequality in (6):

$$F(x_2)-F(x_1) < \gamma(x_2,y')-\gamma(x_1,y')+\frac{\Gamma_3(x_2,y')-\Gamma_3(x_1,y')}{\Gamma_1(x_1,y'')-\Gamma_1(x_2,y'')}[\gamma(x_2,y'')-\gamma(x_1,y'')] = 0.$$

A similar argument shows that $F(x_3) > F(x_2)$, and consequently, F is not QC. $\square$

If $\gamma(x,y)$ is (i) differentiable in x and (ii) has no flat regions,[5] then QC preservation reduces to SC preservation of the derivative $\gamma_x(x,y)$, or equivalently the comparison of absolute risk aversion coefficients (recalling Proposition 1). To see this, decompose γ into any increasing and decreasing portions, namely, $\gamma = \gamma_I + \gamma_D$, where $\gamma_I(x,y)$ and $-\gamma_D(x,y)$ increase in x for all $y \in Y$. The "only if" assertion of the Proposition 1 implies:

Corollary 1. *The aggregation function F in (1) is QC for all μ if (a) $\gamma_I(x,y')$ is more concave than $-\gamma_D(x,y'')$ for all $y',y'' \in Y$, or (b) γ_I and $-\gamma_D$ are twice differentiable in x, with $\gamma_I(x,y')$ more risk averse than $-\gamma_D(x,y'')$ for all $y',y'' \in Y$.*

Proposition 1 assumes neither continuity nor differentiability of $\gamma(x,y)$, and so allows

[5]If $\gamma(x,y)$ is constant for some interval of x, then $\gamma_x(x,y)$ possibly might not be SC in x, and thus one cannot apply SRM to the SC preservation of $\gamma_x(x,y)$. Figure 2 illustrates this point.

aggregation of QC step functions.[6] Non-differentiable functions $\gamma(x, y)$ naturally arise in economics applications. For example, time series data is naturally discrete. We consider three such examples of QC preservation of discontinuous or non-differentiable functions.

EXAMPLE 1 (MEDIAN VOTER THEOREM) Assume majority voting by a population over linearly ordered alternatives. By Black (1948), we can apply the median voter theorem (MVT) to predict the outcome of this game if preferences are single-peaked.[7] Suppose instead that preferences are uncertain independent random draws from a set of single-peaked preferences. If voting occurs before voters' preferences are realized, then by Proposition 1 each voter's expected utility is single-peaked if the decreasing part of every utility function grows faster than the increasing part of any other utility function. Then the MVT applies.

EXAMPLE 2 (QUASI-CONVEX COMPARATIVE STATICS WITH UNCERTAINTY) An agent chooses an action $x \in X$ to maximize utility $U(x, s)$. Here, $s \in S$ is a parameter, and X and S are ordered sets. Milgrom and Shannon (1994) show that the optimizer set increases in s if $[U(x_2, s) - U(x_1, s)]$ is SC in s, for all $x_2 > x_1$. In the same spirit, if $U(x, s)$ is QC in x for all s, then the optimal choice $x^*(s)$, if unique, is quasi-convex in s if $[U(x_2, s) - U(x_1, s)]$ is quasi-convex in s, for all $x_2 > x_1$ (see Lemma 2 below).[8]

Enriching this with uncertainty, define $U(x, s) \equiv \int_Y u(x, s, y) d\mu(y)$. By Theorem 1 in QS, for all $x_2 > x_1$ and $y \in Y$, if $\delta(s, y) \equiv [u(x_2, s, y) - u(x_1, s, y)]$ is SC in s, and if all pairs $\delta(s, y'), \delta(s, y'')$ obey SRM, then $[U(x_2, s) - U(x_1, s)]$ is SC. The optimizer set then increases in s. Instead, assume $U(x, s)$ is QC in x, with a unique maximizer $x^*(s)$ for all s. By Proposition 1, $[U(x_2, s) - U(x_1, s)]$ is quasi-convex in s if $\delta(s, y)$ is quasi-convex in s, and if the increasing part of $\delta(s, y')$ grows proportionally faster than the decreasing part of $\delta(s, y'')$, for every y', y''. By Lemma 2, the optimal choice $x^*(s)$ is quasi-convex in s.

EXAMPLE 3: PROFIT MAXIMIZATION WITH STOCHASTIC PRODUCTION COST. We apply Proposition 1 to preserve QC under some conditions. Let firm revenues $r(x, y)$ be twice differentiable and strictly concave in x. To see how aggregation of discontinuous functions works, assume the firm employs one piecewise linear and discontinuous technology for $x \leq \bar{x}$ and another for $x > \bar{x}$. For example, the firm may use another factory when output exceeds $\bar{x}$, and factory productivity is stochastic. The high output technology has a

[6] Proposition 1 implies that the integral of QC step functions (with multiple jumps and flat areas) is QC for all measures μ if and only if for every pair of functions f, g, whenever f is increasing and g decreasing, (i) they jump at the same points and (ii) the ratio of the size of the jump of g and f is increasing in x.

[7] Black (1948) showed that if all individual preferences are single peaked, then the social preference, as generated by majority rule applied to all pairs of alternatives, is complete and transitive.

[8] For intuition, let U be differentiable. Here, QC is not needed. Assume $U_x(x, s)$ is quasi-convex in s for all x, so that $U_{xs}(x, s)$ is SC in s. Now, $x_s^*(s)$ and $U_{xs}(x^*(s), s)$ share the same sign, by the Envelope Theorem. If $x_s^*(\bar{s}) = U_{xs}(x^*(\bar{s}), \bar{s}) = 0$ at some $\bar{s}$, then $dU_{xs}(x^*(s), s)/ds \geq 0$ at $s = \bar{s}$, as U_{xs} is SC in s. Then $dx_s^*(s)/ds \geq 0$ at $\bar{s}$ since $x_s^*(s)$ and $U_{xs}(x^*(s), s)$ share the same sign. Hence, $x^*(s)$ is quasi-convex.

lower fixed cost than the low output technology in the good state $y = G$. It has a higher fixed and marginal cost than the low output technology in the bad state $y = B$:

$$c(x, G) = c_G x - b\mathbb{1}_{\{x>\bar{x}\}} \quad \text{and} \quad c(x, B) = c_B x + (e + dx)\mathbb{1}_{\{x>\bar{x}\}},$$

where $b, c_G, c_B, d, e > 0$. Then $c(x, G)$ jumps down at $x = \bar{x}$ and $c(x, B)$ jumps up at $x = \bar{x}$, and grows steeper afterwards. Let $\hat{x}_y$ maximize the primary function $\gamma(x, y) \equiv r(x, y) - c(x, y)$, where $\hat{x}_G > \bar{x} > \hat{x}_B$. This ensures $\gamma(x, y)$ is quasi-concave in x.

The firm chooses an output level x to maximize the expected profits $F(x)$ that aggregates the profit function $\gamma(x, y)$ as in (1). By Proposition 1, F is single-peaked for all measures μ provided the ratio of the jump sizes $b/(e + d\bar{x})$ is (i) at least the right limit of the ratio of the slopes $\lim_{x\downarrow\bar{x}} -\gamma_x(x, G)/\gamma_x(x, B)$ and (ii) at most the left limit, namely:

$$\lim_{x\downarrow\bar{x}} \frac{\gamma_x(x, G)}{-\gamma_x(x, B)} \leq \frac{b}{e + d\bar{x}} \leq \lim_{x\uparrow\bar{x}} \frac{\gamma_x(x, G)}{-\gamma_x(x, B)}. \tag{7}$$

For since γ is quasi-concave, $\gamma(x, G)$ is increasing and $\gamma(x, B)$ decreasing near $\bar{x}$. Equation (7) requires that *the ratio of slopes of decreasing and increasing parts* $-\gamma_x(x, G)/\gamma_x(x, B)$ *falls in* x. For the jump ratio $b/(e + d\bar{x}) = -\gamma_x(x, G)/\gamma_x(x, B)$ at $x = \bar{x}$ because $\gamma(x, B)$ and $\gamma(x, G)$ are left continuous at $\bar{x}$. Finally, the gap between the RHS and LHS of (7) vanishes if $d = 0$, whereupon the ratio b/e of jumps is pinned down. The proof is in §B.2.

4 Conclusion: General Ordinal Aggregation via Karlin

We conclude by observing how our aggregation results generalize to higher dimensions. First, SC aggregation very naturally generalizes to n-crossing aggregation. To do so, we use sign-enumerated functions $\Gamma_i(x, y) = |\gamma(x, y)|$ if x lies between the $(i - 1)^{th}$ and i^{th} sign change of $\gamma(x, y)$, and otherwise zero:

$$\Gamma_i(x, y) \equiv |\gamma(x, y)|\mathbb{1}\{\sup_{x_1<\cdots<x_m<x} s(\gamma(x_1, y), \ldots, \gamma(x_m, y), \gamma(x, y)) = i - 1\}.$$

By Lemma 1, *the aggregation function* F *in* (1) *is n-crossing if the kernel in (2) is* TP_{n+1}.

Next, a function changes monotonicity at a *turning point*, from increasing to decreasing, or conversely. Naturally, a function is monotone if and only if it has no turning point, and is QC if it has one turning point, and from increasing to decreasing. Proposition 1 uses Lemma 1 to ensure $S[F(x) - c] \leq 2$ for all c, and so guarantees that the aggregation function F in (1) has at most one turning point. Generalizing this logic, F has at most n

turning points if $S[F(x)-c] \leq n+1$ for all c, assuming the kernel is at least TP_{n+1}.[9]

A Single Crossing Preservation

This section better relates our theory to QS by using the VDP to derive a necessary and sufficient condition for single crossing preservation; we thereby find that the main result in QS is in fact equivalent to the VDP. A function $f(x)$ is *single crossing* (SC) if $f(x_1) \geq 0$ implies $f(x_2) \geq 0$, and $f(x_1) > 0$ implies $f(x_2) > 0$, both for all $x_2 > x_1$. So if f satisfies $S[f] \leq 1$ with the sign sequence $-, +$ when $S[f] = 1$, then f is SC provided *proviso* ($\star$) holds: namely, $f(x) \neq 0$ except at the sign change.

We apply Lemma 1 to the sign-enumerated functions $\Gamma_1(x,y) = \max\{-\gamma(x,y), 0\}$ and $\Gamma_2(x,y) = \max\{\gamma(x,y), 0\}$, respectively, the negative and positive parts of $\gamma(x,y)$.

Corollary 2. *If the primary function $\gamma(x,y)$ is SC in x, for all $y \in Y$, then the aggregation function F in (1) is SC for all positive measures μ if and only if the sign-enumerated functions obey:*

$$\Gamma_1(x_1,y'')\Gamma_2(x_2,y') \geq \Gamma_1(x_2,y'')\Gamma_2(x_1,y') \qquad \textit{for all } y',y'' \in Y \textit{ and } x_2 > x_1. \tag{8}$$

For insight, define the fictitious kernel $k(i,x|y',y'') = -\Gamma_1(x,y'')\mathbb{1}_{\{i=1\}} + \Gamma_2(x,y')\mathbb{1}_{\{i=2\}}$. Then inequality (8) is the assertion that k is TP_2 in (i,x) for any $y',y'' \in Y$, namely that:

$$k(1,x_1|y',y'')k(2,x_2|y',y'') \geq k(1,x_2|y',y'')k(2,x_1|y',y'') \text{ for all } y',y'' \in Y \text{ and } x_2 > x_1.$$

Proof of Corollary 2: ($\Leftarrow$) For any $x_2 > x_1$, integrating (8) over all y', y'' yields

$$\int_Y \Gamma_1(x_1,y)d\mu(y) \int_Y \Gamma_2(x_2,y)d\mu(y) \geq \int_Y \Gamma_1(x_2,y)d\mu(y) \int_Y \Gamma_2(x_1,y)d\mu(y).$$

Thus, $K(i,x)$ is TP_2: By Lemma 1, $F(x) = K(2,x) - K(1,x)$ changes from $-$ to $+$.[10]

($\Rightarrow$) Assume (8) fails for some $y', y'' \in Y$ and $x_2 > x_1$. Then

$$\Gamma_1(x_1,y'')\Gamma_2(x_2,y') < \Gamma_2(x_1,y')\Gamma_1(x_2,y''). \tag{9}$$

[9]For example, one could decompose γ into $n+1$ monotone functions, namely $\gamma(x,y) = \Sigma_{i=2}^{n+2}\Gamma_i(x,y)(-1)^i$, where for $i > 1$, $\Gamma_i(x,y) = \gamma(x,y)(-1)^i$ if x lies between the $(i-2)^{th}$ and $(i-1)^{th}$ turning point of $\gamma(x,y)$, and otherwise is constant. Also, let $\Gamma_1 = c$. Then by Lemma 1, F *has at most n-turning points if the kernel in (2) is TP_{n+2}* for all c.

[10]Proviso (*) holds: Assume $F(x_1) = F(x_3) = 0 \neq F(x_2)$ for some $x_1 < x_2 < x_3$. If $F(x_2) > 0$ then $K(2,x_3)/K(1,x_3) \geq K(2,x_2)/K(1,x_2)$ implies $F(x_3) > 0$. If $F(x_2) < 0$ then $K(2,x_2)/K(1,x_2) \geq K(2,x_1)/K(1,x_1)$ implies $F(x_1) < 0$. So $F(x_2) = 0$.

Since $\gamma(x, y)$ is SC in x, and $\Gamma_1(x_2, y'') > 0$ by (9), we have $\gamma(x, y'') = -\Gamma_1(x, y'')$ for $x \leq x_2$. Likewise, $\Gamma_2(x_1, y') > 0$ implies $\gamma(x, y') = \Gamma_2(x, y')$ for $x \geq x_1$. Put $F(x) \equiv \mu_1\gamma(x, y') + \mu_2\gamma(x, y'')$ where $\mu_1 \equiv \Gamma_1(x_1, y'')$ and $\mu_2 \equiv \Gamma_2(x_1, y')$. Then trivially, $F(x_1) = 0$ while (9) implies $F(x_2) < 0$. Hence, $F(x)$ is not SC. $\square$

QS defined *signed-ratio monotonicity* (SRM) for two SC functions $f(x)$ and $g(x)$ if:

(a) $-g(x_1)/f(x_1) \geq -g(x_2)/f(x_2)$ for all $x_2 > x_1$ at any $x_1 \in X$ with $f(x_1) > 0 > g(x_1)$

(b) $-f(x_1)/g(x_1) \geq -f(x_2)/g(x_2)$ for all $x_2 > x_1$ at any $x_1 \in X$ with $g(x_1) > 0 > f(x_1)$

Theorem 1 in QS asserts that $F(x)$ is SC for all measures μ if and only if $\gamma(x, y')$ and $\gamma(x, y'')$ obey SRM for $y', y'' \in Y$. Since SRM is (8), Corollary 2 yields their Theorem 1.

To complete the circle, we argue that Theorem 1 in QS implies the VDP in the SC case.

Proposition 2. *Theorem 1 in QS is equivalent to the VDP in the SC case.*

Proof: Consider $f(x) = \int_Z K(x, z)h(z)d\sigma(z)$. Write the integrand as $\gamma(x, z) = K(x, z)h(z)$, where h satisfies $S[h] \leq 1$ with sign sequence $-, +$, and K is TP_2. To see that $\gamma(x, z')$ and $\gamma(x, z'')$ satisfy SRM for any $z', z'' \in Z$, suppose not, so that $\gamma(s, z'') > 0 > \gamma(s, z')$ for some z', z'' and s. Since $K \geq 0$, we have $h(z'') > 0 > h(z')$. Then $z'' > z'$ because h changes sign once from $-$ to $+$; therefore, we have the SRM condition $-\gamma(s, z'')/\gamma(s, z') \leq -\gamma(s', z'')/\gamma(s', z')$ for all $s' > s$ because $-h(z'')/h(z') > 0$ and $K(s, z'')/K(s, z') \leq K(s', z'')/K(s', z')$ since K is TP_2. By Theorem 1 in QS, f obeys SC, and so $S[f] \leq 1$. $\square$

B Omitted Proofs

B.1 Proof of Example 2

Lemma 2. *Let $u(x, s)$ be QC with a unique max $x^*(s)$ for all $s \in S$, and $[u(x_2, s) - u(x_1, s)]$ quasi-convex in s if $x_2 > x_1$. For any $s_3 > s_2 > s_1$, if $x^*(s_2) > x^*(s_1)$, then $x^*(s_3) \geq x^*(s_2)$.*

Proof: By definition $u[x^*(s_2), s_2] - u[x^*(s_1), s_2] > 0 > u[x^*(s_2), s_1] - u[x^*(s_1), s_1]$. These two inequalities are strict since $x^*(s_2) > x^*(s_1)$ and the max is unique. Since $s_2 > s_1$ and $u[x^*(s_2), s] - u[x^*(s_1), s]$ is quasi-convex in s, $u[x^*(s_2), s] - u[x^*(s_1), s] > 0$ for all $s \geq s_2$.

Suppose $x^*(s_3) < x^*(s_1)$, then $u(q, s_3)$ is not QC since $u[x^*(s_3), s_3] > u[x^*(s_2), s_3] > u[x^*(s_1), s_3]$ but $x^*(s_3) < x^*(s_1) < x^*(s_2)$. Hence, $x^*(s_3) \geq x^*(s_1)$.

Suppose $x^*(s_3) \in [x^*(s_1), x^*(s_2))$. Then $u[x^*(s_2), s_1] - u[x^*(s_3), s_1] \leq 0$ by the QC of $u(\cdot, s_1)$. Also $u[x^*(s_2), s_2] - u[x^*(s_3), s_2] > 0$, as $x^*(s_2) \neq x^*(s_3)$ and the max of $u(\cdot, s_2)$ is unique. So $u[x^*(s_2), s_3] - u[x^*(s_3), s_3] > 0$ given these two inequalities and the quasi-convexity of $u[x^*(s_2), s] - u[x^*(s_3), s]$. This contradiction proves $x^*(s_3) \geq x^*(s_2)$.

B.2 Proof of Example 3

Since we have assumed $\gamma(x,G)$ and $\gamma(x,B)$ are QC in x with maximizers $\hat{x}_G > \hat{x}_B$, they both are increasing on $[\hat{x}_B,\infty)$ and decreasing on $(-\infty,\hat{x}_G]$. To prove the aggregation function is QC, it suffices to focus on $[\hat{x}_B,\hat{x}_G]$, where $\gamma(x,G)$ and $\gamma(x,B)$ are respectively rising and falling. Then Proposition 1 requires that $-\gamma(x,B)$ grows proportionately faster than $\gamma(x,G)$, namely for any $\hat{x}_B \le x_1 < x_2 < x_3 \le \hat{x}_G$:

$$\frac{\gamma(x_3,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\gamma(x_3,B)} \le \frac{\gamma(x_2,G)-\gamma(x_1,G)}{\gamma(x_1,B)-\gamma(x_2,B)}. \tag{10}$$

Recalling $\gamma(x,y) = r(x,y) - c(x,y)$, where r is strictly concave in x, and c is linear in x except at $x=\bar{x}$, all factors in (10) are positive. Now, $\gamma(x,G)$ is concave and $-\gamma(x,B)$ is convex for $x<\bar{x}$ or $x\ge\bar{x}$. Since $\gamma(x,G)$ is more concave than $-\gamma(x,B)$, the latter grows proportionately faster, proving (10) for $\bar{x} > x_3 > x_2 > x_1$ or $x_3 > x_2 > x_1 \ge \bar{x}$.

Next, for $\bar{x} \in (x_2,x_3)$, we prove that (7) implies (10). First, we find a lower bound for the RHS of (10). Since $\gamma(x,G)$ is more concave than $-\gamma(x,B)$ for $x<\bar{x}$, we have:

$$\frac{\lim_{x\uparrow\bar{x}}\gamma(x,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\lim_{x\uparrow\bar{x}}\gamma(x,B)} \le \frac{\gamma(x_2,G)-\gamma(x_1,G)}{\gamma(x_1,B)-\gamma(x_2,B)}. \tag{11}$$

Since $\gamma(x,G)$ jumps up by b at $x=\bar{x}$, and $\gamma(x,B)$ jumps down by $e+d\bar{x}$ at $x=\bar{x}$, we have $\lim_{x\uparrow\bar{x}}\gamma(x,G)=\gamma(\bar{x},G)-b$ and $\lim_{x\uparrow\bar{x}}\gamma(x,B)=\gamma(\bar{x},B)+e+d\bar{x}$. Hence:

$$\frac{\gamma(x_3,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\gamma(x_3,B)} = \frac{\gamma(x_3,G)-\gamma(x_2,G)+\lim_{x\uparrow\bar{x}}\gamma(x,G)-\gamma(\bar{x},G)+b}{\gamma(x_2,B)-\gamma(x_3,B)-\lim_{x\uparrow\bar{x}}\gamma(x,B)+\gamma(\bar{x},B)+e+d\bar{x}}. \tag{12}$$

To prove (10), it is suffices that the LHS of (11) be an upper bound for (12). To this end, note that:

$$\frac{b}{e+d\bar{x}} \le \lim_{x\uparrow\bar{x}}\frac{\gamma_x(x,G)}{-\gamma_x(x,B)} \le \frac{\lim_{x\uparrow\bar{x}}\gamma(x,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\lim_{x\uparrow\bar{x}}\gamma(x,B)}. \tag{13}$$

The first inequality is (7) and the last is the proportional growth condition.[11] Next,

$$\frac{\gamma(x_3,G)-\gamma(\bar{x},G)}{\gamma(\bar{x},B)-\gamma(x_3,B)} \le \lim_{x\downarrow\bar{x}}\frac{\gamma_x(x,G)}{-\gamma_x(x,B)} \le \lim_{x\uparrow\bar{x}}\frac{\gamma_x(x,G)}{-\gamma_x(x,B)} \le \frac{\lim_{x\uparrow\bar{x}}\gamma(x,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\lim_{x\uparrow\bar{x}}\gamma(x,B)} \tag{14}$$

[11] Since the proportional growth condition holds for $x<\bar{x}$, for any $x_2 \le x \le x+\Delta < \bar{x}$,

$$\frac{\gamma(x+\Delta,G)-\gamma(x,G)}{\gamma(x,B)-\gamma(x+\Delta,B)} \le \frac{\gamma(x,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\gamma(x,B)} \implies \lim_{x\uparrow\bar{x}}\frac{\gamma_x(x,G)}{-\gamma_x(x,B)} \le \frac{\lim_{x\uparrow\bar{x}}\gamma(x,G)-\gamma(x_2,G)}{\gamma(x_2,B)-\lim_{x\uparrow\bar{x}}\gamma(x,B)}.$$

The conclusion is implied by the premise via the limit $\Delta\downarrow 0$ and then $x\uparrow\bar{x}$.

The first and last inequalities are true because the proportional growth condition holds for $x \geq \bar{x}$ and $x < \bar{x}$ respectively. The middle inequality is by (7). Combining (13), (14) and (12), we have[12]

$$\frac{\gamma(x_3, G) - \gamma(x_2, G)}{\gamma(x_2, B) - \gamma(x_3, B)} \leq \frac{\lim_{x \uparrow \bar{x}} \gamma(x, G) - \gamma(x_2, G)}{\gamma(x_2, B) - \lim_{x \uparrow \bar{x}} \gamma(x, B)}.$$

This inequality and (11) imply (10). The case for $\bar{x} \in (x_1, x_2]$ is similar and thus is omitted.

References

Black, D. (1948). On the rationale of group decision-making. *Journal of Political Economy 56*(1), 23–34.

Crouzeix, J.-P. and P. Lindberg (1986). Additively decomposed quasiconvex functions. *Mathematical Programming 35*(1), 42–57.

Debreu, G. and T. Koopmans (1982). Additively decomposed quasiconvex functions. *Mathematical Programming 24*(1), 1–38.

Jewitt, I. (1988). Justifying the first-order approach to principal-agent problems. *Econometrica 56*(5), 1177–1190.

Karlin, S. (1956). Decision theory for Polya type distributions. Case of two actions, I. *Proc. Third Berkeley Symp. on Math. Statist. and Prob., Vol. 1*, 115–128.

Karlin, S. (1968). *Total Positivity*. Number v. 1 in Total Positivity. Stanford University Press.

Karlin, S. and H. Rubin (1955). *The theory of decision procedures for distributions with monotone likelihood ratio*. Technical report. Applied Mathematics and Statistics Laboratory, Stanford University.

Lindsay, B. G. and K. Roeder (1993). Uniqueness of estimation and identifiability in mixture models. *Canadian Journal of Statistics 21*(2), 139–147.

[12] To see why this inequality is true, note that for any $C, D, S, T, X, Y > 0$,

$$\frac{S}{T} \leq \frac{C}{D} \quad \text{and} \quad \frac{X}{Y} \leq \frac{C}{D} \implies \frac{C+S+X}{D+T+Y} \leq \frac{C}{D}.$$

Equivalence follows: $C = \lim_{x \uparrow \bar{x}} \gamma(x, G) - \gamma(x_2, G)$, $D = \gamma(x_2, B) - \lim_{x \uparrow \bar{x}} \gamma(x, B)$, $S = b$, $T = e + d\bar{x}$, $X = \gamma(x_3, G) - \gamma(\bar{x}, G)$ and $Y = \gamma(\bar{x}, B) - \gamma(x_3, B)$. So $C/D \geq S/T$ by (13) and $C/D \geq X/Y$ by (14). Finally $(C+S+X)/(D+T+Y)$ is identical to the RHS of (12).

Milgrom, P. and C. Shannon (1994). Monotone comparative statics. *Econometrica 62*(1), 157–80.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica 32*(1-2), 122–136.

Quah, J. K. and B. Strulovici (2012). Aggregating the single crossing property. *Econometrica 80*(5), 2333–2348.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica 57*(3), 571–587.