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Dynamic Matching and Evolving Reputations

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This paper introduces a general model of matching that includes evolving public Bayesian reputations and stochastic production. Despite productive complementarity, assortative matching robustly fails for high discount factors, unlike in Becker (1973). This failure holds around the highest (lowest) reputation agents for "high skill" ("low skill") technologies. We find that matches of likes eventually dissolve. In another life-cycle finding, young workers are paid less than their marginal product, and old workers more. Also, wages rise with tenure but need not reflect marginal products: information rents produce non-monotone and discontinuous wage profiles.

1. INTRODUCTION

Consider a static Walrasian pairwise matching economy where output depends solely on exogenous abilities. Becker (1973) showed that *positive assortative matching* (PAM) arises when abilities are productive complements. This is the foundational paper in the noncooperative theory of decentralized matching markets, and has established PAM as the benchmark allocation in the matching literature. Shimer and Smith (2000) and Atakan (2006) have since found complementarity conditions under which PAM still obtains in this fixed type framework with random matching and search frictions.

In a static world, productively complementary individuals assortatively match by their expected abilities. We introduce and explore a recursively solvable continuum agent matching model where agents have slowly evolving characteristics. In this dynamic model we prove existence of a steady state equilibrium and the welfare theorems quite generally. We then specialize to a world where all abilities are simply "high" or "low". We assume unobserved abilities, and stochastic but publicly observable output, where the separate contributions to joint production are unseen. Everyone is then summarized by the public posterior chance that he is "high"−namely, his *reputation* is his characteristic. Within this general learning framework we consider two specific models. We focus on the partnership model, in which workers with unobserved abilities are matched in pairs to produce output. In the employment model, these workers are matched one-to-one with jobs whose characteristics are known.

The partnership model can be interpreted literally as a model of production partnerships, or as a parable for production in teams within-firms, or finally as a model of within firm task assignment. Output in many organizations is largely produced by *teams*: academic co-authoring, movie production, advertising, the legal profession, consulting, or team sports. The O-Ring example of Kremer (1993) illustrates the role of stochastic joint production in high-tech industrial production.

The partnership model. Our analysis of the partnership model begins with a two period setting. Becker's result yields PAM in the final period. This yields a *fixed* convex continuation value function. We then deduce that the fixed *expected continuation values* are strictly convex in the reputation of *one's partner*. We show that this induces strict gains from rematching any assortatively matched interior agents with 0 *or* 1 (i.e. surely low or surely high individuals), *or* both, opposing production complementarity. Despite this informational gain to non-assortative matching, PAM will again obtain in the first period with sufficient weight on the current period. However, since the static production losses from non-assortative matching in the first period are bounded, PAM cannot be optimal with sufficient weight on the future (Proposition 2).

Finite horizon models can have drastically different predictions than their infinite horizon counterparts. Is our two period analysis representative of the general setting? While our findings hang in the balance, we rescue a failure of PAM that turns on a trade-off between value convexity due to learning and static input complementarity.

To see where our earlier logic goes wrong, we observe that the two period analysis critically relies on fixed continuation values. With an infinite horizon, the continuation value is endogenous to the discount factor, and in a troubling fashion: as is well known, it "flattens out" with rising patience. So as the discount factor rises to 1, current production and information acquired in a match *both* become vanishingly important. A flattening value function is well understood, but we find a more subtle change. While it is true that the value function becomes less convex for any fixed reputation, it becomes *more* convex in a neighbourhood of the extremes 0 and 1; thus, we are led once again to check whether PAM fails near these extremes. Our analysis requires a very precise characterization of the extremal behaviour of the value function to resolve the knife-edged tradeoff between information and productive efficiency as patience rises.

The paper then turns to a labour economics story. Call the technology *high skill* if matches of one or two "low" agents are statistically similar. For example, the production function in Kremer (1993) (in which project success requires success in all subtasks) is a high skill technology. Proposition 3 shows that efficient matching depends on the nature of the technology: PAM fails for high (low) reputations when production is sufficiently high (low) skill. Not all technologies are high or low skill. The information effect may reinforce the static output effect near 0 and 1, yielding PAM for any level of patience. In general, the PAM failure is quite robust. Proposition 4 shows that for randomly chosen production technologies, the chance of both a high and low skill technology tends to one, as the number of production outcomes grows. We also offer simulation evidence that these conditions are *extremely* likely to hold in practice with few production outcomes.

Unlike other matching models with fixed types, ours affords an economically compelling micro-story as well. While the market is in steady-state, individuals proceed through their life-cycle, and their reputations randomly change, converging towards the underlying true abilities. So, with enough patience, if two genuinely high abilities are paired, then we should expect their reputations to rise as time passes. Eventually, they enter the region where PAM fails, and the partnership will dissolve.

Employment model. We next specialize our model to one where workers are matched to jobs whose types are known. Workers still have unknown abilities revealed over time via stochastic production outcomes. We assume that workers' and firms' types are productive complements, and so ideally should sort by type. But with incomplete information, a worker's job assignment determines both his expected output and the quality of information revealed in production. We then arrive at a much different PAM result: workers near the reputational extremes will always match assortatively (Proposition 6), since the productive effects there are strongest. This difference is the key empirical distinction between the partnership and employment models.

A parsimonious model for labour economics. Our partnership and employment models together provide a single coherent framework for understanding a variety of stylized facts in labour economics.

1. Wages Drift Up. Wages generally rise with work experience. Our model delivers this prediction, since expected values rise over time by Corollary 2, and so on average wages rise. But also consistent with the reality, wages sometimes fall from period-to-period. Both facts are true of our partnership and employment models.

2. Job Tenure, Mobility and Wages. Wages rise with job tenure, separation rates fall with job tenure, and high current wages are correlated with low subsequent mobility (see Jovanovic, 1979; Moscarini, 2005). Just as in MacDonald (1982), our employment model with discrete known jobs matches these stylized facts. To see why, note that workers at the reputational extremes are assortatively matched. Since a worker's wage equals his expected output, these workers receive the highest wages. Finally, over time workers' reputations are pushed to the extremes as their true types are revealed. Thus, the longer a worker is with the same firm, the closer its reputation will be to the extremes and the higher its wage. Finally, the closer a worker's reputation to the extremes, the longer until its type crosses an interior threshold for job changing.

3. Life Cycle Marginal Products versus Wages. Several empirical studies (e.g. Medoff and Abraham, 1980; Hutchens, 1987; Kotlikoff and Gokhale, 1992) have found evidence for an increasing relationship between wages and productivity over the life cycle: young workers earn less than their marginal product and old workers more. In our partnership model, workers at the reputational extremes are paid an informational premium, and others sacrifice for type revelation. But if we follow a cohort of workers over time, their reputations move toward the extremes as their types are revealed. So on average, younger workers will see their wages lag their productivity, while the reverse holds for older workers. Observe how this result in our partnership model is entwined with our PAM failure. With assortative matching, the two partners each receive half the output in wages, and there is no wage productivity gap.

4. Wage Dispersion by Cohort. Huggett *et al.* (2006) find that earnings dispersion across individuals within a cohort increases with age. This is consistent with both our partnership and employment models. Agents who have been around longer should have more accurate reputations than those at the beginning of their careers, and thus their reputations are more dispersed.

Related work. PAM fails in Kremer and Maskin's (1996) complete information matching model−but so does productive complementarity. In Serfes (2005) and Wright (2004), negative assortative matching arises in a principal−agent framework.

There is a small literature of equilibrium matching with incomplete information. Jovanovic (1979) considers a model where slow revelation of information about worker abilities causes turnover. Niederle and Roth (2004) match three key features of our model: complementarity, uncertain types, and publicly revealed signals. Chade (2006) extends Becker's work to uncertain abilities, but assumes private information, reinforcing PAM, by way of a new "acceptance curse". MacDonald (1982) also considers matching with incomplete information. But in his model, the information revelation is invariant to the match. Unlike these papers, we show that Becker's finding robustly unravels given an informational friction that depends on match assignment.

Our model is also related to the learning paper by Easley and Kiefer (1988), who ask when the decision maker eventually learns the true state. Incomplete learning requires that a myopically optimal action be uninformative at some belief. Easley and Kiefer show that no such action is dynamically optimal for a patient enough decision maker. Here, the statically optimal action (PAM) is not chosen given sufficient patience. Bergemann and Välimäki (1996) and Felli and Harris (1996) are related in that an element of the static price is information value, as with our wages.

Paper outline. In Section 2, we set up our general model, define a Pareto optimum and competitive equilibrium, and establish the welfare theorems and existence. Our theory thereby applies both to the efficient and equilibrium analyses; however, our interest in the planner's problem is for the information it provides us about individual agents, since the planner's multipliers are precisely the agents' private present values of wages. In Section 3, we develop Becker's model for workers with uncertain abilities, explore the tradeoff between static complementarity and dynamic information gathering, and prove our PAM failure result. In Section 4, we analyse the employment model. A technical appendix follows.

2. THE MATCHING ECONOMY

2.1. *The static matching model*

We consider a matching model with a continuum of agents, each described by a scalar human capital x belonging to $[0, 1]$. Let $Q(x, y)$ denote the static output of the match of types x and y. We assume that $Q(x, y)$ is symmetric, twice smooth, increasing in x and y, with a nonzero cross partial, lest matching trivialize. As we assume everyone is risk neutral, Q can be either a deterministic output function or the expected output from stochastic production.

A twice differentiable function Q is *strictly supermodular iff* $Q_{12} > 0$, and *strictly submodular* when $Q_{12} < 0$. Although we do not require any special assumptions on Q for our existence and welfare theorems, the following assumption is used in some characterization results.

Assumption 1 (Supermodularity). $Q_{12}(x, y) > 0$.

Assume a distribution G over human capital $x \in [0, 1]$. The social planner maximizes the expected value of output. For now, let $F(x, y)$ be the measure of matches inside $[0, x] \times [0, y]$. As the planner cannot match more of any type than available, he solves:

$$\mathcal{V}(G) = \max_F \int_{[0,1]^2} Q(x, y)F(dx, y)dy \tag{1}$$

$$\text{s.t.} \qquad \text{Feasibility:} \quad F(x, y) \le G(x)\ \forall y. \tag{2}$$

The *matching set* is the support of $F(x, y)$. *Positive assortative matching* (PAM) obtains if the matching set coincides with the 45° line, so that $F(x, y) = G(\min(x, y))$. *Negative assortative* *matching* (NAM) obtains when every reputation x matches only with the opposite reputation $y(x)$ solving $G(y(x)) = 1 - G(x)$. Then:[1]

Proposition 1 (Becker, 1973). *Given supermodularity, PAM solves the planner's static maximization problem.*[2] *NAM is efficient given submodularity.*

In a competitive equilibrium, each worker x chooses the partner y that maximizes his (expected) wage $w(x|y)$, achieving his *value* $v(x)$. Also, wages of matched workers exhaust output, and the market clears. Altogether, a *competitive equilibrium* (CE) is a triple (F, v, w) where F obeys the feasibility constraint (2), while F, v, w satisfy:

- Worker maximization: $v(x) = w(x|y)$ and $v(y) = w(y|x)$ for all $(x, y) \in \text{supp } F$.
- Value maximization: $v(x) = \max_y w(x|y)$.
- Output shares: $w(x|y) + w(y|x) = Q(x, y)$. (3)

Becker proved the welfare theorems which Theorem 3 revisits in a dynamic setting.

Theorem 1 (Becker, 1973). *The First and Second Welfare Theorems obtain, and the competitive equilibrium wage is $w(x \mid y) = Q(x, y) - v(y)$ for any matched pair.*

2.2. *Dynamically evolving human capital*

We now develop our model in a stationary infinite horizon context over periods $0, 1, 2, \ldots$. Crucially, human capital evolves with each match. For instance, when junior and senior colleagues match, each is changed from the experience. We capture these dynamic effects by positing a transition function $\tau(s|x, y)$, which is the sum of the *transition chances* that x updates to at most s, and that y updates to at most s, when x matches with y. Let X^* be the space of matching measures on $[0, 1]^2$, with generic cdf F. For any $z \in [0, 1]$, let $B : X^* \to Z$ be the posterior cdf

$$B(F)(z) = \textstyle\int_0^z \int_{[0,1]^2} \tau(s|x, y)F(dx, dy)ds.$$

Towards a nondegenerate steady state, we assume that agents live to the next period with *survival chance* σ, and to maintain a constant mass 1 of agents, posit a $1 - \sigma$ weight on the inflow cdf $\overline{G}$. To properly align incentives, we assume that the agents' implicit rate of time preference equals the planner's discount factor $\gamma < 1$ scaled by the survival chance, namely $\delta \equiv \sigma\gamma$. To avoid trivialities, $\overline{G}$ does not place all weight on 0 and 1.

Given an initial type cdf G, the planner chooses the matching cdf F in each period to maximize the average present value of output, respecting feasibility. Let $\Phi(G)$ be the *feasibility set* in (2). For any $F \in \Phi(G)$, define the policy operator

$$T_F\mathcal{V}(G) = (1 - \delta) \int\int Q(x, y)F(dx, dy) + \delta\mathcal{V}((1 - \sigma)\overline{G} + \sigma B(F)). \tag{4}$$

1. Our propositions are descriptive matching results, and theorems are technical equilibrium results.

2. Becker proved this for the discrete case. For our purposes, Lorentz (1953) is more appropriate as he proved the formal result in the continuum case (albeit without providing any economic context).

Here, $(1-\sigma)\overline{G}+\sigma B(F)$ is next period's type distribution. Thus, the planner solves for the *Bellman value* $\mathcal{V}$, namely a fixed point of the operator $T\mathcal{V}(G) = \max_{F\in\Phi(G)} T_F\mathcal{V}(G)$.

The planner trades off more output today for a more profitable measure over types tomorrow. This trade-off lies at the heart of our paper. A *steady state Pareto optimum* (PO) is a triple (G, F, v) such that (F, v) solves the planner's problem given G, and $G = (1-\sigma)\overline{G}+\sigma B(F)$. Just as in the analysis of the modified golden rule in growth models, the social planner does not maximize across steady states. Instead, she chooses an optimal matching in each period, after which the steady state requirement is imposed. While our results obtain both in *and* out of steady state, we focus on the steady state for simplicity.

2.3. *Existence and welfare analysis*

Theorem 2 (Pareto optimum). *A steady state Pareto optimum exists.*

The appendix proves this. The first order conditions (FOC) for this problem are:

$$(x, y) \in supp(F) \Rightarrow v(x)+v(y)-(1-\delta)Q(x, y)-\delta\Psi^{v}(x, y) \quad = \quad 0 \qquad (5)$$

$$v(x)+v(y)-(1-\delta)Q(x, y)-\delta\Psi^{v}(x, y) \quad \geq \quad 0, \qquad (6)$$

where $v(x)$ is the multiplier on the constraint (2), i.e. the *shadow value* of an agent x, and $\Psi^{v}(x, y) = \psi^{v}(x|y)+\psi^{v}(y|x)$ is the sum of the *expected continuation values* $\psi^{v}(x|y) = E[v(x')|x, y]$. So the sum of the shadow values in any matched pair (a) equals the planner's total value of matching them, and (b) weakly exceeds their alternative value in other matches.

In a *competitive equilibrium* (CE), let $w(x|y)$ be the wage that agent x earns if matched with y. Anticipating a welfare theorem to come, we overuse notation, letting $v(x)$ denote the maximum discounted sum of wages that x can earn–the *private value*. A *steady state CE* is a 4-tuple (G, F, v, w), where $G = (1-\sigma)\overline{G}+\sigma B(F)$, F obeys constraint (2), wages $w(x|y)$ are output shares (3), dynamic maximization obtains:

- Worker maximization: $$v(x) = \max_{y}[(1-\delta)w(x|y)+\delta\psi^{v}(x|y)], \qquad (7)$$

and finally y is a maximizer of (7) whenever $(x, y) \in supp\ (F)$.

Theorem 3 (Welfare theorems). *If (G, F, v, w) is a steady state CE, then (G, F, v) is a steady state PO. Conversely, if (G, F, v) is a steady state PO, then (G, F, v, w) is a steady state CE, where for all matched pairs (x, y), the wage $w(x|y)$ of x satisfies:*

$$w(x|y) = \overbrace{Q(x, y)-v(y)}^{\textit{static wage}} + \overbrace{\frac{\delta}{1-\delta}[\psi^{v}(y|x)-v(y)]}^{\textit{dynamic rent } (y \textit{ to } x)}. \qquad (8)$$

See how we assert that the planner's shadow values and the private values coincide. These welfare theorems are greatly complicated by the evolution of types. Fortunately, continuation values are linear, and therefore convex, in measures of matched agents.

The competitive wage has two components. First is the *static wage*, or the difference between match output and one's partner's outside option. Second is the *dynamic rent*, or the discounted excess of one's partner's continuation value over his outside option. That the dynamic benefits are publicly observed sustains the welfare theorems, since they can be compensated. For instance, in our Bayesian model the public reputations serve as the types. Here, dynamic rents will be positive by convexity even when identical agents match, and reputations near 0 and 1 will earn greater dynamic rents.

For some insight into why this wage decentralizes the Pareto optimum, consider a pair (x, y) matched in equilibrium. Worker maximization (7) requires that $v(x)$ equal

$$(1-\delta)w(x|y) + \delta\psi^v(x|y) = (1-\delta)\left[Q(x,y) - v(y)\right] + \delta[\psi^v(y|x) - v(y) + \psi^v(x|y)]$$

using our computed wage (8). With some simplification, we get:

$$v(x) + v(y) = (1-\delta)Q(x,y) + \delta\left[\psi^v(x|y) + \psi^v(y|x)\right],$$

which holds if (x, y) are matched in the Pareto optimum, by the planner's FOC (6).

Finally, we consider existence. Theorem 2 proved that a steady state PO exists; also, any such PO can be decentralized as a CE, by Theorem 3. Altogether:

Corollary 1. *There exists a steady state competitive equilibrium.*

2.4. *Values, shadow values, and dynamic rents*

We next exploit the equivalence between the competitive equilibrium and Pareto optimum, and prove that agents' private values $v(x)$ are convex. The convexity of the multipliers is a separate new contribution.

Theorem 4 (MPS and convexity) . *Assume bilinear, strictly supermodular output $Q(x, y)$, with $\int_0^z \tau(s|x, y)ds$ convex in x and y, and convex along the diagonal $y = x$.*
(a) *The planner's value $\mathcal{V}$ strictly rises in mean-preserving spreads (MPS) of types.*
(b) *The shadow value $v(x)$ is* everywhere convex *(i.e. convex and nowhere locally flat).*
(c) *The expected continuation value function $\psi^v(x|y)$ is separately convex in x and y.*

Proof of (a). *$\mathcal{V}$ strictly rises in mean preserving spreads.* Let's consider monotonicity of the planner's value $\mathcal{V}$ in mean preserving spreads:

($\mathcal{P}$) $\mathcal{V}(\hat{G}) \geq \mathcal{V}(G)$ *whenever* $\hat{G}$ is a mean preserving spread of G.

We prove below that if $\mathcal{V}$ obeys $\mathcal{P}$, then $T\mathcal{V}$ obeys $\mathcal{P}$, and because $\mathcal{P}$ is closed under the sup norm, the fixed point $\mathcal{V} = T\mathcal{V}$ obeys $\mathcal{P}$. In fact, we prove that $T\mathcal{V}$ obeys the stronger property $\mathcal{P}_+$, where strict inequality obtains, so that $\mathcal{V}$ strictly rises in MPS.

Let $\hat{G}$ be an MPS of G (the premise of $\mathcal{P}$). Write $\zeta(x) = \hat{G}(x) - G(x)$, where $\int_0^1 x d\zeta(x) = 0$, and ζ does not almost surely vanish. Let $F \in \Phi(G)$ be optimal for G, and define a new matching $\hat{F}(x, y) = F(x, y) + \min(\zeta(x), \zeta(y))$. So $\hat{F}$ differs from F insofar as it places all weight not common to G and $\hat{G}$ along the diagonal. Since $\hat{G}$ is an MPS of G, and $Q(x, x)$ is everywhere convex, being bilinear and strictly supermodular:

$$\int Q d\hat{F} - \int Q dF = \int Q(x,x)d\zeta(x) = \int Q(x,x)d\hat{G}(x) - \int Q(x,x)dG(x) > 0.$$

For the same reason, and since $B(F)$ is a linear operator, we have $B(\hat{F})(s) - B(F)(s) = \int \tau(s|x,x)d\varepsilon(x) = \int \tau(s|x,x)d\hat{G}(x) - \int \tau(s|x,x)dG$. Changing the order of integration:

$$\int_0^z B(\hat{F})(s)ds - \int_0^z B(F)(s)ds = \int \int_0^z \tau(s|x,x)ds\ d\hat{G}(x) - \int \int_0^z \tau(s|x,x)ds\ dG(x)$$

which is non-negative because $\int_0^z \tau(s|x,x)ds$ is convex and $\hat{G}$ is an MPS of G.

Proof of (*b*). *Convexity of the shadow value.* For any x in the support of G, equally spread a small fraction ε of the G distribution near x to $x \pm h$, where $h > 0$ is feasible and arbitrary. The slope of $\mathcal{V}(G) = \int v(x)G(dx)$ in ε is proportional to $[v(x+h) + v(x-h)]/2 - v(x)$, at $\varepsilon = 0$. Since $\mathcal{V}(G)$ rises in *any* MPS of G, this must be strictly positive. So the planner's shadow value $v(x)$ is everywhere convex.

Proof of (*c*)*: Convexity of the continuation shadow value.* For any (x, y) in the support of F, equally spread a small fraction ε of the distribution near (x, y) to $(x \pm h, y)$, where $h > 0$ is feasible and arbitrary. Likewise spread (y, x) to $(y, x \pm h)$. Since $\int_0^z \tau(s|x, y)ds$ is bi-convex, the continuation distribution incurs an MPS, and continuation values weakly rise. As F is symmetric, so is Ψ^v, and the slope of $\mathcal{V}(B(F)) = \int \Psi^v(x, y)F(dx, dy)$ in ε is proportional to $[\Psi(x+h, y) + \Psi(x-h, y)]/2 - \Psi(x, y)$, which must be non-negative. Altogether, Ψ^v is convex in x, and likewise y. ‖

Since the shadow value is everywhere convex, it is less than its continuation.

Corollary 2 (Dynamic rents). *Dynamic rents for interior types are positive, or* $\psi^v(x|y) - v(x) > 0$ *when* $0 < x < 1$ *and* y *is the reputation of* x*'s match partner.*

For a foretaste of the general applicability of our framework, we briefly explore an example with a supermodular integrated transition chance $\int_0^z \tau(s|x, y)ds$. In such an example, by the logic of the last proof, PAM maximizes static payoffs $\int QdF$ *and* the integrated continuation values cdf $\int_0^z B(F)(s)ds$. So the planner's value rises in any MPS *and* PAM constitutes a PO allocation. For instance, assume that individuals pull towards their partner's type in a deterministic way. Specifically, after x matches with y, his type moves to $\alpha x + (1-\alpha)y$. Then the integrated transition cdf equals

$$\int_0^z \tau(s|x, y)ds = \max\{0, z - \alpha x - (1-\alpha)y\} + \max\{0, z - (1-\alpha)x - \alpha y\}.$$

Any maximum of linear functions is supermodular by Topkis 2.6.2(a).

3. REPUTATION IN A PARTNERSHIP MODEL

3.1. *Static production and reputations*

We now specialize to a matching model where each agent can either be "high" ($\mathcal{H}$) or "low" ($\mathcal{L}$). Only nature knows the abilities. There are $N > 1$ possible nonnegative *output levels* q_i. For each pair of matched abilities, there is an implied distribution over output levels. Output q_i is realized by pairs $\{\mathcal{H}, \mathcal{H}\}$, $\{\mathcal{H}, \mathcal{L}\}$, and $\{\mathcal{L}, \mathcal{L}\}$ with respective chances h_i, m_i, and ℓ_i. As probabilities, we have $\sum_i h_i = \sum_i m_i = \sum_i \ell_i = 1$. The *expected outputs* are $H = \sum h_i q_i$, $M = \sum m_i q_i$, and $L = \sum \ell_i q_i$, while we define column vectors $\boldsymbol{h} = (h_i)$, $\boldsymbol{m} = (m_i)$, and $\boldsymbol{\ell} = (\ell_i)$. Stochastic output is essential, as we seek a model in which uncertainty about abilities persists over time; we do not want true abilities revealed after the first period. Figure 1 summarizes.

Each of a continuum $[0, 1]$ of individuals has a publicly observed chance $x \in [0, 1]$ that his ability is $\mathcal{H}$. Call x his *reputation*. So a match between agents with reputations x and y yields output $q_i \geq 0$ $(i = 1, \ldots, N)$ with probability

$$p_i(x, y) = xyh_i + [x(1-y) + y(1-x)]m_i + (1-x)(1-y)\ell_i.$$

The expected output of this match is

$$Q(x, y) = xyH + [x(1-y) + y(1-x)]M + (1-x)(1-y)L.$$

Chance of output q_i

	$\mathcal{L}$	$\mathcal{H}$
$\mathcal{L}$	ℓ_i	m_i
$\mathcal{H}$	m_i	h_i

Expected output

	$\mathcal{L}$	$\mathcal{H}$
$\mathcal{L}$	L	M
$\mathcal{H}$	M	H

FIGURE 1
Match output

Since $q_i > 0$ for all i, we have $Q(x, y) > 0$ and matching is always optimal. As our production function Q is bilinear, we define the constant $\pi \equiv Q_{12}(x, y) = H + L - 2M$. Thus, Assumption 1 simplifies to $\pi > 0$. Here, π is the premium to pairing $\{\mathcal{H}, \mathcal{H}\}$ and $\{\mathcal{L}, \mathcal{L}\}$ rather than matching $\{\mathcal{L}, \mathcal{H}\}$ twice.

Since output is SPM, Proposition 1 implies that PAM obtains in a static matching model. Then we have $v(x) = Q(x, x)/2$, which is strictly convex, as Q is bi-linear and supermodular:

$$Q(x, x) \quad = \quad x^2 H + 2x(1 - x)M + (1 - x)^2 L = \pi x^2 + 2(M - L)x + L.$$

This convexity is crucially exploited in the last period of the two period model below.

Two reinterpretations. One may reinterpret this as a model of within-firm team assignment with unknown worker types. If pairs of workers perform tasks and the firm maximizes the present value of its output, then it solves our planner's problem.

We can also dispense with the assumption that tasks must be performed by groups of workers, but assume workers are employed. We perform this transformation in Section 4.

3.2. *Matching in a two period model*

To build intuition for our infinite horizon results, consider a stylized two period model with payoffs weighted by $1 - \delta \in [0, 1)$ and δ. While $\delta < 1/2$ in a truly two period model with strict time preference, $\delta > 1/2$ obtains if period 2 means "the future" in an infinite horizon model. Thus, $\delta \to 1$ captures increasing patience. The value function varies with the discount factor in the infinite horizon model. But with two periods, we can exploit the *strict convexity* of the final fixed value function. This dodges a hard complication, allowing us to prove a strong impossibility result.

Bayesian updating and continuation values. In a dynamic model, agents x and y produce publicly observed output q_i when matched; their reputations are then updated by Bayes' rule. Agent x's *posterior reputation* is

$$z_i(x, y) \equiv p_i(1, y)x/p_i(x, y).$$

Also, Assumption 1 precludes $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$, and thus the dynamic economy is not a trivial repetition of the static one. For if $\boldsymbol{h} \neq \boldsymbol{m}$ or $\boldsymbol{\ell} \neq \boldsymbol{m}$, then all reputations but 0 and 1 shift with positive chance after each match: $z_i(x, y) \neq x$ for some i, if $x \neq 0, 1$.

By Theorem 4, $v(x)$ and $\psi^v(x|y)$ are strictly convex in x, while $\psi^v(x|y)$ is strictly convex in *one's partner's reputation* y too. Specifically:

$$\psi^v_{yy}(x|y) = \pi \textstyle\sum_i p_i(x, y)[z_{iy}(x, y)]^2 > 0.$$

PAM failure in the two period model. We now deduce an unqualified failure of PAM unique to this setting which cleanly captures the opposition between the value convexity and production supermodularity. For an extreme case, assume everyone cares only about future output. Then type x's match payoff function would be $\psi^v(x|\cdot)$. PAM would then require that $\psi^v(x|y) + \psi^v(y|x)$ be supermodular on the matching set. This requirement cannot be met.

Proposition 2. *Fix* $x \in (0, 1)$. *Given matches* $(0, 0)$, (x, x), *and* $(1, 1)$, *the expected total continuation value is strictly raised by rematching* x *with either* 0 *or* 1.

Since $\psi^v(x|y)$ is strictly convex in y, either $\psi^v(x|0) > \psi^v(x|x)$ or $\psi^v(x|1) > \psi^v(x|x)$. So matching the unknown agent x with either of the known abilities 0 or 1 is dynamically more profitable than assigning him to another x. Since agents 0 and 1 have the same posterior reputation regardless of partner, this implies that either:

$$\psi^v(x|0) + \psi^v(0|x) > \psi^v(x|x) + \psi^v(0|0) \text{ or } \psi^v(x|1) + \psi^v(1|x) > \psi^v(x|x) + \psi^v(1|1). \qquad \|$$

Assume PAM in period zero. Re-match as many of the reputations $x \in (0, 1)$ with 0 or 1 as possible (the choice governed by Proposition 2). The informational gains of this rematching are strictly positive, and swamp the production losses, for large δ.

Corollary 3. *In the two period model, PAM is not an equilibrium for large* $\delta < 1$.

One might venture that extreme agents 0 and 1 are informationally valuable because all match output variance owes to the uncertain ability of the middle type x. But our argument shows only that at least one extreme agent must be informationally valuable: they both need not be. In the numerical example below, the dynamic effect reinforces the static output effect near one extreme and conflicts near the other.

Illustrative example of assortative matching failure. We consider an example technology reminiscent of the O-ring failure in Kremer (1993). Assume that production requires two high abilities, in which case output is produced with chance 1/2. Specifically, assume $(q_1, q_2) = (0, 4)$, $h = (1/2, 1/2)$, and $m = \ell = (1, 0)$. This yields supermodular output, since $\pi = H + L - 2M = 2$. A matched pair (x, y) produces output 4 with chance $xy/2$. Reputation x updates to $z_1(x, y) = (2 - y)x/(2 - xy)$ after output $q_1 = 0$, and to $z_2(x, y) = 1$ after output $q_2 = 4$.

Given PAM in period two, the value of reputation x at the start of the second period is: $v(x) \equiv Q(x, x)/2 = x^2$. Now, agent x's expected continuation value is

$$\psi^v(x|y) \equiv p_1(x, y)v(z_1(x, y)) + p_2(x, y)v(z_2(x, y)) = \left(1 - \frac{xy}{2}\right)\left(\frac{(2 - y)x}{2 - xy}\right)^2 + \frac{xy}{2}.$$

The *present value of the match* (x, y) is $\overline{v}(x, y) \equiv 2(1 - \delta)xy + \delta\Psi^v(x, y)$.

To illustrate Proposition 2, consider the reputation $x = 1/2$. Since $m = \ell$ there is no informational advantage to matching any $x \in (0, 1)$ with $x = 0$. But if x assortatively matches, this *is* dynamically valuable – it may well be a $\{\mathcal{H}, \mathcal{H}\}$ match. Thus, PAM dynamically dominates matching with 0. However, there are informational gains matching with a 1, for example, $\psi^v\left(\frac{1}{2}|1\right) - \psi^v\left(\frac{1}{2}|\frac{1}{2}\right) \approx 0.048$. More generally, whenever $\Psi^v_{12} > 0$, then $\overline{v}$ is supermodular, and PAM is efficient and an equilibrium. So we check along the

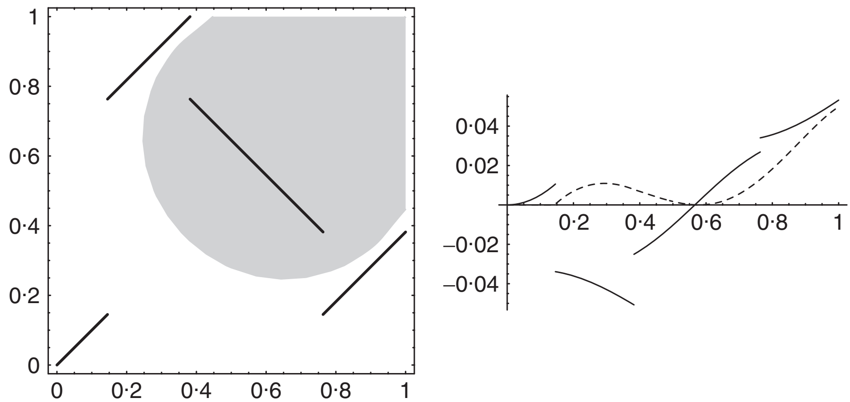

FIGURE 2

Two period example. On the left, we depict the shaded submodular total value region (where $\overline{v}_{xy} < 0$), and the resulting discontinuous optimal matching graph $\mathcal{G} = \{(x, y(x)), 0 \leq x \leq 1\}$ (solid line). On the right, we plot the equilibrium wage function $w(x) \equiv w(x|y(x))$ (solid line). Given the high discount rate $\delta = 0.99$, the wage $w(x|y(x))$ is almost entirely an information rent $\psi^v(y(x)|x) - v(y(x))$–whose discontinuity forces a jump in the wage profile. We superimpose the surplus in optimal values over assortative values. The solution was produced by linear programming with a discrete mesh on [0, 1]

diagonal $y = x$ and find $\Psi^v_{12}(x, x) \gtreqless 0$ for $x \lesseqgtr 0.36$. *Here, learning reinforces the productive supermodular effect for low reputations* x*, but opposes it for high reputations*. Figure 2 depicts the solution for $\delta = 0.99$, for an initial uniform density over reputations and no entry.

Here is an intuition for the shape of the matching set $\mathcal{G}$ in Figure 2. By local optimality considerations, $\mathcal{G}$ is decreasing whenever the match value $\overline{v}(x, y)$ is submodular (shaded region). Next, it cannot exit the supermodular region on a downward slope. Third, by the uniform density on reputations, $\mathcal{G}$ has slope ± 1 whenever continuous.[3]

Over 80% of all agents non-assortatively match, paying or earning an informational rent payment. High reputation agents are willing to match "down" (the solid line in the right-hand panel of Figure 2), as they earn a wage premium for doing so. *The wage profile jumps at each match discontinuity.* Indeed, the information rent in (8) jumps up at the first break point near 0.16, and down at the next two break points near 0.4 and 0.75.

3.3. *Infinite horizon matching by reputation*

In principle, to update the reputations of individuals after any match, one can exploit information about the outcomes of the current matches involving their past partners. This would render our model both intractable and unrealistic, since it would entail complicated output sharing arrangements, involving transfers between past partners. At the same time, we do not want completely anonymous individuals, for that would limit our empirical applications. We adopt a simple compromise:

3. See Kremer and Maskin (1996) for formal characterizations of solutions in a one-shot matching model where match non-supermodular match values induce wage discontinuities.

Assumption 2. *The entire output history of currently matched individuals is observable; however, once a partnership dissolves, only the reputation of each individual is recorded.*

With this assumption, reputation is a sufficient statistic for the information from all previous matches, and yet we may still speak of partnerships in a meaningful sense.

Value convexity. The convexity of the value function in beliefs for a single agent learning problem is well known (see Easley and Kiefer, 1988). Although Theorem 4(b) is new, it admits the standard intuition that information is valuable–only here, its value is to the planner. Why? Suppose that a signal is revealed about the true ability of an agent with reputation x. This resulting random reputation has mean x (so a "fair" gamble). The signal also cannot harm the planner–for she could choose to ignore it. Recall that a decision maker is averse to all fair income gambles iff his utility function is concave. Inverting this logic, the value function is strictly convex since the information is strictly beneficial. For the planner is not indifferent across matches when $\pi = H + L - 2M \neq 0$, since there is a productive reason to prefer assortative or non-assortative matches.

Not only is information about one's own ability valuable, but so too is information about one's partner. The intuition for Theorem 4(c) is one step upstream from value convexity. Given a better signal about x's partner, the match yields a better signal about x. By the Jensen Theorem logic, the function $\psi^v(x|y)$ must be convex in y.

Productive versus informational efficiency. Our two period result obtains because the continuation value function is *fixed*, given PAM in the final period (by Becker); further, it is boundedly and *strictly convex*. Thus, PAM fails with sufficient patience, given our either–or inequality in Proposition 2.

But with no last period, the continuation value function depends on the discount factor in a way that undermines the two period logic of Proposition 2: for as the discount factor δ rises, the value function v_δ flattens out in the limit–hereby indicating the dependence on the discount factor δ. Not only do the static *losses* from PAM vanish, but so too do the dynamic *gains*. We may then have been misled: an infinite horizon model is needed to resolve this race to perfect patience.

A matching is *productively efficient* if it yields the highest current output. It is *dynamically efficient* if no other matching yields a greater continuation output. A necessary condition for either efficiency notion is that no marginal matching change can raise output today or in the future. Suppose we shift from assortatively matching (x, x) and $(x + \Delta, x + \Delta)$ to cross-matching $(x, x + \Delta)$ and $(x + \Delta, x)$. By a second order Taylor Series, the static welfare change is approximately $Q_{12}(x, x)\Delta^2 = \pi\Delta^2$. The dynamic change from such a rematching is approximately $\Psi_{12}^{\delta}(x, x)\Delta^2$. The net welfare change from this matching change is therefore proportional to:

$$\overbrace{(1-\delta)\pi}^{\text{static production losses}} \quad + \quad \overbrace{\delta\Psi_{12}^{\delta}}^{\text{dynamic informational change}} \quad . \tag{9}$$

For PAM to be efficient, this weighted sum must be positive. But our tradeoff is knife-edged in the limit: both the static losses from not matching assortatively and the dynamic gains vanish as $\delta \to 1$, as the value converges upon a linear function: $v_\delta(x) \to x v_\delta(1) + (1 - x)v_\delta(0)$. Thus, the cross partial Ψ_{12}^{δ} vanishes in the limit $\delta \to 1$.

Actually the asymptotic behaviour of the value function is more complex than this logic suggests. While the second derivative $v_\delta''(x)$ at any interior x tends to zero, the integral

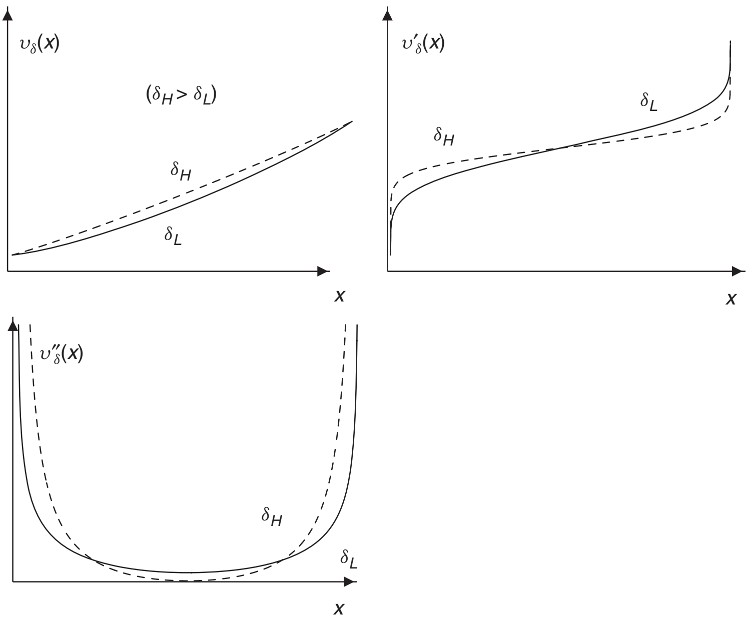

FIGURE 3
Value function v_δ and derivatives v'_δ and v''_δ. This graph depicts the value function flattening, and the convexity explosion near 0, 1: $\lim_{x\to 0} v''_\delta(x) = \lim_{x\to 1} v''_\delta(x) = \infty$

$\int_0^1 v''_\delta(x)dx$ is constant in δ. Perforce, $v''_\delta(x)$ explodes near the extremes 0 and 1 (as in Figure 3). This suggests that we should try to prove our PAM failure near the extremes.[4] There are three logically separate steps that we must take to prove our main result.

- Lemma 1 finds when dynamic and productive efficiency conflict near 0 and 1.
- Proposition 3 finds when dynamic efficiency dominates for large δ.
- Proposition 4 shows that this domination generally occurs for large δ, as $N \uparrow \infty$.

The sign of the information effect assuming PAM. Determining the *optimal value* function in general is an intractable problem. So instead, we derive the PAM value function and then show that PAM is not optimal for the induced value function $v_\delta(x)$, by applying a new finding in Anderson (2009) that if a *fixed policy* generates a convex static payoff, then the second derivative of the value function explodes at a *geometric rate* near extremes 0 and 1. Specifically, Claim 5(a) in Appendix B.1 proves that the PAM value function satisfies:[5]

$$\begin{aligned} v''_\delta(x) &\sim \kappa_\delta x^{-\alpha_\delta} \qquad x \to 0, \quad \text{where } \alpha_\delta \text{ solves } 1 \equiv \delta \textstyle\sum_i \ell_i (m_i/\ell_i)^{2-\alpha_\delta} \\ v''_\delta(x) &\sim \kappa_\delta (1-x)^{-\beta_\delta} \quad x \to 1, \quad \text{where } \beta_\delta \text{ solves } 1 \equiv \delta \textstyle\sum_i h_i (m_i/h_i)^{2-\beta_\delta} \end{aligned} \tag{10}$$

when $\delta < 1$ large enough. We next find when dynamic and productive efficiency conflict near $x = 0$ or 1. This fully exploits (10), and not only the explosive nature.

4. This is only meaningful if the steady state cdf G assigns positive weight to extremal reputations. Since we do not have $h = m = \ell$, whenever $x \notin \{0, 1\}$ matches with x, his type updates down and up with positive chance. Thus as long as $\overline{G}$ does not put all weight on 0 and 1 (as assumed), PAM implies that in any steady state, G will assign positive weight to every open interval of reputations.

5. Standard in asymptotics, we write $a(\delta) \sim b(\delta)$ if $a(\delta)/b(\delta) \to 1$, in the given δ limit, say $\delta \to 1$.

We now introduce an asymmetric correlation function for three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Let:

$$S(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \equiv \sum_i \left[(b_i - a_i) \log \left(\frac{b_i}{c_i} \right) + \frac{b_i^2 - a_i c_i}{c_i} \right].$$

We know of no precedent for this expression. Generally, $S(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \geq 0$ when the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are close. For instance, $S(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b}) = 0$, $S(\boldsymbol{a}, \boldsymbol{a}, \boldsymbol{c}) > 0$, and $S(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}) > 0$.[6]

Lemma 1. *[Dynamic efficiency] There exists $\delta^* < 1$, such that for all $\delta > \delta^*$, PAM is not dynamically efficient near $x = 0$ if $S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) > 0$ and near $x = 1$ if $S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h}) > 0$.*

Proof sketch. Claim 6 in Appendix B.1 uses our approximation (10) to prove that the expected continuation value has cross partial $\Psi_{12}^{\delta}(x, x) \sim [\kappa_\delta/(1-\alpha_\delta)] x^{1-\alpha_\delta} R(\alpha_\delta)$ near $x = 0$, where $\kappa_\delta/(1-\alpha_\delta)$ converges to a positive constant as $\delta \to 1$, and $R(\alpha)$ is a function with $R(1) = 0$ and slope $R'(1) = S(h, m, \ell)$. Thus, $\Psi_{12}^{\delta}(x, x)$ shares the sign of $-S(h, m, \ell)$ near $x = 0$, and $-S(\ell, m, h)$ near $x = 1$. So a conflict between dynamic and productive concerns $\Psi_{12}^{\delta}(x, x) < 0$ arises near $x = 0$ or $x = 1$ iff $S(\cdot) > 0$. ||

For more intuition, assume $m = \ell$–so that $m \neq h$, by Assumption 1. Then the superior $\{\mathcal{H}, \mathcal{H}\}$ matches can be statistically distinguished from the $\{\mathcal{L}, \mathcal{H}\}$ and $\{\mathcal{L}, \mathcal{L}\}$ matches, but the latter two cannot be so nuanced.[7] Consider the signal from an x paired with 0 or 1. Type 0 provides no information, as $\{\mathcal{L}, \mathcal{H}\}$ and $\{\mathcal{L}, \mathcal{L}\}$ yield the same output distribution. So we should not expect PAM failures near $(x, x) = (0, 0)$. But since $S(\boldsymbol{m}, \boldsymbol{m}, \boldsymbol{h}) > 0$, PAM is informationally inefficient near $(x, x) = (1, 1)$.

In the extreme case, $\boldsymbol{m} = \boldsymbol{\ell}$, it takes two "high" agents for stochastically better production; when at least one agent is low ability, the ability of the other agent is irrelevant. With this observation, let's call this a *perfectly high skill* technology. By the same token, $h = m$ yields a *perfectly low skill* technology.

Note that $S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) > 0$ in the perfectly high skill case, while $S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h}) > 0$ in the perfectly low skill case. Thus inspired, we call a technology *high skill* iff $S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) > 0$, and *low skill* iff $S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h}) > 0$. While supermodularity rules out technologies that are simultaneously perfectly high and perfectly low skill (i.e. $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$), a technology can be both high and low skill. We soon argue that such technologies are "common".

The race to perfect patience. We have seen that $S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) > 0$ forces a tradeoff between dynamic and productive efficiency near $x = 0$. We now characterize when the dynamic efficiency dominates for high δ.

Proposition 3 (PAM failure). *There exists $\delta^* < 1$ such that $\forall\ \delta > \delta^*$, in the infinite horizon model, PAM fails in a neighbourhood of* $(0, 0)$ *or respectively* $(1, 1)$ if*:*

$$S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) > 2 \textstyle\sum_i m_i \log(m_i/h_i) \quad \textit{and} \quad S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h}) > 2 \textstyle\sum_i m_i \log(m_i/\ell_i). \qquad (11)$$

6. Indeed, $S(a, a, c) \equiv \sum_i (a_i - c_i) a_i / c_i > \sum_i (a_i - c_i) = 0$, while $S(a, b, a) \equiv \sum_i [(b_i - a_i) \log (b_i/a_i) + (b_i^2/a_i) - a_i] = \sum_i [(b_i - a_i) \log (b_i/a_i) + (b_i^2/a_i) - b_i] = \sum_i [\text{non-negative terms}] + S(b, b, a) > 0$.

7. In Kremer (1993), production success requiring no mistakes by all parties is an excellent example of such a technology. More positively, creative work really identifies whether both parties are talented.

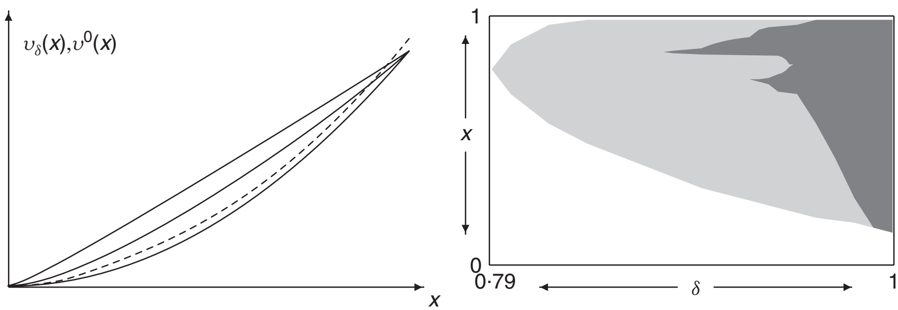

FIGURE 4
Infinite horizon vs. two-period models. In the left panel, the value function v_δ in the infinite horizon model for $\delta \in \{0, 0.9, 0.99\}$ is depicted as a solid line, while the two period model v^0 for $\delta = 1$ is a dashed line. In the right panel, the dark shaded region is the correspondence between δ and the set of reputations x (vertical axis) that do not match assortatively in the infinite horizon model. The lightly shaded region is the same correspondence for the two period model

Proof. Since both terms in (9) vanish, we divide by $-(1-\delta)$, and take the limit:

$$\pi + \lim_{\delta \to 1} \frac{\delta \Psi^\delta_{12}(x, x)}{1-\delta} = \pi + \lim_{\delta \to 1} \frac{\kappa_\delta}{1-\alpha_\delta} \frac{R(\alpha_\delta)}{1-\delta} x^{1-\alpha_\delta} = \pi - \frac{S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell})}{\sum_i m_i \log(m_i/h_i)} \lim_{\delta \to 1} \frac{\kappa_\delta}{1-\alpha_\delta} \tag{12}$$

using l'Hôpital's rule to get $\lim_{\delta \to 1} R(\alpha_\delta)/(1-\delta) = R'(1)\dot{\alpha}_1$, and then $R'(1) = S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell})$ and $\dot{\alpha}_1 = \sum_i m_i \log(m_i/h_i)$ from (10), given $\dot{\alpha}_1 = \partial \alpha_\delta / \partial \delta$ evaluated at $\delta = 1$. If the limit (12) is negative, then PAM fails near $x = 0$. The same expression arises for x near 1 with the analogues $\underline{\kappa}_\delta$, $\underline{\alpha}_\delta$, and $S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h})$. The sum of (12) and its analogue near $x = 1$ is negative when $\lim_{\delta \to 1}[\kappa_\delta/(1-\alpha_\delta) + \underline{\kappa}_\delta/(1-\underline{\alpha}_\delta)] \geq \pi$, given (11). Is this inequality true? Integrating approximation (10),

$$v'_\delta(x) \sim v'(0) + \frac{\kappa_\delta}{1-\alpha_\delta} x^{1-\alpha_\delta} \quad \text{and} \quad v'_\delta(1-x) \sim v'(1) - \frac{\underline{\kappa}_\delta}{1-\underline{\alpha}_\delta} x^{1-\underline{\alpha}_\delta}$$

for small $x \approx 0$. As $\delta \to 1$, v_δ becomes linear, while $v'_\delta(x)$ converges to a constant for any $x \in (0, 1)$. That is, $v'_\delta(x) \to v'_\delta(1-x)$ as $\delta \uparrow 1$ for any $x \in (0, 1)$. This leads to:

$$\lim_{\delta \to 1} \left[\frac{\kappa_\delta}{1-\alpha_\delta} + \frac{\underline{\kappa}_\delta}{1-\underline{\alpha}_\delta} \right] = v'_\delta(1) - v'_\delta(0).$$

Finally, Appendix B.2 proves that $v'_\delta(1) - v'_\delta(0) = \pi$. Thus, PAM fails near 0 or 1. ||

For an idea of how large the deviations from PAM may be, we now extend the earlier two period example to an infinite horizon. In Figure 4, we have graphed the value function in the infinite horizon model for three discount factors $\delta \in \{0, 0.9, 0.99\}$ as solid lines. At $x = 0$, the value functions coincide. They flatten out as $\delta \uparrow 1$. For comparison purposes, we depict the first period expected value function v^0 in the two period model for $\delta = 1$ as a dashed line.

The right panel of Figure 4 shades in the agents for whom PAM fails. Reflecting the diminished convexity with longer horizon models, the PAM failure set is strictly smaller at all discount factors δ in the infinite horizon. PAM obtains in the infinite horizon model for $\delta \leq 0.9$. No $x \geq 0.14$ is matched assortatively at $\delta = 0.99$.

PAM fails for "almost all" production technologies. We have developed sufficient conditions for non-assortative matching at the extremes, and yet we claimed an almost general failure of PAM. How can we justify our assertion? For indeed, with fixed N, the inequalities (11) are *not* always satisfied, and therefore PAM may well obtain for *both* high *and* low reputations. Here is a simple parameterized example of this phenomenon. Let $\boldsymbol{h} = (3\varepsilon^2, 1/2 - 3\varepsilon^2, 1/2)$, $\boldsymbol{m} = (\varepsilon, 1 - 2\varepsilon, \varepsilon)$, and $\boldsymbol{\ell} = (1/2, 1/2 - 3\varepsilon^2, 3\varepsilon^2)$, where $q = (q_1, q_2, q_3) = (0, 0.1, 1)$. For this technology, we have $S(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}) = S(\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{h}) \approx$ (constant) $+ \log(\varepsilon)/2 < 0$ for small ε. For this example, *PAM is optimal even for high* δ *in the infinite horizon model.*

We now assert that this example, while robust for fixed $N = 3$, is vanishingly rare when the number N of production outcomes grows:

Proposition 4. *With technologies* $(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}, q)$ *drawn from an atomless distribution, PAM fails near* 0 and 1 *with chance tending to* 1 *as* $N \to \infty$.

Despite the nature of this result, simulations suggest that PAM failures are extremely likely even for low N. For example, with parameters $(\boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\ell}, q)$ uniformly generated on the unit simplex, we found simultaneous violations of (11) only 43, 18, 5, and 1 out of *one billion* times for the $N = 3, 4, 5, 6$ cases respectively.

Yet again, the *exact* asymptotic form of the value v is critical for Proposition 4. To see this, suppose instead that $v(x) = x \log x - x + constant$. Then $v'(x) = \log x$ and $v''(x) = x^{-1}$. This value function is convex and further v'' is unbounded near $x = 0$; however, v'' lies just outside the geometric family (10), and in fact, Proposition 4 fails.

Long run match dynamics. Our focus until now has been on a failure of PAM *in the large*–on the distribution of matches in an economy. Our model also admits a rich unfolding *micro* story: agents are born and then form and dissolve partnerships as their reputations evolve over time. We finally focus our lens on this subplot with turnover, and the breakup of seemingly successful partnerships. Let's turn to the limit behaviour of an individual's reputation.

First we ensure that no agent ever gets stuck at any reputation.

Assumption 3. *It is not true that* $\dfrac{\ell_i - m_i}{h_i + \ell_i - 2m_i} = c$ $\forall i$ *for some constant* $c \in [0, 1]$.

This housekeeping condition follows from setting $z_i(x, y) \equiv x$, and is generically valid. Also, its failure yields $L - M = y(H - 2M + L) > 0$, or non-monotonic output.

Proposition 5. *Fix an agent with reputation* x^t *at time-*$t = 0, 1, 2, \ldots$ *(with* x^0 *given).*
(*a*) *If he is not eventually matched forever with the same partner, then* $x^t \to 0$ *or* 1 *with time-*0 *chances* $1 - x^0$ *and* x^0*, for generic technologies.*
(*b*) *If he is eventually matched with the same partner (initially* y^0*), then generically,* $x^t \to 0$, 1/2, 1 *with ex ante chances* $(1 - x^0)(1 - y^0)$, $(1 - x^0)y^0 + x^0(1 - y^0)$*, and* $x^0 y^0$.

Indeed, reputations are beliefs about underlying abilities, and thus are martingales: namely, one's current reputation is the expected future reputation. It is then well known that they converge to stationary reputations. Assume first case (a). Intuitively, as long as someone does not have the same-reputation partner each period, any reputation $x \in (0, 1)$ is subject to change as information is revealed. So his ability is eventually revealed: $\mathcal{H}$ or $\mathcal{L}$, or $x = 0$ or $x = 1$.

In case (b), the market's ability to learn an agent's ability is frustrated by its lack of knowledge of his partner's type. The applicable "match state space" here is really $\{\{\mathcal{H}, \mathcal{H}\}$, $\{\mathcal{L}, \mathcal{L}\}$, $\{\mathcal{H}, \mathcal{L}\}$, and $\{\mathcal{L}, \mathcal{H}\}\}$. Since production is symmetric, state $\{\mathcal{H}, \mathcal{L}\}$ is indistinguishable from $\{\mathcal{L}, \mathcal{H}\}$. Thus, if matches are permanent, we may end up with $x^{\infty} = y^{\infty} = 1/2$–that is, it may be clear from observing the (infinite) past output realizations that one person in the match is high and one low, but given that only joint output is observable, there is no way to tell which is which.

Now that we know how individual agents' reputations behave in the long run, we can discuss the micro structure of match dynamics.

Corollary 4. *Assume δ high enough. Then any match of two high (low) agents eventually breaks up with chance tending to 1 as $N \to \infty$.*

Convergence to $(1/2, 1/2)$ obtains for a long-lived match $\{\mathcal{H}, \mathcal{L}\}$. The limit is $(1, 1)$ for $\{\mathcal{H}, \mathcal{H}\}$ and $(0, 0)$ for $\{\mathcal{L}, \mathcal{L}\}$, by Proposition 5. If, in addition, the technology is both high and low skill, then Proposition 3 implies that as we approach $\{\mathcal{H}, \mathcal{H}\}$ or $\{\mathcal{L}, \mathcal{L}\}$, like reputation agents cannot stay matched, and so they must break up.

4. PARTNERSHIPS VERSUS THE EMPLOYMENT MODEL

4.1. *The worker–job benchmark*

In this section, we consider a benchmark model in which workers of uncertain talents are matched to known jobs (or firms). We retain the assumption that workers can be one of two underlying types: $\{\mathcal{H}, \mathcal{L}\}$ with x equal to the chance the worker has type $\mathcal{H}$. We may extend all of our results in the previous sections to a model with a continuum of fixed publicly observable job types $y \in [0, 1]$, as this is little more than a re-labelling of what we already have.[8] Instead we consider in more detail a case that is both closer to the existing literature, and substantively different than our partnership model, and with quite different results. Not only does this model serve as a bridge between our partnership model and the existing assignment literature, it is also a more natural model to consider some stylized facts, like worker mobility.

Related work. Roy (1950) considers a static assignment problem with discrete types of jobs (which he terms sectors), plus a continuum of worker characteristics. As in our discrete jobs benchmark, he assumes costless creation of jobs, thus the one-period version of our discrete jobs benchmark is a simple version of Roy's model.

MacDonald (1982) generalized the Roy model by adding symmetric incomplete information about worker types, and is the closest model to the benchmark we consider here. What he shares with our benchmark: two known jobs and two types of workers with optimal task assignments differing by worker type. Like us, he assumes symmetric incomplete information about each worker's type. At the end of every period a public signal for each worker is revealed. Critically, and different than in our framework, signals of worker type are *uncorrelated* with task assignment, while we have made the natural assumption that production *is* the signal of type. By introducing incomplete information, the distribution over workers is changing over

8. This would yield a dynamic incomplete information version of Sattinger (1979).

time in MacDonald, but the dynamic process is exogenous to the assignment, which implies that each period's equilibrium maximizes static output given the current distribution over worker types.

The model with known jobs. As in our partnership model, assume workers are either type $\mathcal{H}$ or $\mathcal{L}$, and each is distinguished by the public belief x that he is type $\mathcal{H}$. Each worker must be matched to one job (or firm). There are two types of jobs, and there is complete information about job types. We abuse notation and use $\mathcal{H}$ and $\mathcal{L}$ for the types of jobs as well as workers, so that, as with our partnership model there are four underlying types of matches: $\{\mathcal{H}, \mathcal{H}\}$, $\{\mathcal{H}, \mathcal{L}\}$, $\{\mathcal{L}, \mathcal{H}\}$, $\{\mathcal{L}, \mathcal{L}\}$. To further draw a parallel with our earlier results we continue to assume symmetry–that $\{\mathcal{H}, \mathcal{L}\}$ and $\{\mathcal{L}, \mathcal{H}\}$ matches produce identical distributions over output, although now this is clearly a stronger assumption.

We assume the same conditional output distributions as in the partnership model, illustrated in Figure 1. We maintain that these distributions are non-degenerate: $h_i, m_i, \ell_i > 0$ and that matching like types yields strictly higher expected output than cross matching: $\pi \equiv H + L - 2M > 0$. For the market equilibrium, we assume free entry of jobs at zero cost. Thus, in equilibrium workers are paid their marginal products in their jobs. Since job creation is costless, the planner maximizes the discounted value of production for each worker, just as each worker would do in the decentralized market. Thus, the welfare theorems again obtain, and further, the assignment problem can be solved worker by worker as a single agent dynamic learning problem.

4.2. *Comparison of results*

The two period model. We first consider a two period version. In the final period, the assignment solution will be degenerate if output monotonically increases in types: $H > M > L$. In this case all workers will be employed by $\mathcal{H}$ type jobs. To avoid this trivial solution to the static problem, we assume for the purposes of this benchmark model (as in Roy and MacDonald) that: $\min\{H, L\} > M$. As a result both types of jobs will employ workers in period 2. In the final period, we get Roy's solution a threshold reputation $\overline{x} = (L - M)/\pi < 1$, such that all workers with reputation below $\overline{x}$ match with job type $\mathcal{L}$ and all workers with reputation above $\overline{x}$ match with job type $\mathcal{H}$. In this equilibrium the value to a reputation x worker in period 2 is:

$$v(x) = \begin{cases} Mx + L(1-x) & x < \overline{x} \\ Hx + M(1-x) & \overline{x} \leq x \end{cases}.$$

Now consider the first period. Conditional on matching with a type $\mathcal{H}$ job, a reputation x worker updates to $z_i(x, 1)$ with chance $p_i(x, 1)$. Alternatively x can match with a type $\mathcal{L}$ and update to $z_i(x, 0)$ with chance $p_i(x, 0)$. As mentioned, MacDonald (1982) studies the case of assigning workers of unknown types to known jobs when signals of worker productivity are uncorrelated with task assignment. In our model, signals are production outcomes, and we cannot simply assume that signals are exogenous to assignments. In fact, except for certain degenerate sets of parameters, assignments in period 1 will impact the distribution over worker reputations in period 2.[9]

9. For example, assume two output levels, with like types producing high output with chance $\lambda > 1/2$, and unlike types succeeding with chance $1 - \lambda$. Then the distribution over signals is uncorrelated with task assignment, and we get a simple version of MacDonald's model.

The continuation value for a reputation x worker matched to a job of type $\mathcal{H}$ is $\psi^v(x,1) = \sum_i p_i(x,1)v(z_i(x,1))$, and similarly $\psi^v(x,0)$ is the continuation value when matched to $\mathcal{L}$. Then a reputation x worker will be assigned to job type $\mathcal{H}$ *iff*:

$$\begin{aligned}(1-\delta)\,(Hx+M(1-x))+\delta\psi^v(x,1) &\geq (1-\delta)\,(Mx+L(1-x))+\delta\psi^v(x,0) \quad \Rightarrow \\ (1-\delta)\,((H-M)x+(M-L)(1-x)) &\geq \delta\left(\psi^v(x,0)-\psi^v(x,1)\right). \end{aligned} \tag{13}$$

The left hand side is the weighted difference in the static expected payoff from matching the worker with job type $\mathcal{H}$ rather than $\mathcal{L}$ and linearly rises at rate $\pi > 0$. Since we have assumed bounded signals, the right side vanishes for all x near the extremes $\{0, 1\}$. Thus, workers and jobs are matched assortatively at the extremes.

Infinite horizon model. Showing that assortative matching occurs near the extremes $\{0, 1\}$ in an infinite horizon model is not as immediate for high δ. As in the partnership model we must carefully balance the static losses from not matching assortatively with the (potential) dynamic gains to see how this trade off plays out for high δ. We carry out this analysis near $x = 0$. The analysis near $x = 1$ is symmetric.

First we establish (in the Appendix) the analogue to our earlier result on dynamic efficiency near the extremes in the partnership model (Lemma 1).

Lemma 2. *There exists $\delta^* < 1$ such that for all $\delta > \delta^*$, matching workers having reputations near $x = 0$ with the type $\mathcal{L}$ job maximizes continuation values* iff*:*

$$\sum_i m_i \log\left(\frac{m_i}{\ell_i}\right) > \sum_i h_i \log\left(\frac{h_i}{m_i}\right). \tag{14}$$

In the partnership model, static and dynamic efficiency are generally at odds as the number of productive outcomes grow. In the employment model this is no longer true. Indeed, inequality (14) and its converse are equally likely if parameters are drawn uniformly over the simplex for all N. Thus, the a.s. failure of PAM that obtained in the partnership model, cannot obtain in this employment model benchmark.

Nevertheless, we can still ask (as we did in the partnership model) whether dynamic efficiency dominates static efficiency for high δ. Or put another way: when inequality (14) is reversed, can we show that all x near 0 match with job type $\mathcal{H}$ for high enough δ? In the continuum case we showed PAM failed by considering an infinitesimal rematching at the extremes. Of course, in the discrete jobs model we cannot do such a marginal rematching. Instead, our approach in the Appendix is to compare the static loss from matching with $\mathcal{H}$ rather than $\mathcal{L}$ to the dynamic gain from such a rematching for x near the extremes $\{0, 1\}$. It turns out that for all values of δ, the static loss of non-assortative matching dominates any dynamic gain. Specifically:

Proposition 6. *In the employment model, assortative matching obtains for all reputations in an open interval around each extreme reputation at all discount factors $\delta < 1$.*

Altogether, the partnership model that we have introduced radically differs from the analogous employment model. In the partnership model, failures of assortative matching robustly obtain at the extreme reputations and thus assortative matches eventually break up. Conversely, in the employment model, assortative matching fails for interior reputations, and all workers are assortatively matched in the long run.

5. CONCLUSION

We have developed a general dynamic matching model in which the characteristics of agents stochastically evolve depending on their chosen match partner. We have shown that a steady state competitive equilibrium exists and coincides with a competitive equilibrium in which agents choose their partner.

Within this general framework, we have considered two applications in which the characteristics that evolve can be interpreted as reputations. In the partnership model, we found that contrary to Becker (1973), production complementarity no longer implies global PAM. We instead find a conflict between productive and dynamic efficiency. Given enough patience, PAM cannot arise in a stylized two period model, while PAM may well be dynamically efficient for high or low types. We have argued, however, that it cannot *globally* be dynamically best when agents are patient enough: *either* high *or* low reputation agents will match non-assortatively, not necessarily both. What matters is a new statistically-based condition on the production technology alone that is completely unrelated to supermodularity. Our proof also relies on a knife-edge trade-off between dynamic and productive efficiency, as δ races up to 1.

In the employment model, we have shown that informational concerns dominate productive concerns for workers with reputations near some interior cutoff, while productive concerns dominate near extreme reputations. Thus, unlike in the partnership model, workers with extreme reputations will be assortatively matched in the employment model.

This paper offers both theoretical and applied insights. First, we have developed a proof by counterfactual for the failure of PAM by exploiting the convexity of shadow values. Second, we have shown that our learning model provides a single coherent framework for understanding a host of time series and cross-sectional properties of the labour market, ranging from job tenure to wage growth. These properties can therefore be seen as owing to the factors identified here. Finally, we also identify a new phenomenon–the efficient break-ups of matched stars (like the Beatles) in industries with high skill technology–on which more empirical work is needed.

APPENDIX A. EXISTENCE, WELFARE THEOREMS, AND VALUES

Here, we assume that the social planner chooses the measure over matches μ, where $\mu(X \times Y)$ is the mass of matches (x, y) in the product space $[0, 1]^2$, where $x \in X$ and $y \in Y$, for measurable sets $X, Y \subset [0, 1]$. Let $\lambda_G(A)$ be the measure of any measurable sets $A \subset [0, 1]$ associated with cdf G (see Theorem 12.4 in Billingsley (1995) for existence and uniqueness of such a measure).

A.1. *Steady-state Pareto optima: proof of Theorem 2*

Equip[10] $W \equiv \mathcal{L}_\infty([0, 1]^2)$ with the standard norm topology. The dual W^* of W is the space of bounded measures on $[0, 1]^2$, in which our joint cdf's F dwell. Endow W^* with the weak* topology. Let Z be the space of cdfs on $[0, 1]$ with the sup-norm (discrepancy metric) defined on their measures, i.e. $||G - \hat{G}|| = \sup_A |\lambda_G(A) - \lambda_{\hat{G}}(A)|$. Let $\Phi : Z \rightrightarrows W^*$ be the correspondence from distributions $G \in Z$ into feasible matchings given (2):

$$\Phi(G) = \{\mu \in W^* : \lambda_G(A) \geq \mu(A \times [0, 1]) \ \forall A \text{ measurable}\}. \tag{A1}$$

Claim 1. *The correspondence Φ given by (A1) is continuous and compact valued.*

Proof. By Alaoglu's Theorem (see Royden, 1988, Section 10.6), if $\Phi(G) \subset W^*$ is weak* closed, bounded, and convex, then $\Phi(G)$ is weak* compact. Convexity and boundedness are immediate. For weak-* closed, let $\mathbb{I}_Y$

10. We are indebted to Ennio Stacchetti for providing the key insights for this proof.

be the indicator function of the set Y. Let $\mathbb{B}^A_{[a,b]} = \{\mu : a \leq \int \mathbb{I}_{A\times[0,1]} d\mu \leq b\}$, and likewise define notation for open intervals and half-open and half-closed intervals. Note that $\Phi(G) = \bigcap_A \mathbb{B}^A_{[0,\lambda_G(A)]}$ and that $\mathbb{I}_{A\times[0,1]} \in W$. By definition, $\mathbb{B}^A_{[0,\lambda_G(A)]}$ is weak* closed, and thus $\Phi(G)$ is weak* closed.

We show that this correspondence is upper and lower hemi-continuous (u.h.c. and l.h.c.). Now Φ is pointwise closed, and we can assume without loss of generality that it maps into a compact subset of W^*, say with upper bound $M < \infty$. Thus, u.h.c. follows if Φ has the closed graph property–i.e. if for any $G \in Z$: $\mu \notin \Phi(G)$ implies that $\mathcal{O} \cap \Phi(G) = \emptyset$ for some open set $\mathcal{O}$ containing μ. But $\mu \notin \Phi(G)$ if $\mu > \lambda_G(A)$ for some A. The result follows from continuity of $\lambda_G(A) = \int \mathbb{I}_{A\times[0,1]} d\mu$ in μ.

Recall that a correspondence Φ is l.h.c. at $G \in Z$ if for every open set $\mathcal{O}$ in W^* with $\mathcal{O} \cap \Phi(G) \neq \varnothing$, there exists $\eta > 0$ such that $\mathcal{O} \cap \Phi(\hat{G}) \neq \varnothing$ for all $\hat{G}$ with $\|G - \hat{G}\| < \eta$. We need only consider (basis) open sets of the form $\mathcal{O} = \bigcap_{k=1}^m \mathbb{B}^{A_k}_{(a_k,b_k)}$. Pick $\mu \in \mathcal{O} \cap \Phi(G)$. Define μ_ϵ by $\mu_\epsilon(A \times [0,1]) \equiv \mu(A \times [0,1]) - \epsilon\lambda(A)$ for all A. We claim that there exists $\epsilon, \eta > 0$ such that $\mu_\epsilon \in \mathcal{O} \cap \Phi(\hat{G})$ for all $\hat{G} \in Z$ with $\|\hat{G} - G\| < \eta$. Pick any such $\hat{G}$. Easily, $\mu_\epsilon \in \mathcal{O}$ for small $\epsilon > 0$. We show that $\mu_\epsilon \in \Phi(\hat{G})$ for any $\eta \leq \epsilon$, namely $\lambda_{\hat{G}}(A) \geq \mu_\epsilon(A \times [0,1])$ for all A. Since $\|G - \hat{G}\| < \eta$ and $\lambda(A) \leq 1$, this follows if $|\lambda_G(A) - \lambda_{\hat{G}}(A)| < \eta\lambda(A)$ for all A, or simply if $\lambda_{\hat{G}}(A) > \lambda_G(A) - \eta\lambda(A)$. But $\lambda_G(A) - \eta\lambda(A) \geq \mu(A \times [0,1]) - \epsilon\lambda(A)$ given $\lambda_G(A) \geq \mu(A \times [0,1])$ and $\eta \leq \epsilon$. ‖

Using the norm $\|\mathcal{V}\| = \sup_{\|G\|\leq 1} |\mathcal{V}(G)|$ on value functions, define:

$$V = \{\mathcal{V} : Z \to \mathbb{R} : \mathcal{V} \text{ is homogeneous of degree 1, continuous, and } \|\mathcal{V}\| < \infty\}.$$

The planner's present value of output

$$\Gamma(\mathcal{V}, \mu) = (1-\delta) \int Q d\mu + \gamma \mathcal{V}((1-\sigma)\overline{G} + \sigma B(\mu)).$$

For $G \in Z$, then define the Bellman operator T on V by

$$T\mathcal{V}(G) = \max_{\mu \in \Phi(G)} \Gamma(\mathcal{V}, \mu).$$

Claim 2. *T: $V \to V$.*

Proof. The mapping clearly preserves boundedness and homogeneity. We now show that T preserves continuity. First, $\Gamma(\mathcal{V}, \mu)$ is weak* continuous in $(\mathcal{V}, \mu) \in V \times W^*$. Indeed, $\mu \mapsto \int Q d\mu$ and $\mu \mapsto B(\mu)$ are bounded linear operators (recall $\rho \in W^*$), and thus are weak* continuous on W^*. For each $\mathcal{V} \in V$, the composition $\mathcal{V}(B(\mu))$ is continuous in μ. Thus, $(\mathcal{V}, \mu) \mapsto \Gamma(\mathcal{V}, \mu)$ is continuous. Also, the constraint correspondence is continuous and compact-valued by Claim 1. Then $T\mathcal{V}$ is continuous by the generalization of Berge's Theorem of the Maximum in Robinson and Day (1974). ‖

Claim 3. *For any cdf G, a Pareto optimal value $\mathcal{V}$ and matching measure μ exists.*

Proof. First, the Bellman operator T is a contraction. Indeed, T is monotonic and $T(\mathcal{V} + c) = T\mathcal{V} + \gamma c$, where $0 < \gamma < 1$ and c is real. Thus, T is a contraction by Blackwell's Theorem, has a unique fixed point $\mathcal{V}$ in V by the Banach Fixed Point Theorem. So $\mathcal{V}$ is continuous, as is the composition $\mathcal{V}(B(\mu))$. Thus, the maximizer μ of the continuous function $\Gamma(\mathcal{V}, \mu)$ on the compact constraint set $\Phi(G)$ exists. ‖

Claim 4. *There exists a cdf G and a matching measure μ that is a steady-state PO.*

Proof. Define the correspondence $T^* : Z \to W^*$ by $T^*(G) = \arg\max_{\mu \in \Phi(G)} \Gamma(\mathcal{V}, \mu)$, where $G \in Z$. Let $\Theta : W^* \to Z$ be the function capturing the transition equation: $\Theta(\mu) = (1-\sigma)\overline{G} + \sigma B(\mu)$, for $\mu \in W^*$. If the map $T*^\circ \Theta : W^* \to W^*$ has a fixed point $G^* = T^*(G^*)$, then we can assume a constant optimal matching measure $\mu^* \in \Phi(G^*)$.

By an extension of the Kakutani Fixed Point Theorem in (1981, Section 10), it suffices that $T*^\circ \Theta$ be nonempty, convex-valued, closed-valued, and u.h.c. Claim 3 yields non-emptiness. Now, T^* is u.h.c. and closed-valued by Robinson and Day (1974). It is convex-valued, as Γ is linear in μ and the constraint set $\Phi(G)$ is convex. Claim 2 proved Θ continuous. As a composition of a continuous function and a u.h.c. correspondence, $T*^\circ \Theta$ is u.h.c. Also, Θ is linear in μ, and preserves closedness and convexity; thus, $T*^\circ \Theta$ is convex-valued and closed-valued. It has a fixed point $G^* = T^*(G^*)$. ‖

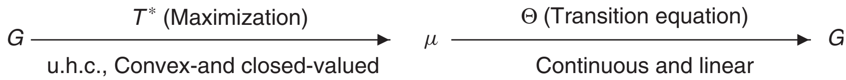

FIGURE A1

Existence. This schematic illustrates the proof of steady state existence

A.2. *Welfare theorems: proof of Theorem 3*

A. First welfare theorem. Define $\langle Q, F\rangle \equiv (1-\delta)\sum_{t=0}^{\infty}\gamma^t \int_{[0,1]^2} Q(x, y)F(dx, y)dy$. Assume that (F, v, w) is a CE, but F is not a PO. Thus, there exists feasible $\hat{F}$ with $\langle Q, \hat{F}\rangle > \langle Q, F\rangle$. Define $w^y(x, y) = w(y|x)$. By definition of a competitive equilibrium and (3), we have $\langle w + w^y, F\rangle = \langle Q, F\rangle$ and $\langle w + w^y, \hat{F}\rangle = \langle Q, \hat{F}\rangle$. Hence, $\langle w + w^y, \hat{F}\rangle > \langle w + w^y, F\rangle$. By symmetry, $\langle w, \hat{F}\rangle = \langle w^y, \hat{F}\rangle$, and so $\langle w, \hat{F}\rangle > \langle w, F\rangle$.

Let $\hat{G}$ be the density associated with the matching $\hat{F}$, and let $\rho_t(z, x, F)$ be the chance that agent x at time 0 updates to z at time t. By worker maximization (7),

$$\sum_t \delta^t \int \int_{z\in \operatorname{supp} G} w(z|y)\rho_t(z, x, F)dF(z|y)dy \geq \sum_t \delta^t \int \int_{z\in \operatorname{supp} G} w(z|y)\rho_t(z, x, \hat{F})d\hat{F}(z|y)dy.$$

Integrate over x to get $\langle w, F\rangle \geq \langle w, \hat{F}\rangle$, contrary to $\langle w, \hat{F}\rangle > \langle w, F\rangle$. Thus, F is a PO.

To prove that v is a multiplier in the planner's problem for the given (efficient) F, we show that (F, v) satisfies the planner's FOC. Take any matched pair (x, y). If we sum the worker maximization conditions (7) for x and y we obtain:

$$v(x) + v(y) = (1-\delta)(w(x|y) + w(y|x)) + \delta\Psi^v(x, y).$$

Since $w(x|y) + w(y|x) = Q(x, y)$, the planner's FOC (6) is satisfied for this matched pair. Now take any (x, y) (not necessarily matched). Worker maximization (7) implies:

$$v(x) \geq (1-\delta)w(x|y) + \delta\psi^v(x|y) \quad \text{and} \quad v(y) \geq (1-\delta)w(y|x) + \delta\psi^v(y|x).$$

Summing these two inequalities and applying (3) yields:

$$v(x) + v(y) \geq (1-\delta)(w(x|y) + w(y|x)) + \delta\Psi^v(x, y) = (1-\delta)Q(x, y) + \delta\Psi^v(x, y). \qquad \|$$

B. Second welfare theorem. Let (F, v) be a PO. Assume that the pairs (x, y) and $(\hat{x}, \hat{y})$ are matched in the PO, but there does not exist a competitive equilibrium in which these pairs are matched. Let $V(x, y) \equiv (1-\delta)Q(x, y) + \delta\Psi^v(x, y)$. By definition of PO, we have:

$$V(x, y) + V(\hat{x}, \hat{y}) \geq V(x, \hat{y}) + V(\hat{x}, y).$$

As this holds for *any* matched pairs, output shares w exist so that (F, v, w) is a CE. $\|$

APPENDIX B. NON-ASSORTATIVE MATCHING

B.1. *Asymptotic analysis for Lemma 1 and Proposition 3*

We proceed by contradiction, assuming PAM and using the implied value function v_δ. Indeed, given $v'_\delta(0) = M - L$ and $v'_\delta(1) = H - M$ fixed, $\int_0^1 v''_\delta(x)dx$ is constant in δ. Since the value function flattens ($\lim_{\delta\to 1} v''_\delta(x) = 0$) for any interior $x \in (0, 1)$, convexity must accumulate at 0 and 1, as in Figure 3. Curiously, we now show that $\delta\sum_i m_i^2/\ell_i > 1$ suffices near $x = 0$, so that the convexity explosion may occur far from $\delta = 1$: for instance, if $\boldsymbol{\ell} = (0.01, 0.99)$ and $\boldsymbol{m} = \boldsymbol{h} = (0.99, 0.01)$, then $\delta > (\sum_i m_i^2/h_i)^{-1} \approx 1/98$ works.

Claim 5. *Assume PAM and $\delta\sum_i m_i^2/\ell_i > 1$. Define $\alpha_\delta \in (0, 1)$ by (10). Then:*

(*a*) $v''_\delta(x) \sim \kappa_\delta x^{-\alpha_\delta}$ *near $x = 0$, where $\kappa_\delta > 0$*
(*b*) $\lim_{\delta\uparrow 1} \kappa_\delta/(1-\alpha_\delta) = \kappa$*, where $\kappa > 0$.*

Likewise, for $\delta < 1$ big enough that $\delta \sum_i m_i^2/h_i > 1$, we have $v''_\delta(1-x) \sim \underline{\kappa}_\delta x^{-\beta_\delta}$ near $x = 1$ for $\beta_\delta \in (0, 1)$ given by (10); further, $\underline{\kappa}_\delta/(1-\beta_\delta) \to \underline{\kappa}$ as $\beta \uparrow 1$, for some $\underline{\kappa} > 0$.

Part (a) provides a functional form for the second derivative near $x = 0$ at fixed δ. Essentially, v'' geometrically blows up as fast as it can and still leave v' integrable.

Proof. By Anderson (2007),[11] for any twice continuously differentiable policy function $a(x)$, the second derivative $v''(x)$ obeys part (a) if $v(x)$ is convex, the static output function $Q(x, a(x))$ is strictly convex, and $\delta \sum_i m_i^2/\ell_i > 1$ holds. These assumptions obtain here, since $a(x) = x$, and $Q(x, x)$ is strictly convex.

Part (b) also follows from Anderson (2007). Beyond the assumptions needed for part (a), the derivative of the policy function must be finite at $x = 0$, and $|p_{iy}(x, y)| < \infty$. Since $a(x) = x$ and $p_i(x, y)$ is linear, these assumptions are met. ‖

Define $R(\alpha) \equiv \sum_i (h_i - m_i)(m_i/\ell_i)^{1-\alpha} + (1-\alpha)\sum_i [(h_i\ell_i - m_i^2)/\ell_i](m_i/\ell_i)^{1-\alpha}$.

Claim 6 (Diagonal cross partial). *$\Psi^\delta_{12}(x, x) \sim \kappa_\delta R(\alpha_\delta) x^{1-\alpha_\delta}/(1-\alpha_\delta)$ near $x = 0$, where $R(1) = 0$ and $R'(1) = S(h, m, \ell)$.*

Proof. Computing $\Psi^\delta_{12}(x, x)$ we have:

$$\theta_\delta(x) \equiv \Psi^\delta_{12}(x, x) = \textstyle\sum_i (\pi_i v_\delta(z_i) + a_i(x) v'_\delta(z_i) + b_i(x) v''_\delta(z_i)), \tag{B1}$$

where $\pi_i = h_i - 2m_i + \ell_i$, $a_i(x) = h_i - m_i + O(x)$, $b_i(x) = [(h_i\ell_i - m_i^2)m_i/\ell_i^2]x + O(x^2)$.[12]

We can integrate our asymptotic approximation from Claim 5, written as $v''_\delta(x) = \kappa_\delta x^{-\alpha_\delta}(1 + o(1))$, to yield expressions for v_δ and v'_δ in (B1). Making this substitution and $z_i(x) = (m_i/\ell_i)x + O(x^2)$, the terms in $\theta_\delta(x)$ are:

$$\begin{aligned}
\sum_i \pi_i v_\delta(z_i) &= \sum_i \pi_i \left[\frac{\kappa_\delta}{(1-\alpha_\delta)(2-\alpha_\delta)} z_i^{2-\alpha_\delta} + v'_\delta(0) z_i + v_\delta(0) + o(x^{2-\alpha_\delta})\right] \\
&= \left(\textstyle\sum_i \pi_i (m_i/\ell_i)\right) v'_\delta(0) x (1 + o(1)) \\
\sum_i a_i(x) v'_\delta(z_i) &= \sum_i (h_i - m_i + O(x)) \left[v'_\delta(0) + \frac{\kappa_\delta}{1-\alpha_\delta} z_i^{1-\alpha_\delta}(1 + o(1))\right] \\
&= \frac{\kappa_\delta}{1-\alpha_\delta} \sum_i (h_i - m_i) \left(\frac{m_i}{\ell_i}\right)^{1-\alpha_\delta} x^{1-\alpha_\delta}(1 + o(1)) \\
\sum_i b_i(x) v''_\delta(z_i) &= \kappa_\delta \sum_i \left[\frac{(h_i\ell_i - m_i^2)m_i}{\ell_i^2} x + O(x^2)\right] \left(\frac{m_i}{\ell_i} x\right)^{-\alpha_\delta} (1 + o(1)) \\
&= \kappa_\delta \sum_i \frac{h_i\ell_i - m_i^2}{\ell_i} \left(\frac{m_i}{\ell_i}\right)^{1-\alpha_\delta} x^{1-\alpha_\delta}(1 + o(1)).
\end{aligned}$$

Thus,

$$\theta_\delta(x) = \frac{\kappa_\delta}{1-\alpha_\delta} R(\alpha_\delta) x^{1-\alpha_\delta}(1 + o(1)).$$

We need to know that this term is negative and bounded away from zero as $\delta \to 1$. We have already shown that $\kappa_\delta > 0$ for $\delta < 1$ (Claim 5 (a)). Thus, we need only show now that $R(\alpha) < 0$ for large $\alpha < 1$, since $\alpha_1 = 1$ and α_δ is increasing in δ near 1. Given $R(1) = 0$, we are done since $R'(1) > 0$, for then $\theta_\delta(x) < 0$ for δ near 1 and small x. ‖

11. We can translate to the Anderson (2007) framework by letting the action $a(x)$ be the reputation of the agent to match with x, and $h_i y + m_i(1-y)$ and $m_i y + \ell_i(1-y)$ be the probabilities of realizing signal q_i given action y in the high and low state of the world, respectively.

12. $\phi(x) = o(g(x))$ iff $\lim_{x\to 0} \phi(x)/g(x) = 0$ while $\phi(x) = O(g(x))$ if $\limsup_{x\to 0} |\phi(x)/g(x)| < \infty$.

B.2. *Proof that PAM implies $v'_\delta(1) - v'_\delta(0) = \pi$*

PAM yields $v_\delta(x) = (1-\delta)Q(x,x)/2 + \delta \sum_i v_\delta(z_i(x,x))$. Anderson (2007) proves that v'_δ exists and is continuous on $[0,\varepsilon]$, for some $\varepsilon > 0$. Differentiation yields:

$$v'_\delta(0) = (1-\delta)Q'(0,0)/2 + \delta v'_\delta(0)$$

since $z_i(0,0) = 0$, $\sum_i p'_i(0,0) = 0$ and $\sum_i z'_i(0,0)p_i(0,0) = 1$. Thus, $v'_\delta(0) = Q'(0,0)/2$. Similarly, $v'_\delta(1) = Q'(1,1)/2$, and so $v'_\delta(1) - v'_\delta(0) = [Q'(1,1) - Q'(0,0)]/2 = \pi$. ‖

B.3. *PAM almost always fails: proof of Proposition 4*

As we have shown, PAM fails near $x = 0$ iff: $\pi - S(h,m,\ell)\dot{\alpha}_1\kappa < 0$, where $\kappa = \lim_{\delta\to 1}\kappa_\delta/(1-\alpha_\delta) > 0$. Thus, we need the measure of parameters (h,m,ℓ) for which this inequality fails (the opposite weak inequality holds) to vanish as N increases.

We cannot apply a Law of Large Numbers, since the summands are dependent (e.g. $\sum_i h_i = 1$), and the domain $(\Delta_N)^3$ changes in N. But if we can show that under a uniform distribution over parameters the probability that PAM fails converges to 1, then the measure of the parameters $(\boldsymbol{h},\boldsymbol{m},\boldsymbol{\ell})$ for which PAM fails also converges to 1. Thus we assume that the probabilities $\boldsymbol{h},\boldsymbol{m},\boldsymbol{\ell}$ are each uniformly distributed on Δ_N, i.e. the density Δ_N is $(N-1)!$ and has marginals (λ is Lebesgue measure) $d\lambda_i(h_i^N) = (N-1)(1-h_i^N)^{N-2}$ and $d\lambda_{ij}(h_i^N, h_j^N) = (N-1)(N-2)(1-h_i^N-h_j^N)^{N-3}$.

Define the random variables, $S_N = S(\boldsymbol{h},\boldsymbol{m},\boldsymbol{\ell})$ and $A_N = 1/\dot{\alpha}_1 = \sum_i m_i^N \log(m_i^N/\ell_i^N)$. We need $Pr[S_N/A_N > \pi/c] \to 1$. We shall establish this in two steps:

Step 1. $A_N \to 1$ a.s. as $N \to \infty$.

Define $\chi_i^N = m_i^N \log(m_i^N/\ell_i^N)$. Direct calculation with the densities yields: $E[\sum_i \chi_i^N] = NE\chi_i = (N-1)/N$. If we can show $var(\sum_i \chi_i^N)) \to 0$, we are done.

$$\text{var}\left(\sum_i \chi_i^N\right) = N\text{var}(\chi_i^N) + N(N+1)\text{cov}(\chi_i^N, \chi_j^N).$$

By direct calculation, $\text{cov}(\chi_i^N, \chi_j^N) \le 0$. Alternatively, each (m_i^N, m_j^N) pair is only correlated by the restriction $\sum_i m_i^N = 1$, and thus negatively (likewise (ℓ_i^N, ℓ_j^N)). This negative correlation and the monotonicity of χ_i^N in m_i^N and ℓ_i^N imply $\text{cov}(\chi_i^N, \chi_j^N) \le 0$.

So, $N\text{var}(\chi_i^N) \to 0$ is sufficient to establish the result. Substituting from above: $\text{var}(\chi_i^N) = E[(\chi_i^N)^2] - (N-1)^2/N^4$, where direct calculation yields for large N:

$$E[(\chi_i^N)^2] \propto \frac{\Gamma[N]}{\Gamma[N+2]}\left(\frac{\Gamma'[N]}{\Gamma[N]}\right)^2$$

where $\Gamma(N) \equiv \int_0^\infty s^{N-1}e^{-s}ds$ is the gamma function, i.e. $\Gamma'(N) = \int_0^\infty \log(s)s^{N-1}e^{-s}ds$. Since the '$\psi$ function' obeys $\Gamma'(N)/\Gamma(N) \sim \log N$ as $N \to \infty$, we have:

$$NE[(\chi_i^N)^2] \propto \frac{N\Gamma[N]}{\Gamma[N+2]}(\log N)^2.$$

This converges to 0 as $N \to \infty$. So $\text{var}(\sum_i \chi_i^N) \to 0$, and Step 1 is true.

Step 2. $\Pr[S_N/N \ge 2] \to 1$.

Define $s_N^1 = (1/N)\sum_i \log(m_i^N/\ell_i^N)(m_i^N - h_i^N)$ and $s_N^2 = (1/N)\sum_i (m_i^N)^2/\ell_i^N$, so that $S_N/N = s_N^1 + s_N^2$.

Claim 7. *The sum s_N^1 vanishes in probability: $s_N^1 \Rightarrow 0$.*

Proof. First $Es_N^1 \to 0$. Given the above densities, $Es_N^1 = ((N-1)/N^2)(\varsigma + \Gamma'(N)/\Gamma(N))$, where ς is Euler's constant. So $Es_N^1 > 0$.

Then by routine calculations, $\text{var}(s_N^1) \to 0$. Thus, $Pr[|s_N^1 - Es_N^1| \ge \varepsilon] \to 0\ \forall\varepsilon$. ‖

Claim 8. *The sum s_N^2 converges in probability to some constant ≥ 2.*

Proof. Define $\tilde{\ell}_i^N = \ell_i^N$ if $\ell_i^N \ge 1/N^2$ and $1/N^2$ otherwise. Let $\tilde{s}_N^2 = s_N^2$ with ℓ_i^N replaced with $\tilde{\ell}_i^N$, and note that $s_N^2 \ge \tilde{s}_N^2$. It suffices to prove $\tilde{s}_N^2 \Rightarrow 2$.

We first claim that $E\tilde{s}_N^2 \to 2$. Indeed, if $\rho^N = 1-(1-1/N^2)^{N-1}$ is the chance that $\ell_i \le 1/N^2$, then $E(1/\tilde{\ell}_i^N) = N^2\rho^N + \int_{1/N^2}^1 (N-1)(1-s)^{N-2}/s ds \le N^2$. Letting $\rho^N \to 1$ yields $E(1/\tilde{\ell}_i^N) = N^2 + o(N^2)$. Finally, $E(m_i^N)^2 = 2/N(N+1)$, so that $E\tilde{s}_N^2 \to 2$.

It thus suffices that $Pr[|\tilde{s}_N^2 - E\tilde{s}_N^2| \ge \varepsilon] \to 0\ \forall\varepsilon$. By similar reasoning as above, $E(1/(\tilde{\ell}_i^N)^2) = N^4 + o(N^4)$, and $E((1/\tilde{\ell}_i^N)(1/\tilde{\ell}_j^N)) = N^4 + o(N^4)$. Also, $E((m_i^N)^4) = 24/(N(N+1)(N+2)(N+3))$, and $E((m_i^N)^2(m_j^N)^2) = 4/(N(N+1)(N+2)(N+3))$. Then the variance of $\tilde{s}_N^2$ equals

$$\frac{1}{N}E\frac{(m_i^N)^4}{(\tilde{\ell}_i^N)^2} + \frac{N-1}{N}E\frac{(m_i^N m_j^N)^2}{\tilde{\ell}_i^N\tilde{\ell}_j^N} - \left(E\frac{(m_i^N)^2}{\tilde{\ell}_i^N}\right)^2 = \frac{4N^2(N-1)}{(N+1)^2(N+2)(N+3)} + o(N)$$

using the independence of m^N from $\tilde{\ell}^N$–i.e., $\text{var}(\tilde{s}_N^2) \to 0$. ||

B.4. *Long run match dynamics: proof of Proposition 5*

Case (a). By Easley and Kiefer (1988), the reputation may only converge to a potentially confounding point, where it is unchanged. If $x \neq 0$, then $z_i(x,y) = x$ *iff* $p_i(1,y) = p_i(x,y)$. Since $\partial p_i(x,y)/\partial x$ is constant in x, this requires $\partial p_i(x,y)/\partial x = (h_i + \ell_i - 2m_i)y + m_i - \ell_i = 0$, or $y = (m_i - \ell_i)/(h_i + \ell_i - 2m_i)\ \forall\ i$, contrary to Assumption 2. As there are no interior potentially confounding beliefs, the long run density g_∞ has support $\{0, 1\}$. Reputation being a martingale, the weights are $(1 - x_0, x_0)$. ||

Case (b). Let $P_{\mathcal{HH}}, P_{\mathcal{HL}}, P_{\mathcal{LH}}, P_{\mathcal{LL}}$ be the current beliefs over the possible state space $\{\{\mathcal{H},\mathcal{H}\}, \{\mathcal{H},\mathcal{L}\}, \{\mathcal{L},\mathcal{L}\}, (\mathcal{L},\mathcal{L})\}$. Let $\hat{P}_{\mathcal{HH}} \equiv h_i P_{\mathcal{HH}}/(h_i + m_i(P_{\mathcal{HL}} + P_{\mathcal{LH}}) + \ell_i P_{\mathcal{LL}})$ be the updated chance that the state is $\{\mathcal{H},\mathcal{H}\}$, and define similar updated beliefs $\hat{P}_{\mathcal{HL}}, \hat{P}_{\mathcal{LH}}, \hat{P}_{\mathcal{LL}}$. Note that $\hat{P}_{ii} = P_{ii}$ iff $P_{ii} = 1$, while $\hat{P}_{ij} = P_{ij}$ iff $P_{ij} = 1$ or $P_{ij} = P_{ji} = 1/2$. Thus, these are the only potentially confounding beliefs. ||

APPENDIX C. ANALYSIS FOR DISCRETE JOBS BENCHMARK

C.1. *Extremal approximation given matching*

Near the extremes $x = 0$ and $x = 1$ the same functional asymptotic approximation for $v(x)$ holds as in the partnership model when we assumed PAM. Near 0, the value function $v(x)$ is approximately equal to a linear function plus $x^{\alpha_\theta(\delta)}$ if the worker is matched to a type $\theta \in \{\mathcal{H}, \mathcal{L}\}$ firm where the $\alpha_\theta(\delta)$ solve:

$$1 \equiv \delta\sum_i m_i\left(\frac{h_i}{m_i}\right)^{\alpha_{\mathcal{H}}(\delta)} \quad\text{and}\quad 1 \equiv \delta\sum_i \ell_i\left(\frac{m_i}{\ell_i}\right)^{\alpha_{\mathcal{L}}(\delta)}.$$

Static payoffs are trivially convex in x, and static payoffs $z_i(x,y)$, and $p_i(x,y)$ are all twice differentiable in x. Further, beliefs are a martingale and near $x = 0$ the belief process is approximately linear, satisfying:

$$\begin{array}{lll} z_i(x,1) \sim (h_i/m_i)x & p_i(x,1) \sim m_i & x \to 0 \\ z_i(x,0) \sim (m_i/\ell_i)x & p_i(x,0) \sim \ell_i & x \to 0. \end{array}$$

Together these assumptions yield some $\delta^* < 1$ such that the desired match contingent value approximations hold with $\alpha_\theta(\delta) \in (1,2)$ continuously falling in δ for all $\delta \in (\delta^*, 1)$ and $\alpha_\theta(1) = 1$ by Lemma 2 and Proposition 1 in Anderson (2007).

C.2. *Extremal ordering of values for high δ*

We have shown that when matching with firm type θ, the worker's value function is approximately linear plus $x^{\alpha_\theta(\delta)}$ near $x = 0$. Thus, for low enough x the value to matching with a type $\mathcal{H}$ job will be higher than matching with a type $\mathcal{L}$ job *iff* $\alpha_{\mathcal{H}}(\delta) < \alpha_{\mathcal{L}}(\delta)$. Since we have $\alpha_{\mathcal{H}}(1) = \alpha_{\mathcal{L}}(1) = 1$ and $\alpha'_\theta(\delta) < 0$, for δ close to 1:

$$|\alpha'_{\mathcal{H}}(1)| < |\alpha'_{\mathcal{L}}(1)| \Leftrightarrow \alpha_{\mathcal{H}}(\delta) < \alpha_{\mathcal{L}}(\delta).$$

By the Implicit Function Theorem:

$$\alpha'_{\mathcal{H}}(1) = -\left(\sum_i h_i \log\left(\frac{h_i}{m_i}\right)\right)^{-1} \quad\text{and}\quad \alpha'_{\mathcal{L}}(1) = -\left(\sum_i m_i \log\left(\frac{m_i}{\ell_i}\right)\right)^{-1}$$

and so:

$$|\alpha'_{\mathcal{H}}(1)| < |\alpha'_{\mathcal{L}}(1)| \Leftrightarrow \sum_i h_i \log\left(\frac{h_i}{m_i}\right) > \sum_i m_i \log\left(\frac{m_i}{\ell_i}\right).$$

C.3. *PAM still obtains at extremes with high δ*

We now show that the static effect dominates near $x = 0$ for δ arbitrarily close to 1. Matching with job type $\mathcal{H}$ rather than job type $\mathcal{L}$ results in a static loss valued at:

$$(1-\delta)\left((H-M)x + (M-L)(1-x)\right)$$

which converges to 0 as $\delta \to 1$ and to a positive constant as $x \to 0$ for any fixed $\delta < 1$.

Given our asymptotic approximation, the dynamic change in value is approximately:

$$\Delta\psi(x,\delta) \equiv \delta\left[x^{\alpha_{\mathcal{H}}(\delta)} \sum_i m_i \left(\frac{h_i}{m_i}\right)^{\alpha_{\mathcal{H}}(\delta)} - x^{\alpha_{\mathcal{L}}(\delta)} \sum_i \ell_i \left(\frac{m_i}{\ell_i}\right)^{\alpha_{\mathcal{L}}(\delta)}\right].$$

But since $\alpha_\theta(1) = 1$, $\lim_{\delta\to 1} \Delta\psi(x,\delta) = 0$, static and dynamic effects vanish as $\delta \to 1$.

To determine the rate of convergence, we normalize the static and dynamic effect, dividing both by $1-\delta$. The static loss is then simply a linear function of x, converging to $(M-L) > 0$ as $x \to 0$. The dynamic effect is more complicated. By L'Hopital, we have:

$$\lim_{\delta\to 1} \frac{\Delta\psi(x,\delta)}{1-\delta} = \lim_{\delta\to 1} -\Delta\psi_\delta(x,\delta),$$

where $\lim_{\delta\to 1} \Delta\psi_\delta(x,\delta)$ equals

$$\begin{aligned} & \alpha'_{\mathcal{H}}(1)\left[x\log x + x\sum_i h_i \log\left(\frac{h_i}{m_i}\right)\right] - \alpha'_{\mathcal{L}}(1)\left[x\log x + x\sum_i m_i \log\left(\frac{m_i}{\ell_i}\right)\right] \\ = \quad & x\log x\left[\frac{1}{\sum_i m_i \log\left(\frac{m_i}{\ell_i}\right)} - \frac{1}{\sum_i h_i \log\left(\frac{h_i}{m_i}\right)}\right] \end{aligned}$$

which is $k\ x\log x$ for some constant k. So even if the dynamic gain to matching x near 0 with $\mathcal{H}$ rather than $\mathcal{L}$ is positive, assortative matching obtains for all x such that:

$$k\ x\log x < (H-M)x + (M-L)(1-x).$$

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